PROCEEDINGS BOOK

of

MICOPAM 2025

The 8th Mediterranean International Conference of Pure & Applied Mathematics and Related Areas

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on the occasion of his 81st birth anniversary

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Proceedings Book of the 8th Mediterranean International Conference of Pure & Applied Mathematics and Related Areas (MICOPAM 2025)

DEDICATED TO PROFESSOR MANUEL LÓPEZ-PELLICER ON THE OCCASION OF HIS 81ST BIRTH ANNIVERSARY

CONFERENCE VENUE

University of Osijek

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Osijek, 2025

ABOUT THE CONFERENCE

The MICOPAM conference series began with its first event in Antalya, Turkey, in 2018, with the next edition held in Paris, France, in 2019. The third conference, which was postponed in 2020 due to the coronavirus pandemic, was held in Antalya, Turkey, as MICOPAM 2020-2021, together with the fourth one. The fifth conference was also held in Antalya, Turkey, in 2022. The sixth conference was held in Paris, France, in 2023. The seventh conference was also held in Antalya, Turkey, in 2024. Over the last seven years, this conference has brought together researchers from all over the world, who work in the fields of pure and applied mathematics and related areas.

The 8th Mediterranean International Conference of Pure & Applied Mathematics and Related Areas (MICOPAM 2025) was held at the School of Applied Mathematics and Informatics, University of Osijek, in Osijek, Croatia, over five days, from September 8 to 12, 2025, and was dedicated to the esteemed mathematician Professor Manuel López-Pellicer on the occasion of his 81st birthday.

The aim of the MICOPAM 2025 conference was to bring together leading scientists in the fields of pure and applied mathematics and related areas to present their research, exchange new ideas, discuss challenging issues, foster future collaborations and interact with each other.

The MICOPAM 2025 conference took place in a hybrid form with both physical and virtual (online) participation, over five days from September 8 to 12, 2025. The conference was attended by 65 participants from 20 different countries. Among them, three participants attended the conference as listeners, and only one participant gave two presentations.

During the five days of MICOPAM 2025, the 65 participants delivered a total of 63 distinct presentations, representing 20 different countries [Albania (1), Algeria (1), Chile (1), Croatia (11), Czech Republic (3), France (1), Hungary (1), Iran (1), Iraq (1), Japan (1), Mexico (1), Poland (2), Portugal (1), Russia (1), Saudi Arabia (6), Serbia (4), South Africa (1), Spain (2), Taiwan (1), Turkey (22)].

In addition to a great number of excellent presentations, three participants attended the conference as listeners, supporting the conference with their presence.

The MICOPAM 2025 conference welcomed speakers whose talks or poster presentations are mainly related to the following areas: Mathematical Analysis, Algebra and Analytic Number

Theory, Combinatorics and Probability, Pure and Applied Mathematics and Related Areas, Mathematical Statistics and Its applications, Recent Advances in General Inequalities, Mathematical Physics, Fractional Calculus and Its Applications, Polynomials and Orthogonal Systems, Special Numbers and Special Functions, *q*-analysis and Its Applications, Approximation Theory and Optimization, Extremal Problems and Inequalities, Integral Transformations, Equations and Operational Calculus, Differential Equations and Their Applications, Geometry and Its Applications, Numerical Methods and Algorithms, Scientific Computation, Mathematical Methods and Computation in Engineering, Mathematical Geosciences, *p*-adic Numbers and *p*-adic Analysis, and Their Applications, Mathematical Modeling, Marketing, Business and Their Applications, Mathematical Methods for Engineering Applications.

In this context, the oral and poster presentations at this conference were mainly related not only to the areas mentioned above, but also to their applications in various fields of mathematics and related areas.

Further details about the MICOPAM 2025 conference are given as follows:

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Dora Pokaz (University of Zagreb, Croatia)
Mihaela Ribičić Penava (University of Osijek, Croatia)
Yilmaz Simsek (Akdeniz University, Turkey)

INVITED SPEAKERS of MICOPAM 2025

(Listed alphabetically by the speakers' last name)

Manuel López-Pellicer (Universitat Politècnica de València, Spain)
Abdelmejid Bayad (Université Paris-Saclay, France)
Gradimir V. Milovanović (Serbian Academy of Sciences and Arts, Serbia)
Julije Jakšetić (University of Zagreb, Croatia)
Santiago Emmanuel Moll-López (Universitat Politècnica de València, Spain)
Yilmaz Simsek (Akdeniz University, Turkey)
He made a Welcome Speech as the Head of the Organizing Committee of MICOPAM 2025.

Foreword written by the Editors

We, the Editors, are honored to write the foreword for this book, which presents the papers from the MICOPAM 2025 conference, dedicated to the 81st birthday of the esteemed mathematician Professor Manuel López-Pellicer.

Professor Manuel López-Pellicer was born in Valencia in 1944. He obtained his PhD degree in Mathematics from the University of Valencia, the first thesis supervised by Prof. M. Valdivia (1970) with the Cañada-Blanch doctorate award. He has the highest number of descendants among Valdivia disciples, 58, followed by four Valdivia disciples with 33, 28, 15, and 13 descendants, as recorded in the Mathematics Genealogy Project. Some of his current research areas are as follows: topology, topological vector spaces, non-regular Grothendieck topologies, functional analysis, K-analytic and Suslin spaces, Fréchet space, bounded scalar measures on Banach space, bounded finitely additive measures, etc. He is member of Spanish Royal Academy of Sciences (since 1998) and secretary of its Mathematics Section (2000-2007). Since 2004, he is Editor-in-Chief of Revista de la Real Academia de Ciencias Exactas Físicas y Naturales Serie A-Matemáticas, founded in 2001.

Here is a brief history of the “MICOPAM 2025 (Mediterranean International Conference of Pure & Applied Mathematics and Related Areas)” conference. The first edition of the MICOPAM conference series was held in Antalya, Turkey, in 2018. Since 2018, it has been organized regularly by Professor Yilmaz Simsek and his colleagues. Over the last seven years, this conference has brought together researchers from all over the world, who work on pure and applied mathematics and related areas.

This year, with the help of Professor Yilmaz Simsek and his team (Mustafa Alkan, Ayse Yilmaz Ceylan, Rahime Dere, Neslihan Kilar, Irem Kucukoglu and Ortaç Öneş), Professor Dora Pokaz (University of Zagreb, Croatia) and Professor Mihaela Ribičić Penava (University of Osijek, Croatia) and their team (Matea Đumić, University of Osijek, Croatia, Ljiljana Primorac Gajčić, University of Osijek, Croatia, Kristina Krulić Himmelreich, University of Zagreb, Croatia, Sanja Kovač, University of Zagreb, Croatia, Dragana Jankov Maširević, University of Osijek, Croatia, and Sanja Tipurić Spužević, University of Split, Croatia, also the staff of the School of Applied Mathematics and Informatics of the University of Osijek, the MICOPAM 2025 conference was held in Osijek, CROATIA, from September 8 to 12, 2025, and was by dedicated to the 81st birthday of Professor Manuel López-Pellicer.

This conference will be continued by Professor Yilmaz Simsek and his highly capable team. Thanks to their efforts, series MICOPAM has secured its place among the world’s respected conference and will continue to produce annual proceedings with ISBN numbers. Therefore, Professor Yilmaz Simsek would like to sincerely thank the following MICOPAM 2025’s Editorial Board members:

- Professor Mustafa Alkan, Professor Irem Kucukoglu, Professor Ortaç Öneş, Professor Dora Pokaz, and Professor Mihaela Ribičić Penava.

Professor Yilmaz Simsek would also like to express his sincere gratitude not only to his team, but also to the following Scientific Committee members:

- Elvan Akin, USA; Mustafa Alkan, Turkey; Hacer Ozden Ayna, Turkey; Abdelmejid Bayad, France; Naim L. Braha, Republic of Kosova; Tomislav Burić, Croatia; Nenad P. Cakić, Serbia; Ismail Naci Cangul, Turkey; Clemente Cesarano, Italy; Ahmet Sinan Cevik, Turkey;

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The MICOPAM 2025 conference welcomed speakers whose talks or poster presentations are mainly related to the following areas:

- Mathematical Analysis, Algebra and Analytic Number Theory, Combinatorics and Probability, Pure and Applied Mathematics and Related Areas, Mathematical Statistics and Its Applications, Recent Advances in General Inequalities, Mathematical Physics, Fractional Calculus and Its Applications, Polynomials and Orthogonal Systems, Special Numbers and Special Functions, q-analysis and Its Applications, Approximation Theory and Optimization, Extremal Problems and Inequalities, Integral Transformations, Equations and Operational Calculus, Differential Equations and Their Applications, Geometry and Its Applications, Numerical Methods and Algorithms, Scientific Computation, Mathematical Methods and Computation in Engineering, Mathematical Geosciences, p-adic Numbers, p-adic Analysis and Their Applications, Mathematical Modeling, Marketing, Business and Their Applications, Mathematical Methods for Engineering Applications.

This conference series, which has covered the above topics since its inception, continues to offer a wide range of activities, bringing together researchers from relevant fields around the world.

Professor Manuel López-Pellicer, who is the world's leading scientist, has collaborated with Professor Yilmaz Simsek for many years. Therefore, this year the MICOPAM 2025 conference was organized under the leadership of Professor Simsek, and dedicated to celebrating Professor Manuel López-Pellicer's 81st birthday.

A brief description of the contents of the "Proceedings Book of MICOPAM 2025" is provided below:

The first section of the MICOPAM 2025 Proceedings Book includes the Foreword written by the Editors, the Opening Ceremony Talk by Professor Yilmaz Simsek, including a brief biography of Professor Manuel López-Pellicer, the Opening Ceremony Talk by Professor Mihaela Ribičić Penava, the Opening Ceremony Talk by Professor Dora Pokaz, and some information about the MICOPAM 2025 conference, including the names of invited speakers, the names of committee members, and other details.

The rest of this volume includes all contributed talks and their corresponding manuscripts.

In this regard, we would like to thank all speakers and participants for their valuable contributions. We also express our sincere gratitude to all members of the Scientific Committee and all members of the Organizing Committee for their efforts in ensuring the success of both the conference and this book.

Finally, we are delighted to have celebrated the 81st birthday of the highly esteemed scientist Professor Manuel López-Pellicer, to whom this book is dedicated. In recognition of Professor Manuel López-Pellicer's great contribution to mathematics, we would like to extend our deepest respect and congratulations on his 81st birthday, with our very best wishes. We hope that the years ahead will be happy, fruitful, successful and filled with health and time spent with his loved ones.

We would also like to sincerely thank the following invited speakers for their contributions to the success of this conference:

- Abdelmejid Bayad, Université Paris Saclay, France
- Gradimir V. Milovanović, Serbian Academy of Sciences and Arts, Serbia
- Julije Jakśetić, University of Zagreb, Croatia
- Santiago Emmanuel Moll-López, Universitat Politécnica de València, Spain

Moreover, we would like to thank the Dean, Professor Kristian Sabo, the Dean's Office, and the staff for providing us with all the faculty's resources.

Consequently, we wish to extend our heartfelt thanks to everyone who contributed to the successful realization of this conference.

Editors of the Proceedings Book of MICOPAM 2025

Professor Mustafa Alkan
Professor Irem Kucukoglu
Professor Ortaç Öneş
Professor Dora Pokaz
Professor Mihaela Ribičić Penava
Professor Yilmaz Simsek

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Opening Ceremony Talk by Professor Yilmaz Simsek

(Welcome Speech as the Head of the Organizing Committee of MICOPAM 2025)

Plato was an ancient classical Greek philosopher, mathematician, and the founder of the Academy of Athens, the first institution of higher education in the Western world.

NUMBER IS THE BEGINNING OF EVERYTHING (In many cultures, especially Babylonian, Indo-Arabic, and Pythagorean, number is the beginning of everything.) At the entrance to Plato's school, he declared, "No One Unfamiliar with Mathematics May Enter."

Thanks to this sacred scientific journey that began with Plato, and thanks to the collaborative work of sciences such as physics, chemistry, biology, and engineering, especially mathematics, we understand the movements of our Earth and atmosphere, attempt to understand diseases, natural disasters, and other phenomena, and seek solutions to them.

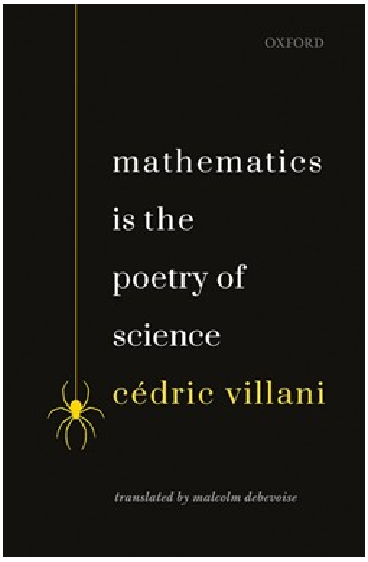
Today, researchers are deciphering DNA codes and can now delve into the roots of life. With supercomputers (quantum computers), we can accomplish tasks in seconds that would take tens of thousands of people thousands of years to complete. With the help of these computers, we utilize immense possibilities, such as communication, information transfer, and the internet, that all of humanity must possess.

According to the famous Austrian psychiatrist Alfred Adler, founder of the school of Individual Psychology: "Mathematics is the single, unique language of science."

Because mathematics is:

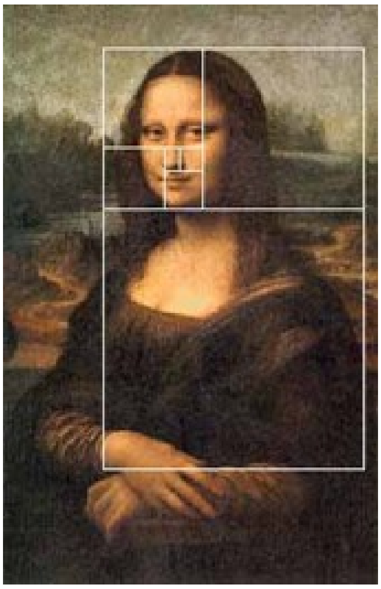
A language: (Alfred Adler: Mathematics is pure language - the language of science. It is unique among languages in its ability to provide precise expression for every thought or concept that can be formulated in its terms.)

A science:

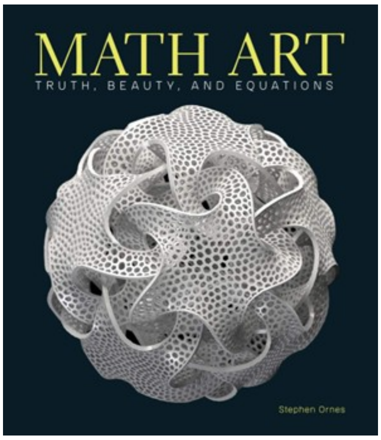


The cover page of the book titled "Mathematics is the Poetry of Science" by Cedric Villani

Art: Mathematics and art are closely related. This means that mathematics can be defined as an art inspired by beauty and creativity. In addition to its theoretical nature, mathematics is used extensively in arts such as music, dance, painting, architecture, sculpture, and textiles.



For the images mentioned in this talk, refer to:
`https://en.wikipedia.org/wiki/Mathematics_and_art`



The cover page of the book titled "Math Art: Truth, Beauty, and Equations" by Stephen Ornes

It has no limits in its scope or depth.

It is also indispensable in daily life.

It is magnificent, robust, reliable, and universal, transcending national boundaries and passing down through the ages. Thus, mathematics is the greatest common value created by the human mind.

I am a scientist who is devoted to mathematics.

After my family, mathematics is my biggest source of life. That's why I've published numerous articles, book chapters, and other works, and I continue to do so. I've mentored numerous graduate students.

To serve mathematics, I founded both the conference series MICOPAM (The Mediterranean International Conference of Pure & Applied Mathematics and Related Areas) in 2018 and the symposium series GFSNP (Symposium on Generating Functions of Special Numbers and Polynomials and Their Applications) in 2010. These conferences and symposiums continue to serve all researchers and will continue to do so.

In addition, our journal the Montes Taurus Journal of Pure and Applied Mathematics (MTJPAM) which I founded in 2018 and whose first issue was published its first issue in 2019, started its publication free of charge; see:

- Montes Taurus Journal of Pure and Applied Mathematics (MTJPAM) (ISSN: 2687-4814) – [Indexed in Elsevier – Scopus Database, Index Copernicus International (ICI Journals Master List), Google Scholar, Semantic Scholar, CiteFactor, Journal Factor (JF), and General Impact Factor (GIF)].

Mathematics, which is grounded in both deep thinking and creativity, and at the same time represents the art of disciplined thinking, has captivated many researchers today. There are many researchers at this meeting who embody this understanding, including my dear friend and mentor, Professor Manuel López-Pellicer.

That is why our MICOPAM 2025 conference is dedicated this year to celebrating Professor Manuel's 81st birthday.

I will now briefly present information about Professor Manuel's life.

1. He was born in Valencia in 1944. He obtained his PhD degree in Mathematics from the University of Valencia, the first thesis supervised by Prof. M. Valdivia (1970) with the Cañada-Blanch doctorate award. He has the highest number of descendants among Valdivia disciples, 58, followed by four Valdivia disciples with 33, 28, 15, and 13 descendants, as recorded in the Mathematics Genealogy Project.

2. He was a secondary school teacher (1968-75) with many brilliant students who took part in Maths Olympiads. He was Full Professor in the Applied Mathematics Department at the Polytechnic University of Valencia, UPV, (1978-2015), its director (1979-1984 and 1986-1997), and ICE director (1984-1986). Professor Manuel López-Pellicer served on the Valencian School Council (1986–2002) and on its Permanent Commission as a member of Recognized Prestige (2002–2005). He received the Teaching Excellence Award (2002–2003) and was a member of the IUMPA Scientific Committee (2004–2015). He also served as University Ombudsman (2006–2015) and as Mathematics Coordinator (1978–2020). On the website commemorating the 50th anniversary of UPV (December 6, 2018), he is mentioned among the speakers: "Rector Francisco Mora and Manuel López-Pellicer."

3. He supervised 11 PhD students and has authored 206 publications (901 citations, 310 since 2018, h=13, i10=23, according to Google Scholar). His major works include two monographs: Metrizable Barrelled Spaces (Longman, 1995, 238 pp.) and Descriptive Topology in Selected Topics of Functional Analysis (Springer, 2011, 493 pp.). He also contributed The Contribution of Jorge Juan to the Earth's Shape Problem for the 2019 ICIAM, the quadrennial world congress, which commissions one paper per issue on a distinguished mathematical contribution from the host country.

4. Professor Manuel López-Pellicer has been a member of the Spanish Royal Academy of Sciences since 1998 and served as secretary of its Mathematics Section (2000-2007). Since 2004, he has been Editor-in-Chief of Revista de la Real Academia de Ciencias Exactas Físicas y Naturales Serie A-Matemáticas, founded in 2001. It is the only Spanish mathematics journal in the top decile in each of the last three JCRs published (29/330-2020, 26/332-2021, and 15/329-2022, published on June 28, 2023). In the Academy's Scientific Culture dissemination program, four of the five mathematical conferences with the most views are by Professor López-Pellicer. The two most viewed are "Ramanujan, a Great Mathematician…" and "Riemann Hypothesis…", which had 105,978 and 55,201 views, respectively, on March 4, 2024 over five years, and 88,695 and 48,284 views, respectively, on March 13, 2023.

5 most important publications:

1. Ferrando, Juan C.; López-Pellicer, Manuel. Strong barrelledness properties in $\ell^\infty(XX, \mathcal{A}, \mathcal{A})$ and bounded finite additive measures. Mathematische Annalen 287(4) (1990), 727–736. Citations: 8 in MathSciNet, 33 in Google Scholar. The journal ranked in position 24/118 (JCR 1990).

2. López-Pellicer, Manuel. Webs and bounded finitely additive measures. Journal of Mathematical Analysis and Applications 210 (1) (1997), 257–267. Citations: 13 in MathSciNet, 34 in Google Scholar. The journal ranked in position 64/136 (JCR 1997).

3. Ferrando, Juan C.; Kąkol, Jerzy; López-Pellicer, Manuel; Saxon, Stephen. A. Tightness and distinguished Fréchet spaces. Journal of Mathematical Analysis and Applications 324 (2) (2006), 862–881. Citations: 31 in MathSciNet, 48 in Google Scholar. The journal ranked in position 46/186 (JCR 2006).

4. Kąkol, Jerzy; López-Pellicer, Manuel. On Valdivia strong version of Nikodym boundedness property. Journal of Mathematical Analysis and Applications 446 (1) (2017), 1–17. Citations: 7 in MathSciNet (2 citations by L. Zdomsky and others in 2019 and 2023), 19 in Google Scholar. The journal ranked in position 53/309 (JCR 2017).

5. Kąkol, Jerzy; Kubiś, Wiesław; López-Pellicer, Manuel. Descriptive Topology in Selected Topics of Functional Analysis. Developments in Mathematics, 24. Springer, New York, 2011. xii+493 pp. ISBN: 978-1-4614-0528-3; e-ISBN 978-1-4614-0529-0. Citations:120 (MathSciNet), 189 (Google Scholar). Books downloaded: Over 20,000. Chapters downloaded: 19,193 (2011-2021). Descriptive Topology in Selected Topics of Functional Analysis, Second Edition - Updated and Expanded, 831 pp., delivered to Springer in December 10, 2023.

Summary of his main contributions and the impact of his work.

1. Summary of the main contributions with some results:

The main objective of López Pellicer's research is focused on innovative development and application of topological methods to gain more knowledge about several classes of spaces in functional analysis and operators between them. Jointly with specialists from Spain, Israel, Poland and the United States, we successfully extended several results about distinguished Fréchet spaces, (DF)-, (LB)- and (LF)- spaces, spaces of measures, analytic, K-analytic and Suslin spaces, providing solutions to several longstanding open problems, some described in **(1A)-(1D)**.

(1A) In 1964, Kömura constructed a topological vector space, tvs, (E,U) whose k-topology Uk is regular, but (E, Uk) is not a tvs, where Uk is the finest topology having the same compact subsets as U. In 1974, Professor López Pellicer obtained the first example of a completely regular topological space (X,U) whose k-topology Uk is not regular, hence for (Cc(X)*, weak*) the k-topology (weak*)k is not regular and it proved the existence of non-regular Grothendieck topologies Tf.

(1B) In 2006, we characterized that a Fréchet space E is distinguished if its strong dual F has countable tightness, and got several topological properties related with cardinality assumptions. With our inequality for the distance to a Fréchet space E of points of the weak* closure in E** of a bounded subset of E, in 2013, we obtained a quantitative version of Krein's theorem for Fréchet spaces, extending the versions of Hajek, Marciszewski and Zizler for Banach spaces.

(1C) In 2008, we found a very large class S of locally convex spaces whose duals endowed with the weak* or with the precompact-open topologies are quasi-Suslin, unifying properties studied with different methods for subclasses of S. Professor López Pellicer's quasi-Suslin space X, for which $X \times X$ is not quasi-Suslin, rectifies a classic productivity error (2010). For X compact, $C(X)$ is weak-K-analytic if and only if $Cp(X)$ is K-analytic (Talagrand). We add if and only if $C(X)$ is K-analytic with topology of point convergence on a Gδ-dense subset of X. Moreover, in 2017, for a Tychonoff space X, we extended Talagrand equivalence to $Cb(X)$ in 2017. For Tychonoff spaces X and Y, the isomorphism of topological rings $Cp(X)$ and $Cp(Y)$ implies the homeomorphism of X and Y (Nagata). In 2014, we found that isomorphism of topological vector spaces $Cp(X)$ and $Cp(Y)$ only implies that X and Y share certain topological properties, such as Lindelöf Σ or K-analytic, getting similar results if $Cc(X)$ and $Cc(Y)$ are isomorphic.

(1D) Let ba($\mathcal{A}\mathcal{A}$) be the Banach space of finitely additive bounded scalar measures defined on an algebra $\mathcal{A}\mathcal{A}$ of subsets endowed with the variation norm. A subset $\mathcal{B}$ of $\mathcal{A}\mathcal{A}$ has property N if $\mathcal{B}$-pointwise boundednes in ba($\mathcal{A}\mathcal{A}$) implies boundedness in ba($\mathcal{A}\mathcal{A}$). $\mathcal{B}$ has property G if $\mathcal{B}$-pointwise convergence of a bounded sequence in ba($\mathcal{A}\mathcal{A}$) implies its weak convergence. $\mathcal{B}$ has the property sN (sG) if every countably increasing covering of $\mathcal{B}$ has an element with property N (G).$\mathcal{B}$ has property wN (wG) if each web in $\mathcal{B}$ has a string whose elements have property N (G). These properties enable us to improve deep results of measure theory. If $\mathcal{A}\mathcal{A}$ is an α-algebra, then $\mathcal{A}\mathcal{A}$ has properties N (Nikodým-Grothendieck), G (Grothendieck), sN (Valdivia), wN and wG (López-Pellicer 1997, improving results of 1990 and 2021). Let $\mathcal{A}$ be an algebra. Valdivia asked if property N of $\mathcal{A}\mathcal{A}$ implies that $\mathcal{A}\mathcal{A}$ has property sN (2013). He got an affirmative answer for a class of rings of subsets in 2019. Our feeling is that the general solution to this Valdivia open problem could be related with an axiomatic system.

2. Impact of his work:

(2A) Impact of the monograph and international relations. The described main goal of his research, explored also in recent papers by Banakh, Gabriyelyan, Gartside, Tsaban, Zdomsky and several of his 58 descendents, among others, lead finally to the monograph Descriptive Topology in Selected Topics of Functional Analysis, the first monograph to approach functional analysis from the perspective of descriptive topology. It gathers and extends several results starting from early papers of Corson until recent results, including techniques of transfinite decompositions into smaller subspaces to get properties of non-separable spaces. The second edition of the monograph was suggested by Springer due to its popularity among specialists. The extension of the first edition by another 350 pages, and incorporating new results from recent research required very intensive international cooperation with scientific centers in Austria, Israel, Italy, Poland, Spain, Ukraine and United States. In February 2023, Professor López-Pellicer visited the prestigious the Kurt Gödel Research Center at the University of Vienna to discuss the final work of this extension with Professors Kakol, Sobota and Zdomsky.

(2B) Impact of his research work in Smolyanov's monograph. The monograph by V.I. Bogachev and O.G. Smolyanov, Topological Vector Spaces and Their Applications, Springer 2017, X+456 pp., doi 10.1007/978-3-319-57117-1, was an initiative of V.I. Sobolev (see pp. X), whose spaces facilitate the modern treatment of differential equations, essential in science and technology. M. Valdivia, A. Grothendieck and M. Talagrand have 9, 6 and 6 references, respectively, making them the three authors with the most references in Smolyanov's monograph; O.G. Smolyanov (a co-author with 23), S.A. Shkarin (with 12) and López-Pellicer (with 11 references).

(2C) Social impact of his work in RACSAM. At the beginning of his 20 years work (since 2004) as Editor-in-Chief, its diffusion was very limited and local. Professor López-Pellicer oversaw its inclusion in the JCR in 2009 as well as its publication and distribution by Springer in 2011. Following its most recent JCR rankings—29/330, 26/332 and 15/329—for Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales, Matemáticas, it is widely recognized that the jurnal has strengthened the international presence of both the Spanish Academy of Sciences as well and Spanish mathematics. The lastest JCR was released on June 28, 2023, fter which the Academy published the announcement: "The RAC Mathematics Journal in the Top 15 in the World". Professor López-Pellicer is currently managing the renewal of the agreement with Springer for the publication of RACSAM, covering the period from January 1, 2024 to December 31, 2034. RACSAM is also the only top decile Spanish mathematics journal in the most recent Chinese ranking by Impact Factor (54/558, December 27, 2023). The number of submissions has grown significantly, from 101 papers in 2011 to 1,363 in 2023.

(2D) Educational impact of his work. Manuel Valdivia, who placed equal importance on research and teaching to guarantee continuity, also inspired Professor López-Pellicer's educational work. In addition to supervising 11 PhD students, mentoring 58 academic descendants, and delivering four of the five mathematical conferences with the most views in the Academy of Sciences (see Populares), Professor López-Pellicer was recently invited to the prestigious "Invited Lecture Series" at Adam Mickiewicz University (Poznan, January 23, 2024). This series invites recognized mathematicians to give lectures to specialists and students for didactic purposes. In 2023, lectures were given by Professors G. Godefroy (Paris), M. Fabian (Prague) and A. Leiderman (Beer-Sheva).

In addition, the following special issue of the journal **Montes Taurus Journal of Pure and Applied Mathematics (MTJPAM)** has been dedicated to his 81st birthday:

- `Special Issue`:
  - Recent Developments in Contemporary Mathematics and Their Applications "Dedicated to Professor Manuel López-Pellicer on the occasion of his 81st anniversary".
- `Special Issue Editors`: Hari Mohan Srivastava and Yilmaz Simsek.
- `Submission Deadline`: October 31, 2025.

In this special dedicated issue, I have published the following article, which is very up-to-date and full of applicable results:

- Y. Simsek, New generating functions and formulas for Bernstein-Stancu basis functions and their applications, Montes Taurus J. Pure Appl. Math. 7 (3), 1-11, 2025; Article ID: MTJPAM-D-24-00054.

- One of the main aims of this article is to construct new generating functions for the Bernstein-Stancu basis functions, with the help of hypergeometric series, the descending factorial polynomials, and the ascending factorial polynomials. Another main aim is to reveal novel definitions and formulas of the Bernstein-Stancu basis functions with the help of the Euler gamma function and beta functions. Furthermore, by blending the higher order Bernoulli polynomials, the Lah numbers and the Stirling numbers represented by the descending factorial polynomials, and the ascending factorial polynomials, we derive many new identities and relations of the Bernstein-Stancu basis functions.
- Finally, recurrence relations, derivative formulas for the Bernstein-Stancu basis functions are also given. Finally, we give Bézier-type curves in terms of control points and the generalized Bernstein-Stancu basis functions.

For more details about the article mentioned above, see: `https://mtjpamjournal.com/papers/article_id_mtjpam-d-24-00054`.

I wish my dear friend Professor Manuel a long and healthy life. I hope he continues to make significant contributions to mathematics. I sincerely wish him and his family happiness, health, and well-being.

I extend my sincere gratitude to Professor Dora Pokaz and Professor Mihaela Ribičić Penava, and their team for organizing this conference. I extend my gratitude not only to the Local Organizing Committee (Matea Đumić, University of Osijek, Croatia; Ljiljana Primorac Gajčić, University of Osijek, Croatia; Kristina Krulić Himmelreich, University of Zagreb, Croatia; Sanja Kovač, University of Zagreb, Croatia; Dragana Jankov Maširević, University of Osijek, Croatia; Dora Pokaz, University of Zagreb, Croatia; Mihaela Ribičić Penava, University of Osijek, Croatia; Sanja Tipurić Spužević, University of Split, Croatia; but also to the heads of the Organizing Committee (Mustafa Alkan, Akdeniz University, Turkey; Ayse Yilmaz Ceylan, Akdeniz University, Turkey; Rahime Dere, Alanya Alaaddin Keykubat University, Turkey; Neslihan Kilar, Niğde Ömer Halisdemir University, Turkey; Irem Kucukoglu, Alanya Alaaddin Keykubat University, Turkey; Ortaç Öneş, Akdeniz University, Turkey; Dora Pokaz, University of Zagreb, Croatia; Mihaela Ribičić Penava, University of Osijek, Croatia).

I extend my sincere gratitude to the University of Osijek rectorate and the Dean of the School of Applied Mathematics and Informatics for providing these opportunities at the university.

I extend my sincere gratitude to the invited speakers at this conference:

- Abdelmejid Bayad, Université Paris Saclay, France,
- Gradimir V. Milovanović, Serbian Academy of Sciences and Arts, Serbia,
- Julije Jakśetić, University of Zagreb, Croatia,
- Santiago Emmanuel Moll-López, Universitat Politécnica de València, Spain.

I extend my sincere gratitude to all the committee members.

I extend my sincere gratitude to the participants for their valuable presentations and support.

I would like to thank my esteemed wife Saniye and my precious daughters Burcin and Buket for the morale, support and love they have given me.

I sincerely believe that the presentations from this conference, together with the forthcoming ISBN-numbered proceedings, will benefit researchers worldwide.

I sincerely wish you all a fruitful conference, full of stimulating and productive discussions.

I wish you all a healthy and productive day filled with mathematical results.

Professor Yilmaz Simsek

8 September 2025, Osijek

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Opening Ceremony Talk by Professor Mihaela Ribičić Penava

Ladies and Gentlemen, Distinguished Guests, Dear Colleagues,

It is my great honor and privilege to welcome all of you, on behalf of both the Organizing Committee of the 8th Mediterranean International Conference of Pure & Applied Mathematics and Related Areas dedicated to Professor Manuel López-Pellicer on the occasion of his 81st birthday and on my own behalf.

First of all, let me express my sincere gratitude to Professor Simsek for giving us the opportunity to host this important event in Croatia, at the School of Applied Mathematics and Informatics.

During the last conference in Antalya, Professor Simsek, in a relaxed conversation, as indeed most conversations with him usually are, suggested that we might organize the next conference in Croatia. Professor Dora Pokaz and I gladly agreed and accepted the honor of hosting the 8th edition of the conference here in Osijek. Together with our local team, we have worked hard to prepare this event, and we sincerely hope that in the days ahead we will justify the trust placed in us when the organization of the conference was entrusted to our care.

It is also my great pleasure to extend heartfelt thanks to Professor Manuel López-Pellicer for graciously accepting that this conference be organized in his honor here at the School of Applied Mathematics and Informatics. This is both a privilege and an honor for our Faculty and our University.

Let me also warmly thank all our other invited speakers for accepting the invitation to attend this conference and deliver their lectures.

- Professor Gradimir Milovanović, from the Serbian Academy of Sciences and Arts, Serbia.
- Professor Abdelmejid Bayad, from Université Paris-Saclay, France.
- Professor Santiago Emmanuel Moll-López, from Universitat Politècnica de València, Spain.
- Professor Julije Jakšetić, from the University of Zagreb, Croatia.

Allow me to thank the Dean of the School of Applied Mathematics and Informatics, Professor Kristian Sabo, who is with us today and will address us a little later, for his generous support in organizing this conference. He gave us his approval to host MICOPAM 2025 here in Osijek, and he has also been helpful by providing valuable organizational advice and assisting with visa arrangements. Thank you very much, Professor Sabo.

My gratitude also extends to the members of the Organizing and Scientific Committees for their dedicated and diligent work, some of whom worked tirelessly throughout the summer — especially Professor Irem Kucukoglu, whose kindness and availability were remarkable — and above all, to the members of the Local Organizing Committee from my Faculty:

Professor Ljiljana Primorac Gajčić, Professor Mateja Đumić and Professor Dragana Jankov Maširević.

Without their support, the organization of the conference at the local level would not have been possible.

My heartfelt gratitude goes to all of you, the participants, for taking part in the 8th MICOPAM Conference. We are truly delighted to see so many researchers who have come to Osijek from different parts of the world to present their latest findings, exchange innovative ideas, and foster future collaborations.

I wish you all inspiring lectures, fruitful discussions, and many pleasant moments during your stay, and I truly hope you will enjoy your time here in Osijek.

With these words, I am pleased and honored to declare the conference open.

Welcome!

Professor Mihaela Ribičić Penava

`8 September 2025, Osijek`

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Opening Ceremony Talk by Professor Dora Pokaz

Ladies and Gentlemen, Distinguished Colleagues, Dear Friends,

It is a true honor and a great joy to welcome you all to MICOPAM 2025 – the 8th Mediterranean International Conference of Pure & Applied Mathematics and Related Areas, here in the beautiful city of Osijek. On behalf of the co-organizers, I extend my warmest greetings to each and every one of you.

First and foremost, I wish to express my deepest gratitude to Professor Yilmaz Simsek (Akdeniz University, Türkiye), Head of the Organizing Committee and the founder of these conferences. It is truly an honor that our acquaintance has grown into a meaningful friendship over the years. Getting to know you better has been a privilege, and I am deeply grateful for the warmth and kindness you and your family have shown. Your integrity, wisdom, and genuine care for others are an inspiration not only to me, but to many of us gathered here. The way your family lives by example reminds us of the values that truly matter. Thank you, Saniye, Buket, and Burcin, for being here today. My dear friend Yilmaz, your leadership and dedication have been instrumental in bringing this conference to life—thank you sincerely.

This year's conference is dedicated to Professor Manuel López-Pellicer (Universitat Politècnica de València, Spain), whose remarkable contributions to mathematics and distinguished career we have the privilege of celebrating this week. Professor López, it is an honor to recognize your outstanding achievements and your lasting impact on the mathematical community.

I would also like to sincerely thank the University of Osijek for its generous hospitality in hosting this important event. Your support has provided us with the perfect setting for academic exchange and collaboration.

Allow me to extend a special welcome to our invited speakers, whose expertise and insights will undoubtedly enrich our discussions:

- Abdelmejid Bayad, Université Paris-Saclay, France
- Gradimir V. Milovanović, Serbian Academy of Sciences and Arts, Serbia
- Santiago Emmanuel Moll-López, Universitat Politècnica de València, Spain
- Julije Jakšetić, University of Zagreb, Croatia

Professors Bayad and Milovanović have been with MICOPAM from the very beginning, playing a key role in its foundation and growth. We are deeply grateful for their unwavering support and invaluable contributions. Professors Moll-López and Jakšetić have joined us as invited speakers for the first time, and their presence is a testament to the way MICOPAM continues to grow and expand its scientific family.

Thank you all in advance for your contributions—through your lectures, presentations, and open discussions.

And, of course, a warm welcome to all participants, with special appreciation for those who have traveled long distances to join us here in Osijek. Your presence enriches the spirit of MICOPAM and makes this gathering truly special.

For me personally, this moment carries deep meaning. I have been fortunate to witness the journey of MICOPAM from its very beginning: at the first conference in Antalya in 2018, the online edition during the pandemic, as an invited speaker in Paris in 2021, and again in Antalya in 2024. Each time, I was inspired not only by the mathematics we shared, but also by the friendships and sense of belonging that MICOPAM has fostered.

Let me also acknowledge our close connection with GFSNP 2025 – the 15th Symposium on Generating Functions of Special Numbers and Polynomials and Their Applications, held this June at Gaziantep University. I am deeply grateful for the invitation to participate as an invited speaker and for the generous hospitality extended to me. As an affiliated event, GFSNP is an important part of our extended scientific family.

That is why I am especially pleased that today we are able to host the MICOPAM "family" here in Croatia – a country known for its distinguished scientists and inventors, the birthplace of the cravat, the origin of the mechanical pencil, and today, home to one of the fastest electric cars in the world.

Names such as Ruđer Bošković, Franjo Hanaman, Ivan Vučetić, Lavoslav Ružička, and Vladimir Prelog may not be immediately familiar to all of you. But if you look them up, you will discover remarkable stories: Bošković, one of the most outstanding mathematicians and physicists of his time; Hanaman, co-inventor of the tungsten light bulb filament; Vučetić, a pioneer of fingerprint identification in forensic science; and Ružička and Prelog, Nobel Prize laureates in chemistry, who both received part of their education here in Osijek.

The cravat, as many of you may know, originated with Croatian soldiers in the 17th century. Their distinctive scarves became fashionable across Europe and eventually gave rise to the modern tie. The mechanical pencil was invented in Zagreb in 1906 by Slavoljub Eduard Penkala, leaving a lasting mark on education and science worldwide. And more recently, Croatia has gained global recognition for the Rimac Nevera – the fastest electric hypercar in the world. This long list of inventors and innovators reflects Croatia's tradition of creativity and ingenuity—values that we proudly share with you today as a community of mathematicians.

Dear colleagues, I truly hope you will enjoy not only the scientific program of this conference, but also the warmth and culture of Osijek and the hospitality of Croatia. May this gathering bring new ideas, fruitful collaborations, and lasting friendships.

Welcome once again to MICOPAM 2025. It is my great pleasure to greet you all here in Osijek.

Thank you.

Professor Dora Pokaz

8 September 2025, Osijek

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The Proceedings Book of 8th Mediterranean International Conference of Pure & Applied Mathematics and Related Areas (MICOPAM 2025)

December 15, 2025

Contents

1 INVITED SPEAKERS' PAPERS

Web properties of measures on algebras of subsets

Salvador López-Alfonso [1], *Manuel López-Pellicer* [*2] *and Santiago Moll-López* [3]

Let $\mathcal{A}$ be an algebra of subsets of a set Ω, i.e., $\emptyset \in \mathcal{A}$ and for every $B \in \mathcal{A}$ and $C \in \mathcal{A}$ we have that $B \cup C \in \mathcal{A}$ and that $\Omega \backslash B \in \mathcal{A}$. For each $B \in \mathcal{A}$ we denote by e_B the characteristic function of the subset B, i.e., $e_B(x) = 1$, if $x \in B$, and $e_B(x) = 0$, if $x \in \Omega \backslash B$.

Let $L(\mathcal{A})$ be the real (or complex) normed space formed by the linear span of $\{e_B, B \in \mathcal{A}\}$ endowed with the supremum norm, defined by $\| f \|_\infty = \sup\{|f(x)| : x \in \Omega\}$, for $f \in L(\mathcal{A})$. The topological dual of $L(\mathcal{A})$ is isometric to the Banach space $ba(\mathcal{A})$ of real (or complex) bounded finitely additive measures defined on $\mathcal{A}$ endowed with the variation norm, which is a norm equivalent to supremum norm. Hence each measure $\mu \in ba(\mathcal{A})$ verifies that for every $B \in \mathcal{A}$ and $C \in \mathcal{A}$ with $B \cap C = \emptyset$ we have that

1. $\mu(B) \in \mathbb{R}$ (or $\mu(B) \in \mathbb{C}$, respectively),
2. $\mu(B \cup C) = \mu(B) + \mu(C)$ and
3. $\sup\{|\mu(D)| : D \in \mathcal{A}\}$ is finite.

Let $\mathcal{B}$ be a subset of $\mathcal{A}$. The subset $\mathcal{B}$ is a Nikodym set for $ba(\mathcal{A})$, or $\mathcal{B}$ has property (N), if each $\mathcal{B}$-pointwise bounded subset M of $ba(\mathcal{A})$ is bounded in $ba(\mathcal{A})$, in brief, $\mathcal{B}$-pointwise boundedness of a subset $M \subset ba(\mathcal{A})$ imply its uniform boundedness, i.e.

$$\big(\forall C \in \mathcal{B}\big) \ \sup\{\, |\mu(C)| : \mu \in M \,\} < \infty \implies \sup\{\, |\mu(C)| : \mu \in M,\ C \in \mathcal{A} \,\} < \infty.$$

$\mathcal{B}$ has property (G) $[(VHS)]$ if for each bounded sequence [if for each sequence] in $ba(\mathcal{A})$ the $\mathcal{B}$-pointwise convergence implies its weak convergence.

$\mathcal{B}$ has property (sN) $[(sG)$ or $(sVHS)]$ if for every increasing covering $\{\mathcal{B}_n : n \in \mathbb{N}\}$ of $\mathcal{B}$ there exist $p \in \mathbb{N}$ such that $\mathcal{B}_p$ has property (N) $[(G)$ or $(VHS)]$, respectively.

It is said that $\{\mathcal{B}_{n_1 n_2 \cdots n_m} : n_i \in \mathbb{N}, 1 \le i \le m, m \in \mathbb{N}\}$ is an increasing web of $\mathcal{B}$ if $\{\mathcal{B}_n : n \in \mathbb{N}\}$ is an increasing covering of $\mathcal{B}$ and for each $(n_1 n_2 \cdots n_{m-1}) \in \mathbb{N}^{m-1}$, with $m > 1$, the sequence $\{\mathcal{B}_{n_1 n_2 \cdots n_m} : n_m \in \mathbb{N}\}$ is an increasing covering of $\mathcal{B}_{n_1 n_2 \cdots n_{m-1}}$.

Finally, $\mathcal{B}$ has property (wN) $[(wG)$ or $(wVHS)]$ if for every increasing web $\{\mathcal{B}_{n_1 n_2 \cdots n_m} : n_i \in \mathbb{N}, 1 \le i \le m, m \in \mathbb{N}\}$ of $\mathcal{B}$ there exists a sequence $(p_n : n \in \mathbb{N})$ such that each $\mathcal{B}_{p_1 p_2 \cdots p_n}$ has property (N) $[(G)$ or $(VHS)]$, respectively, for every $n \in \mathbb{N}$.

An algebra $\mathcal{A}$ of subsets of a set Ω is a σ-algebra of subsets if $\mathcal{A}$ contains the union of every countable family of elements of $\mathcal{A}$.

The classical theorems of Nikodym-Grothendieck, Valdivia, Grothendieck and Vitali-Hahn-Saks say, respectively, that every σ-algebra $\mathcal{A}$ of subsets of a set Ω verifies that $\mathcal{A}$ has the properties (N), (sN), (G) and (VHS).

2020 MSC: 28A60, 46G10

Keywords: Algebra and σ-algebra of subsets, bounded finitely additive scalar measure, Strong and web Nikodým (Grothendieck and Vitali-Hahn-Saks properties)

Conclusion

The aims of this talk are the presentation of the following results:

1. Every σ-algebra $\mathcal{A}$ of subsets of a set Ω has property (wN).

2. A subset $\mathcal{B}$ of an algebra $\mathcal{A}$ has property $(wWHS)$ if and only if $\mathcal{B}$ has property (wN) and $\mathcal{A}$ has property (G).

3. Hence, every σ-algebra $\mathcal{A}$ of subsets of a set Ω has properties (wG) and $(wWHS)$.

4. Additionally, some open questions will be considered.

Acknowledgments

The first and third authors dedicate this paper to the second author Professor Manuel López-Pellicer on the occasion of his 81st anniversary.

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Dedekind sums: Progress and problems

Abdelmejid Bayad

Under the influence of B. Riemann, R. Dedekind was interested in the behavior of the function $\eta(z)$, defined by

$$\eta(z) = e^{\frac{\pi i z}{12}} \prod_{n=1}^{\infty} \left(1 - e^{2\pi i n z}\right), \ \Im(z) > 0,$$

It should be noted that previously, Jacobi and Hermite had already considered this function in their work. However, it is R. Dedekind who studied it the most. More specifically, he examined the action of the modular group $\mathrm{SL}_2(\mathbb{Z})$ on the Poincaré half-plane. Precisely, under the action of the matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ with $c \neq 0$. He discovered an important formula involving the sums

$$s(d,c) = \sum_{k=1}^{|c|} \left(\left(\frac{k}{c}\right)\right)\left(\left(\frac{kd}{c}\right)\right)$$

commonly referred to as Dedekind sums, where $((x))$ denotes the "sawtooth" function. These sums play a crucial role in number theory. In this presentation, we will explore both historical results and recent advances regarding this topic. We will highlight the fundamental arithmetic properties of these sums, notably Dedekind's reciprocity law, as well as notions of density and equidistribution modulo 1. We will also establish connections between these sums and the special values of the partial zeta function associated with a number field, as well as the Euler class in cohomology. If time permits, I will also address three analogues of the Dedekind sums. These analogues represent multidimensional and elliptic generalizations that broaden our understanding of the properties and applications of Dedekind sums in more complex contexts. Furthermore, I will present some open problems that remain fascinating in number theory.

Acknowledgments

This paper is dedicated to Professor Manuel López-Pellicer on the occasion of his 81st birth anniversary.

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Gaussian and anti-Gaussian quadrature formulas for a class of nonstandard complex Jacobi weights

Gradimir V. Milovanović

This lecture is a continuation of my previous lecture from the 15th Symposium - GFSNP 2025 (Gaziantep University, Turkey), in which the focus was on the generating functions of the non-standard complex Jacobi polynomials on [0,1] (with complex parameters in the weight function). Here we consider the corresponding quadrature formulas of Gaussian and anti-Gaussian type, including their construction, distribution of nodes and error analysis, as well as some applications of such kind of quadratures.

2020 MSC: 33C20, 33C45, 33C90, 41A55, 65D30, 65D32

Keywords: Jacobi polynomials, orthogonality, complex moment functional, three-term recurrence relation, zeros, Jacobi matrix, Gaussian quadrature formula, anti-Gaussian formula

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Combinatorial insights into sums of floor functions

Julije Jakšetić

In this presentation, we first prove several new identities in elementary number theory by combinatorial and visual methods. We then present a unified approach to these identities using the floor function. In the final part of the talk, we provide algebraic proofs of the statements under somewhat more general assumptions, examples and applications to Fibonacci sequence and we present further generalizations and open problems.

2020 MSC: 11A25

KEYWORDS: Floor function, involutions, Fibonacci sequence

Introduction

The motivation for this presentation is the result (see [3])

$$\sum_{k=1}^{\infty} \left\lfloor \frac{r}{k} \right\rfloor^2 = \sum_{k=1}^{\infty} (2k-1) \left\lfloor \frac{r}{k} \right\rfloor$$

and the following theorem (see [1, p. 251]).

Theorem 1. *Let $a < b$ and $c < d$ be positive real numbers and let $f : [a, b] \to [c, d]$ be a continuous, bijective, and increasing function. Then*

$$\sum_{a \leqslant k \leqslant b} \lfloor f(k) \rfloor + \sum_{c \leqslant k \leqslant d} \left\lfloor f^{-1}(k) \right\rfloor - n\left(G_f\right) = \lfloor b \rfloor \lfloor d \rfloor - \alpha(a)\alpha(c), \tag{1}$$

where k is an integer, $n\left(G_f\right)$ is the number of points with nonnegative integer coordinates on the graph of f, and $\alpha : \mathbb{R} \to \mathbb{Z}$ is defined by

$$\alpha(x) = \begin{cases} \lfloor x \rfloor & \text{if } x \in \mathbb{R}\backslash\mathbb{Z} \\ 0 & \text{if } x = 0 \\ x - 1 & \text{if } x \in \mathbb{Z}\backslash\{0\} \end{cases}$$

Then , using combinatorics, we derive a recurrence for sums of powers

$$S_m(n) = 1^m + 2^m + \cdots + n^m,$$

and we discuss the result of [2].

Main results

In the main part, we give a combinatorial treatment of a result for which (1) is the special case corresponding to involutions. We then extend the result to Fibonacci numbers using the golden ratio.

Conclusion

We discuss relaxed hypotheses of the main theorem, as well as further generalizations and open problems.

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Applications of topological data analysis across science and engineering

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Complex systems in science and engineering often produce high-dimensional, heterogeneous data in which non-linear patterns remain hidden to traditional statistical methods. Topological Data Analysis (TDA) provides a robust mathematical framework to explore such data through its shape and connectivity rather than predefined assumptions or linear projections. This work introduces the theoretical foundations of TDA, from metric and simplicial constructions to persistent homology and the Mapper algorithm, and illustrates their integration with optimization techniques. We show how these methods uncover latent structures, transition pathways, and subpopulations across diverse domains including orbital mechanics, health sciences, and materials research.

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Keywords: Topological data analysis, mapper, persistent homology, particle swarm optimization, orbital mechanics, sleep, heavy menstrual bleeding, microbiome, perovskite solar cells

Introduction

The growing complexity of modern datasets—characterized by high dimensionality, noise, and heterogeneity—poses significant challenges for conventional analytical tools. Methods such as Principal Component Analysis, clustering, or regression are powerful but rely on assumptions of linearity or continuity that often fail to capture intricate manifolds or disconnected subspaces within the data. Topological Data Analysis (TDA) emerged from algebraic topology as a mathematically rigorous approach to describe and quantify the *shape of data*. Rather than focusing on pairwise relationships or average trends, TDA extracts invariant features that persist across scales and are stable under perturbations.

At its foundation, TDA treats data as a finite subset of a metric space (M, ρ). The topology of this data can be inferred by constructing simplicial complexes such as the Vietoris–Rips or Čech complexes, which approximate the underlying continuous shape of the data. Persistent Homology (PH) then studies how topological features—connected components, loops, and voids—appear and disappear as the scale parameter grows. These features are encoded in barcodes or persistence diagrams, providing multi-scale signatures that distinguish meaningful structure from noise.

While PH offers a quantitative and stable summary of the data's topology, the **Mapper algorithm** complements it with an interpretable, graphical representation. Mapper projects data through a chosen *filter* or *lens* (e.g., density, PCA coordinate, or physical variable), covers the range of the filter with overlapping intervals, clusters data locally within each interval, and connects overlapping clusters. The result is a simplicial complex—often visualized as a graph—that captures the structure and connectivity of the dataset. Each node represents a local cluster, edges indicate overlap, and the overall topology of the graph reveals branches, loops, or bridges that correspond to transitions or heterogeneity within the data.

One of the key advantages of TDA is its **stability under noise and sampling**. Theorems by Chazal and colleagues have shown that persistent homology is continuous with respect to perturbations in the input data under the Hausdorff and Gromov–Hausdorff distances [1]. Consequently, small changes in measurement or sampling do not drastically alter the inferred topological features, providing robustness that is essential for scientific datasets. Moreover, TDA does not require prior dimensionality reduction, allowing it to detect patterns invisible to PCA or clustering methods.

TDA has found increasing applications across domains. In biology, it enables the classification of microbiome structures, gene networks, and brain connectivity patterns [3, 4]. In chemistry and materials science, it has been used to characterize porous materials, crystalline structures, and high-throughput design spaces [5]. In physics and engineering, it supports model reduction, stability analysis, and trajectory optimization [2]. Despite its mathematical depth, TDA integrates naturally with data-driven approaches, bridging geometry, statistics, and machine learning through the language of topology.

Results and Applications

- *Orbital Mechanics (Mapper + PSO).* We integrated Mapper with Particle Swarm Optimization (PSO) to study intercept trajectories for interstellar objects such as ‘Oumuamua. Classical PSO, while effective, tends to converge prematurely to a single basin of attraction. By analyzing the swarm’s sampled solutions with Mapper, we identified multiple disconnected “corridors” on the pork-chop surface—regions of near-optimal trajectories connected by smooth transitions. The hybrid approach guided the swarm to explore under-sampled regions and maintain diversity, improving both convergence speed and coverage. Quantitatively, the PSO+Mapper combination achieved equivalent minimum Δv with 30–40% fewer iterations, illustrating how topology-guided optimization enhances global search strategies.

- *Health Sciences: Sleep Patterns.* TDA was applied to datasets exploring sleep quality, stress, and lifestyle variables. Mapper revealed hidden subpopulations invisible to linear analysis, such as “short sleepers–resilient” versus “short sleepers–vulnerable”, and non-linear transitions linking stress, social media exposure, and sleep architecture. The resulting topological graphs showed loops and bridges connecting behavioral and physiological dimensions, suggesting that stress and lifestyle modulate resilience to sleep disruption. These findings demonstrate TDA’s ability to uncover latent relationships beyond conventional statistical boundaries.

- *Health Sciences: Heavy Menstrual Bleeding (HMB).* We further applied Mapper to datasets integrating clinical, lifestyle, and microbiome data related to heavy menstrual bleeding. Using filters based on clinical burden (PBAC or quality-of-life indices) and microbiome axes (e.g., *Lactobacillus* dominance vs. dysbiosis), Mapper identified distinct subgroups and transition paths between normal and HMB phenotypes. Branches corresponded to inflammatory, metabolic–hormonal, and vaginal-microbiota profiles, revealing potential mechanistic links between stress, sleep quality, and symptom severity. These topological patterns provide a foundation for hypothesis-driven research and support personalized approaches in menstrual health.

- *Materials Research (Perovskite Solar Cells).* In datasets combining composition, processing parameters, and performance metrics, Mapper graphs displayed distinct branches representing chemical or structural families. Loops captured alternative formulations achieving similar efficiencies through different synthesis routes, while sparse interconnecting edges indicated unexplored but promising design corridors. Persistent homology quantified the stability of these branches across parameter scales. Together, these analyses support data-driven experimental design, improving understanding of efficiency–stability trade-offs and accelerating discovery in materials science.

Conclusions

Topological Data Analysis provides a unifying geometric framework that complements classical and machine learning methods by emphasizing the structure and continuity of data. The combination of Mapper and Persistent Homology offers both interpretability and mathematical rigor, enabling the identification of subgroups, transition pathways, and invariant features across scientific domains. Coupling TDA with optimization algorithms, such as PSO, extends its reach from descriptive analysis to prescriptive applications, improving exploration and decision-making in complex search spaces. Future work will focus on automating lens selection, quantifying uncertainty in Mapper graphs, and integrating topology-informed feedback loops for adaptive optimization and hypothesis generation.

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Applications of the Bernstein polynomials related to topological and functional analysis techniques

Yilmaz Simsek

In [23], we constructed new generating functions and formulas for Bernstein-Stancu basis functions and their applications. We also gave some remarks and observations for the Bernstein polynomials related to topological and functional analysis techniques. Saye [16] studied and investigated an algorithm that recursively partitions topology into hyperrectangular sub-cells until simplified, with a particular topology test leveraging Bernstein polynomial properties. Saye gave some applications of this algorithm. Moreover, in [23], we defined the Bèzier-type curves and spline-type curves in terms of the Bernstein-Stancu polynomials. We gave some questions about their effects, distinctions from classical curves, advantages, disadvantages, and topological properties. The aim of this presentation is not only to derive derivative formulas for these curves, but also to study and investigate solutions to these problems.

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KEYWORDS: Generating functions, Gamma and beta functions, Bernoulli numbers and polynomials, special sequences and polynomials, recurrences, curves, factorials

Introduction

The Bèzier type curves and B-spline curves, expressed in terms of the Bernstein basis functions and the Bernstein polynomials, have found many applications in approximation theory, probability distribution functions, computer-aided geometric design, etc. (*cf.* [1]-[23]).

The motivation of this paper is to give partial derivatives of Bèzier type curves in terms of the generalized Bernstein-Stancu basis functions and the Bernoulli polynomials of higher order. We give some remarks and observations about questions given in [23], related to effects, distinctions from classical curves, advantages, disadvantages, and topological properties.

Some standard notations for this paper are given as follows:

Let $\mathbb{Z}$, $\mathbb{R}$, and $\mathbb{C}$ denote the sets of integers, real numbers, and complex numbers, respectively.

$\mathbb{N} = \{1, 2, 3, \ldots\}$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

The generalized falling factorial (or descending factorial polynomials) $w^{(m\theta)}$, which is a homogeneous polynomial in w and θ of degree m, with increment θ defined by

$$w^{(m\theta)} = \prod_{v=0}^{m-1} (w - v\theta)$$

for $m \in \mathbb{N}$, with the convention $w^{(0\theta)} = 1$ (*cf.* [2], [11], [10], [15, Eq. (2.11)], [24]). If $\theta \neq 0$, we have

$$w^{(m\theta)} = \theta^m \left(\theta^{-1} w\right)^{(m)},$$

with

$$w^{(m0)} = w^m.$$

The ascending factorial polynomials are defined by

$$w_{(m\theta)} = \prod_{v=0}^{m-1} (w + v\theta)$$

for $m \in \mathbb{N}$, with the convention $w_{(0\theta)} = 1$ and $w_{(m0)} = w^m$ (*cf.* [15, Eq. (2.11)], [23]). It is also well known that $w_{(m)}$ denotes the Pochhammer symbol (also known as the rising factorial polynomial), which is also given in terms of the Euler gamma function:

$$w_{(m)} = \prod_{v=0}^{m-1} (w + v), \tag{1}$$

(*cf.* [15], [20]).

The Bernstein-Stancu polynomials are defined by

$$S_v^{<b>} [g] (w) = \sum_{m=0}^{v} P_{v,m}^{<b>} (w) \, g\left(\frac{m}{v}\right),$$

where $w \in \mathbb{C}$ and $b \geqslant 0$ and $P_{v,m}^{<b>} (w)$ denote the Bernstein-Stancu basis functions, which are defined by

$$P_{v,m}^{<b>} (w) = \frac{\binom{v}{m}}{\prod\limits_{c=0}^{v-1} (1 + cb)} \prod_{c=0}^{m-1} (w + cb) \prod_{c=0}^{v-m-1} (1 - w + cb), \tag{2}$$

(*cf.* [5, p. 68]).

In [23], we constructed new generating functions for the Bernstein-Stancu polynomials with the aid of the hypergeometric series.

The function $P_{v,m}^{<b>} (w)$ is given in terms of the ascending factorial polynomials as follows:

$$P_{v,m}^{<b>} (w) = \binom{v}{m} \frac{w_{(mb)} (1 - w)_{((v-m)b)}}{1_{(vb)}}, \tag{3}$$

where $m, v \in \mathbb{N}_0$ with $v \geqslant m$ and $w \in \mathbb{C}$ (*cf.* [23]).

With the aid of the following formula:

$$y^{(v)} = \sum_{j=0}^{v} \binom{v}{j} \frac{j}{v} y^j B_{v-j}^{(v)}, \tag{4}$$

where $B_n^{(v)}$ denotes the Bernoulli numbers of order v defined by

$$\left(\frac{t}{e^t - 1}\right)^v = \sum_{n=0}^{\infty} B_n^{(v)} \frac{t^n}{n!},$$

(*cf.* [15]), we gave derivative formulas and recurrence relations for the function $P_{v,m}^{<b>} (w)$ in terms of the higher-order Bernoulli polynomials. One of them is given by the following theorem:

Theorem 1. *(cf. [23]) Let $m, v \in \mathbb{N}_0$ with $v \geqslant m$ and $w \in \mathbb{C}$. For $b > 0$, we have*

$$P_{v,m}^{<b>}(w) = \sum_{s=1}^{m} \sum_{d=1}^{v-m} (-1)^s \binom{m-1}{s-1} \binom{v-m-1}{d-1} \binom{v}{m} \frac{w^s (w-1)^d B_{m-s}^{(m)} B_{v-m-d}^{(v-m)}}{b^{s+d} \left(-\frac{1}{b}\right)_{(v)}}, \tag{5}$$

which satisfies

$$\sum_{m=0}^{v} P_{v,m}^{<b>}(w) = 1.$$

We gave the following generating function for the polynomials $P_{v,m}^{<b>}(w)$:

Theorem 2. *(cf. [23]) Let $m \in \mathbb{N}_0$ and $w \in \mathbb{C}$. For $b > 0$, we have*

$$\begin{aligned} F_s(t, w; m, b) &= \frac{\left(\frac{w}{b}\right)_m}{m! \left(\frac{1}{b}\right)_m} t^m \, {}_1F_1 \left(\begin{matrix} \frac{1-w}{b} \\ \frac{1}{b} + m \end{matrix} ; t \right) \quad (6) \\ &= \sum_{v=0}^{\infty} P_{v,m}^{<b>}(w) \frac{t^v}{v!}. \end{aligned}$$

Note that there is one generating function for each value of m and b.

We need the following formula:

$$\frac{d^k}{dw^k} \left\{ x^{(m)} \right\} = m^{(k)} B_{m-k}^{(m+1)}(x+1), \tag{7}$$

where $B_n^{(v)}(x)$ denotes the Bernoulli numbers of order v defined by

$$\left(\frac{t}{e^t - 1} \right)^v e^{tx} = \sum_{n=0}^{\infty} B_n^{(v)}(x) \frac{t^n}{n!},$$

(*cf.* [15]).

Bèzier type curves in terms of the functions $P_{v,m}^{<b>}(w)$ and their partial derivative formulas

In [23], we constructed the following Bèzier type curves in terms of control points and generalized Bernstein-Stancu basis functions:

$$B(w; v, m; b) = \sum_{m=0}^{v} Q_m P_{v,m}^{<b>}(w),$$

where Q_m, $m \in \{0, 1, 2, \ldots, v\}$, denote the control points (*cf.* [23]).

We now give a derivative formula for $B(w; v, m; b)$ as follows:

By taking derivative:

$$\frac{\partial}{\partial w} \{B(w; v, m; b)\} = \sum_{m=0}^{v} Q_m \frac{\partial}{\partial w} \left\{ P_{v,m}^{<b>}(w) \right\}, \tag{8}$$

we get the following theorem:

Theorem 3. *Let $m, v \in \mathbb{N}$ with $v \geqslant m$ and $w \in \mathbb{C}$. For $b > 0$, we have*

$$\begin{aligned}\frac{\partial}{\partial w}\{B(w;v,m;b)\} &= \frac{1}{b}\sum_{m=0}^{v}\left(Q_m H_{m-1}\left(\frac{w}{b}\right)\right)P_{v,m}^{<b>}(w) \\ &\quad -\frac{1}{b}\sum_{m=0}^{v}\left(Q_m H_{v-m-1}\left(\frac{1-w}{b}\right)\right)P_{v,m}^{<b>}(w).\end{aligned}$$

Using (8), we also get the following formula for $\frac{\partial}{\partial w}\{B(w;v,m;b)\}$:

Theorem 4. *Let $k \in \mathbb{N}$. Let $m, v \in \mathbb{N}_0$ with $v \geqslant m$ and $w \in \mathbb{C}$. For $b > 0$, we have*

$$\begin{aligned}\frac{\partial^k}{\partial w^k}\{B(w;v,m;b)\} &= \sum_{m=0}^{v} Q_m \sum_{d=0}^{k} \frac{\binom{v}{m}\left(-\frac{1}{b}\right)^k (-1)^k \binom{k}{d} m^{(d)} m^{(k-d)}}{m!(v-m)!\binom{-\frac{1}{b}}{v}} \\ &\quad \times B_{m-d}^{(m+1)}\left(-\frac{w}{b}+1\right) B_{v-m+d-k}^{(v-m+1)}\left(\frac{w-1}{b}+1\right).\end{aligned}$$

In [14], Li et al. described the topology during subdivision of the Bézier curves using angular convergence and homeomorphism (between a Bézier curve and its control polygon under subdivision). Here, it is shown that the de Casteljau algorithm is a subdivision algorithm associated to the Bézier curves which recursively generate control polygons more closely approximating the curve under the well-known Hausdorff distance, which is defined by:

Let U and V be two non-empty subsets of a metric space. The Hausdorff distance $\mu(U, V)$ is given by

$$\mu(U,V) = \max\left\{\sup_{u\in U}\inf_{v\in V} d(u,v), \sup_{v\in V}\inf_{u\in U} d(u,v)\right\}.$$

It is also shown that topological characteristics, such as homeomorphism and ambient isotopy, exist between an initial geometric model and its approximation.

For detailed properties of these Bézier curves, see also (*cf.* [4], [5], [7], [8], [9], [12], [14], [17], [23]).

In [23], we gave the following comments and observations:

To appeal to a diverse range of researchers, Bernstein polynomials have recently gained significant traction across various scientific domains. Notably, they have found applications in classical analysis, functional analysis, approximation theory, topology, geometry, numerical analysis, algebra, linear algebra, probability and statistics, generator functions, algorithm theory, geometric design, medicine, economics, etc. For instance, Saye [16] devised an algorithm that recursively partitions topology into hyperrectangular sub-cells until simplified, with a particular topology test leveraging Bernstein polynomial properties. Numerous applications of this algorithm have been demonstrated.

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2 CONTRIBUTED PAPERS

Dynamic EWMA control chart with variable sample size for monitoring process dispersion

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In this study, we suggested a innovative approach by introducing Exponential Weighted Moving Average (EWMA) control chart utilizing Variable Sample Size (VSS) for monitoring the process dispersion. The proposed methodology utilized an integer linear functional to dynamically adjust sample sizes according to the EWMA statistic. We reveal the superiority of our recommended control chart by extensive simulations to existing EWMA control chart using Fixed sample size (FSS). The suggested VEWMA chart is more sensitively detection improvement, a decrease in the false alarm rate, and overall more effective than the existing methods. These findings provide additional justification for the basic notion that process control statistical tools needed to be dynamic, as the manufacturing process itself was dynamic. The results suggest the importance in introducing adaptive SPC methods in dynamic manufacturing environments. A real data application is performed to evaluate the validity and optimal performance of our proposed chart.

Keywords: Bayesian approach, adaptive EWMA, variable sample size, ARL, SDRL, control charts

Acknowledgments

This paper is dedicated to Professor Manuel López-Pellicer on the occasion of his 81st birth anniversary.

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Optimisation of project solutions by incorporating probabilistic transitions and multi-criteria evaluation

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The aim of the scientific research in this article is to determine the applicability of Markov decision models for decision-making in computer query systems under uncertainty and multi-criteria optimization using container virtualization as an example. The main tasks are to define the container environment problem, defining key performance indicators, implement three planning strategies, and analyze the results of experimental modeling. Current research is empirical and demonstrates the effectiveness of the Markov decision models (MDM), which are based on adaptive planning. Comparative analysis measures the values of response time, resource application, economic effectiveness and relatedness to QoS for approving adaptive planning as the best solution in balancing the effectiveness, cost and quality of service opposite to traditional cycling planning and threshold-based autoscaling. The hypothesis of MDM's applicability to the distributed computer system has been approved by experimental studies in Kubernetes. At the end of the study, an economic evaluation of the adaptive MDM strategy is presented.

2020 MSC: 60-06, 60J06, 68-06, 68W06

Keywords: Optimization, Markov decision models, container virtualization, probabilistic transitions, multi-criteria evaluation

Introduction

Markov decision models (MDMs) are a particular category of Markov processes, in which a decision maker intervenes to guide the evolution of a system at discrete moments in time [4]. Markov's theory offers a robust framework for probabilistic transitions and multicriteria evaluation [7]. A Markov decision process is characterized by the following elements:

- States (S) represent the possible configurations of the system at a given moment;
- Actions (A) indicate the set of decisions that can be implemented in each state;
- Transition probabilities (P) express the probability of moving from one state to another based on the action taken;
- Rewards (R) represent the benefits or costs associated with each action performed;
- Policy (π) is a function that associates an action to each state, orienting the decision-making process over time.

In the context of modern computing, containers offer an isolated environment for the execution of applications, while sharing the resources of the underlying system [8]. This characteristic makes them particularly suitable for dynamic resource allocation [9]. However, efficiently managing request traffic in containerized environments involves several challenges, including [4]:

- Dynamic workload variations. The volume of requests can change rapidly, requiring continuous adaptation in resource allocation;
- Quality of Service (QoS) constraints. Applications must meet specific performance requirements, such as low latency and high availability;
- Cost-consumption trade-offs. Balancing cost-efficiency with performance is a complex, multi-objective decision problem.

Markov decision processes provide a structured way to model decision problems where outcomes are uncertain and depend on probabilistic rules [5].

Methods

In Kubernetes, metrics are collected via Metrics Server (collecting data on CPU, memory and transmitting it to the horizontal autoscaling (HPA) controller), Prometheus (a monitoring system that collects user metrics from applications, nodes and pods), Kube-state-metrics / cAdvisor (information about the state of cluster objects and containers) [2]. The simulation was performed using a Kubernetes-based container orchestration system, where the request traffic was simulated based on real workload traces for the following parameters: dynamic workload fluctuations, resource usage by checking resource consumption by nodes and pods, comparison of allocated and used resources, determining QoS SLA (response time $<$ 200 ms) through the request match rate (The system should provide a response time (sending a request and receiving a response) of no more than 200 milliseconds in most cases). Three scheduling strategies were compared in the simulation:

- Traditional cycling planning scheduling – distributes requests evenly across available containers without regard to system state [2];
- Threshold-based auto scaling – containers are scaled based on predefined CPU and memory thresholds [9];
- Adaptive MDM-based scheduling – decisions are made dynamically based on a learned MDM policy [6].

Each metric was collected as follows, the results are presented in Table 1:

- Average response time implemented through Prometheus tools `histogram_quantile (0.95, rate(http_request_duration_seconds_bucket[5m]))]` and Incoming message log checks [`kubectl logs -l app=my-app -tail=100 | grep "response_time"`]
- Resource usage by checking resource consumption by nodes and pods [`kubectl top nodes` `kubectl top pods`] and in Prometheus [`rate(container_cpu _usage_seconds_total[5m])`]
- Comparison of allocated and used resources: `sum(container_memory_usage_bytes) / sum(container_spec_memory_limit_bytes)`;

- Definition of QoS SLA (response time < 200 ms). Request matching rate: `count(http_requests_ total{status="200"}> 180ms) /count(http_requests_total) * 100`

Results

The MDM approach reduced the average response time by 28% compared to threshold-based scheduling and 44% compared to cycling planning scheduling. Higher resource utilization – by dynamically adjusting container allocation, the MDM approach achieved 85% resource utilization by optimizing computing power. Improved cost-effectiveness – The MDM policy balanced power consumption and performance, resulting in a 20% improvement in cost-effectiveness compared to traditional methods. Better QoS compliance is demonstrated by the fact that the system met 97% of the QoS requirements under varying request loads, outperforming other approaches.

Metrics	Cycling planning	Threshold-based autoscaling	Adaptive planning with MDM
Average response time (ms)	320	250	180
Resource Utilization (%)	65	78	85
Cost effectiveness (relative)	1.0	0.85	1.2
QoS Compliance (%)	88	92	97

Table 1: Comparative analysis

The numerical analysis of the candlestick chart (Figure 1) in the context of optimization with probabilistic transitions and multi-criteria evaluation assumes the opening (response time) and closing (QoS) values, which allow one to directly measure two key objectives: latency minimization and service efficiency. A low opening (such as 180 ms of MDM) combined with a high closing (97%) indicates a strongly preferable performance-driven alternative.

The comparison between maximum value (resource usage) and minimum value (economic efficiency) reflects the trade-off between resources employed and costs incurred: large divergences between these extremes highlight less balanced solutions. The chart therefore allows us to visually quantify the dispersion inside the candlestick, which represents the compromise area between criteria.

To estimate this dispersion simply, we calculate the numerical difference between these two values for each method:

- Traditional cycling planning: the maximum value is 65%, while the minimum is 1.0 (scaled on a scale of 100 it would be 100%). Considering both on a 0–100 scale, the dispersion is about 65 – 100 = –35 points (in absolute value 35pt), which indicates a large gap between cost and efficiency.
- Auto-scaling on thresholds: the dispersion is 78 – 85 = –7 points (in absolute value 7pt), so a much smaller trade-off between use and cost.
- Adaptive MDM: the dispersion is even lower, with 85 – 120 = –35 points (125 if we considered a 0–100 scale, but on a relative scale the difference between 85% resources and 1.2 efficiency is about 35pt in comparative terms).

Numerically, a lower dispersion indicates a better cohesion between criteria—that is, the fact that optimizing on one criterion does not occur at the expense of quality on another. In this case, auto-scaling shows the lowest dispersion (about 7pt), suggesting a more balanced system, while MDM and traditional cycling planning show higher dispersions, therefore more marked trade-offs.

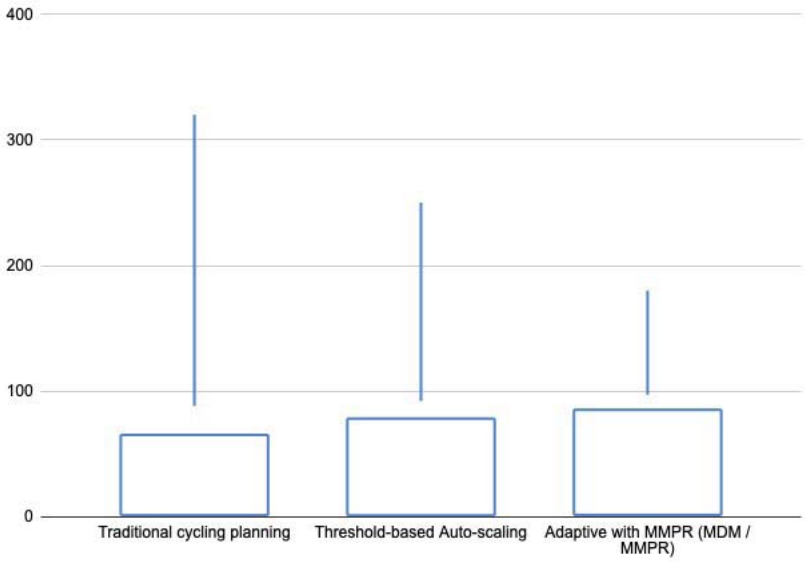

Figure 1: Comparative analysis

Discussion

In the comparative analysis of the three scheduling models, a progressive optimization of performances clearly emerges: the cycling planning, adopted as a baseline, presents rather high response times (320 ms), a resource utilization of about two-thirds of the available capacity, a standard economic efficiency (1.0) and a quality of service of 88%. The introduction of threshold auto-scaling significantly reduces latency (250 ms) and improves QoS compliance (92%), but introduces an increase in resource utilization (78%) and a slight decrease in economic efficiency (0.85), suggesting additional costs related to scalability. The best performing model is the MDM-based adaptive scheduling model, which combines a minimum response time (180 ms) with an optimal resource utilization (85%), reaching an economic efficiency higher than the baseline (1.2) and a QoS compliance of 97%. These results demonstrate how the adaptive approach can effectively combine performance improvement and cost control, while ensuring a highly reliable service in terms of quality.

From an application point of view, the choice of the scheduling model must take into account the trade-off between different objectives: if a context allows moderate latencies and has stringent constraints on operating costs, cycling planning can still be adequate; in the case of medium-high QoS requirements but without ensuring maximum performance, auto-scaling represents a balanced choice; However, in mission-critical environments (financial, healthcare, and real-time), the MDM-based adaptive model is far superior, resulting in the best choice due to its ability to reduce latencies, maximize resource efficiency, contain costs, and maintain very high levels of compliance.

To complete the study, it would be useful to extend the analysis considering horizontal and vertical scalability, to further evaluate the total cost of ownership (TCO) over time, and to integrate complementary metrics such as throughput, temporal stability, energy consumption, and fault tolerance. These insights would provide a more robust validation and a stronger basis for any real-world monitoring at operational scale. By integrating the evaluation of horizontal and vertical scaling into table 1, it's possible to get a more complete picture of the total cost of ownership (TCO) over time. Vertical scaling typically requires a higher upfront investment (CapEx) to upgrade existing nodes, while horizontal scaling distributes the CapEx across multiple servers, providing greater elasticity and fault tolerance. In the cloud context, horizontal scaling is particularly advantageous due to pay-as-you-go models and the ability to turn resources on or off based on demand.

For each of the three scheduling strategies based on the Table 2 (cycling planning, auto-scaling, adaptive MDM), you can estimate TCO components as follows: cycling planning may have low upfront costs and lower operating expenses, but is limited in terms of vertical scaling; auto-scaling introduces extra costs for load balancing and monitoring while improving operational efficiency; MDM is ideal for robust horizontal scaling, but requires distributed infrastructure and more orchestration [1]. To complete the TCO, integrating parameters such as productivity (throughput in operations per second), temporal stability (described by the variance or standard deviation of response times), energy consumption (calculated through PUE or direct measurements) and fault tolerance (percentage of failures sustained without interruption), makes the comparison more closely aligned with operational reality.

In Table 2, for the translation of CapEx and OpEx, the rate of 79.07 rub. per eu was used, the average current rate for today. CapEx for cycling planning (100,000 eu) became ~7.91 million rub., for auto-scaling (150,000 eu) ~11.86 million rub., and for MDM (200,000 eu) ~15.81 million rub. OpEx (50,000 eu, 120,000 eu, 150,000 eu) respectively = 3.95 million rub., 9.49 million rub. and 11.86 million rub.

Strategy	Scaling Type	CapEx (5 years), rub.	OpEx (5 years), rub.
Traditional Cycling Planning	Vertical	9 293 000	4 464 500
Auto-Scaling	Horizontal	13 939 500	11 151 600
MDM Adaptive	Advanced Horizontal	18 586 000	13 939 500

Table 2: TCO

The vertical model (cycling planning) offers the lowest CapEx and OpEx, but remains rigid and physically bounded. Horizontal strategies (auto-scaling and MDM) have higher CapEx and OpEx, justified by resilience, elasticity and superior performance. In particular, MDM, while costing more, is strategically justified in mission-critical environments due to higher returns in uptime days, QoS and adaptability.

Traditional cycling planning, while the most cost-effective in terms of CapEx and OpEx, has an average response time of 320 milliseconds and resource utilization of 65%, indicating lower efficiency than other strategies. Threshold-based autoscaling provides improved performance with an average response time of 250 milliseconds and resource utilization of 78%, but at a higher cost, with higher CapEx and OpEx than traditional cycling planning. Adaptive strategy with MDM, while the most cost-effective in terms of CapEx and OpEx, provides the best performance with an average response time of 180 milliseconds, resource utilization of 85%, and QoS compliance of 97%, suggesting better resource optimization and higher service reliability.

Conclusion

The study demonstrates the effectiveness of Markov decision models in optimizing design decisions in containerized environments. Using adaptive scheduling based on MDM, request traffic can be effectively managed even under uncertainty and fluctuating workloads.

MDMs outperform traditional scheduling approaches in handling dynamic request traffic by optimizing response time, resource utilization, and cost efficiency. Multi-objective optimization is necessary to balance QoS, performance, and operational costs in cloud computing environments.

The choice of scaling strategy depends on the specific priorities of the project. If the primary goal is to minimize costs, traditional cyclicling planning may be the appropriate choice, although it comes with trade-offs in performance and resource utilization. Threshold-based autoscaling represents a trade-off between cost and performance, offering improvements over traditional cyclic scheduling, but at a higher cost. Adaptive strategy with MDM, although incurring the highest costs, offers the best performance and reliability, making it ideal for projects where quality of service and resource efficiency are priorities. Therefore, the decision should be based on a careful assessment of the specific needs of the project, balancing cost, performance, and service requirements.

Future research should explore the integration of deep learning with real-time monitoring to further improve decision making in cloud infrastructures.

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New operators on covering approximation spaces via ideals

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Rough set theory is a method for solving the problem of uncertainty. So both it and its generalizations have been studied by many people. In this study, we introduce a new method for spaces obtained by adding the concept of an ideal to covering approximation spaces, which are among the generalized rough set models. We obtain some basic properties of the approximations related to it.

2020 MSC: 54A05, 54C10, 54F15

KEYWORDS: Ideal, covering approximation spaces, j-neighborhoods

Introduction and Preliminaries

The most important problem encountered in data analysis is to make decisions by using information that is incomplete or insufficient. One of the methods presented for this purpose is rough set theory. The theory focuses on eliminating uncertainty by Pawlak [14], [15] using lower (respectively upper) approximation operators. The fundamental concepts of the theory are approximation operators defined by means of equivalence classes obtained from given equivalence relation [10] such that it is reflexive, symmetric and transitive. However because the conditions for equivalence relations are difficult, it is generalized by different authors such as [8], [11], [12], [13], [16], [17] in the literature. Yao [17] used right neighborhoods instead of equivalence classes. Abd El-Monsef et al. [1] introduced j-neighborhoods concepts on neighborhood spaces. Amer et al. [3] gave new approximations in these spaces. Atef et al. [4] generalized of rough set models on neighborhood space. Al-shami et al. [2] introduced new types neighborhoods and constructed a new topology on them.

Another important topic in mathematics is the notion of an ideal proposed by Kuratowski [9], which has the hereditary and finite additivity properties of a subfamily of the power set of a given set. Studies on rough sets theory have continued using the concept of an ideal (Kandil et al. [7], Hosny [5], Hosny [6], Amer et al. [3]). Idealized types of rough sets theory are important as the boundary region is reduced, and the accuracy measure is enhanced.

This study consists of two sections. Section 1 comprises the introduction and preliminaries. In Section 2, firstly we define E_j-neighborhoods for any generalized covering approximation spaces. Then, we introduce $\mathcal{I}$-E_j-approximations by using the E_j-neighborhoods for any generalized covering approximation spaces via ideal. We obtain some properties of them. Let's recall some essential necassary notions for this paper.

Definition 1. *([17]) (1) Let A be any binary relation on Z, where Z be a finite non-empty set. The pair (Z, A) is called the Approximation Space (shortly, AS).*

(2) Let $(Z,\ A)$ be any AS. For any $z \in Z$, the successor and predecessor are defined as follows, respectively:

$$zA = A_s(z) = \{y \in Z : (z, y) \in A\},$$

$$Az = A_p(z) = \{y \in Z : (y, z) \in A\}.$$

In literature, different upper and lower approximations from obtained A_n for $n \in \{r, l\}$ are found.

Definition 2. *([18]) Let Z be a domain of discourse. Then a subfamily $\mathcal{K}$ of $\mathcal{P}(Z)$ is called a covering of Z if the union of non-empty elements of $\mathcal{K}$ is equal to Z.*

The pair $(Z, \mathcal{K})$ is called covering approximation space.

According to Definition 2 it is obvious that any partition of Z is certainly a covering of Z and any covering approximation space is a generalization of Pawlak Approximation Space.

The next two definition, four important concepts are recalled.

Definition 3. *([1]) Let $Z \neq \varnothing$ be a finite set and A be a binary relation on Z.*

The right cover(resp. left cover) of Z are defined as follows:

(1) $\mathcal{K}_r = \{zA : Z$ is equal to the union of all $zA\}$,

(2) $\mathcal{K}_l = \{Az : Z$ is equal to the union of all $Az\}$.

The triple $(Z, A, \mathcal{K}_n)$ for $n \in \{r, l\}$ is called generalized covering approximation space.

Definition 4. *([1]) Let $(Z, A, \mathcal{K}_n)$ be a $\mathcal{G}_n$-CAS for $n \in \{r, l\}$. The j-neighborhoods for every $z \in Z$ and every $j \in J = \{r, l, i, u\}$ (shortly, $N_j(z)$) are defined as follows:*

(1) $N_r(z) = \cap\{T \in \mathcal{K}_r : z \in T\}$ (i.e., right neighborhood),

(2) $N_l(z) = \cap\{T \in \mathcal{K}_l : z \in T\}$ (i.e., left neighborhood),

(3) $N_i(z) = N_r(z) \cap N_l(z)$ (i.e., intersection neighborhood),

(4) $N_u(z) = N_r(z) \cup N_l(z)$ (i.e., union neighborhood),

New j-neighborhoods and New Approximation Operators Depend on Them using Ideals

In this section firstly, to introduce new approximation operators we define new j-neighborhoods using Definition 4. Then we give some properties of these operators.

Definition 5. *Let $(Z, A, \mathcal{K}_n)$ be a $\mathcal{G}_n$-CAS for $n \in \{r, l\}$. The j-neighborhoods for every $z \in Z$ and every $j \in J = \{r, l, i, u\}$ (shortly, $N_j(z)$) are defined as follows: (shortly, $E_j(z)$) are defined as follows:*

(1) $E_r(z) = \{x \in Z : N_r(x) \cap N_r(z) \neq \varnothing\}$ (i.e., E_r-neighborhood),

(2) $E_l(z) = \{x \in Z : N_l(x) \cap N_l(z) \neq \varnothing\}$ (i.e., E_l-neighborhood),

(3) $E_i(z) = E_r(z) \cap E_l(z)$ (i.e., E_i-neighborhood),

(4) $E_u(z) = E_r(z) \cup E_l(z)$ (i.e., E_u-neighborhood),

Example 6. *Let* $(Z, A, \mathcal{K}_n)$ *be an* $\mathcal{G}_n$*-CAS such that* $Z = \{z_1, z_2, z_3, z_4\}$ *and* $A = \{(z_1, z_1), (z_1, z_3), (z_1, z_4), (z_2, z_1), (z_2, z_3), (z_3, z_1), (z_3, z_2), (z_3, z_4), (z_4, z_2)\}$*. Since* $\mathcal{K}_r = \{\{z_1, z_3, z_4\}, \{z_1, z_3\}, \{z_1, z_2, z_4\}, \{z_2\}\}$*, we have* $N_r(z_1) = \{z_1\}$*,* $N_r(z_2) = \{z_2\}$*,* $N_r(z_3) = \{z_1, z_3\}$*,* $N_r(z_4) = \{z_1, z_4\}$ *for* $j = r$*. Similarly, we obtain that* $E_r(z_1) = \{z_1, z_3, z_4\}$*,* $E_r(z_2) = \{z_2\}$*,* $E_r(z_3) = \{z_1, z_3, z_4\}$*,* $E_r(z_4) = \{z_1, z_3, z_4\}$*.*

Definition 7. *Let* $(Z, A, \mathcal{K}_n)$ *be a* $\mathcal{G}_n$*-CAS for* $n \in \{r, l\}$ *and* $\mathcal{I}$ *be an ideal on Z. The quaternary* $(Z, A, \mathcal{K}_n, \mathcal{I})$ *is said to be ideal generalized covering approximation space (shortly,* $\mathcal{I}\mathcal{G}_n$*-CAS).*

Definition 8. *Let* $(Z, A, \mathcal{K}_n, \mathcal{I})$ *be an* $\mathcal{I}\mathcal{G}_n$*-CAS. The* $\mathcal{I}\underline{E}_j$*-lower approximations and* $\mathcal{I}\overline{E}_j$*-upper approximations of* X *for* $j \in J = \{r, l, i, u\}$ *and* $X \subseteq Z$ *are defined as follows:*

(1) $\mathcal{I}\underline{E}_j(X) = \{z \in Z : (E_j(z) \cap X^c) \in \mathcal{I}\}$*,*

(2) $\mathcal{I}\overline{E}_j(X) = \{z \in Z : (E_j(z) \cap X) \in \mathcal{I}\}$*.*

Theorem 9. *Let* $(Z, A, \mathcal{K}_n, \mathcal{I})$ *be an* $\mathcal{I}\mathcal{G}_n$*-CAS,* X *and* Y *be two subsets of Z. Then, for every* $j \in J$*, the next properties are hold:*

(1) $\mathcal{I}\underline{E}_j(Z) = Z$ *and* $\mathcal{I}\overline{E}_j(\varnothing) = \varnothing$*,*

(2) If $X \subseteq Y$*, then* $\mathcal{I}\underline{E}_j(X) \subseteq \mathcal{I}\underline{E}_j(Y)$ *and* $\mathcal{I}\overline{E}_j(X) \subseteq \mathcal{I}\overline{E}_j(Y)$*,*

(3) $\mathcal{I}\underline{E}_j(X) \cap \mathcal{I}\underline{E}_j(Y) = \mathcal{I}\underline{E}_j(X \cup Y)$ *and* $\mathcal{I}\overline{E}_j(X) \cup \mathcal{I}\overline{E}_j(Y) = \mathcal{I}\overline{E}_j(X \cup Y)$*,*

(4) $\mathcal{I}\underline{E}_j(X) \cap \mathcal{I}\underline{E}_j(Y) \subseteq \mathcal{I}\underline{E}_j(X \cup Y)$*,* $\mathcal{I}\overline{E}_j(X \cap Y) \subseteq \mathcal{I}\overline{E}_j(X) \cap \mathcal{I}\overline{E}_j(Y)$ *and*

(5) $\mathcal{I}\underline{E}_j(X^c) = (\mathcal{I}\overline{E}_j(X))^c$ *and* $\mathcal{I}\overline{E}_j(X^c) = (\mathcal{I}\underline{E}_j(X))^c$*,*

(6) If $X^c \in \mathcal{I}$*, then* $\mathcal{I}\underline{E}_j(X) = Z$ *and if* $X \in \mathcal{I}$*, then* $\mathcal{I}\overline{E}_j(X)) = \varnothing$*.*

Proof. (1) Since $Z^c = \varnothing$ and $\varnothing \in \mathcal{I}$, we have $\mathcal{I}\underline{E}_j(Z) = Z$. Similarly, $\mathcal{I}\overline{E}_j(\varnothing) = \varnothing$ is obtained.

(2) Let $z \in \mathcal{I}\underline{E}_j(X)$. Then, $(E_j(z) \cap X^c) \in \mathcal{I}$. Since $X \subseteq Y$, $(E_j(z) \cap Y^c) \in \mathcal{I}$ and we have $z \in \mathcal{I}\underline{E}_j(Y)$. Hence $\mathcal{I}\underline{E}_j(X) \subseteq \mathcal{I}\underline{E}_j(Y)$. Similarly, we obtain $\mathcal{I}\overline{E}_j(X) \subseteq \mathcal{I}\overline{E}_j(Y)$ using Definition 7(2).

(3) $\mathcal{I}\underline{E}_j(X \cap Y) \subseteq (\mathcal{I}\underline{E}_j(X) \cap \mathcal{I}\underline{E}_j(Y))$ is clear from intersection property. Now, let $z \in (\mathcal{I}\underline{E}_j(X) \cap \mathcal{I}\underline{E}_j(Y))$. So, $(E_j(z) \cap X^c) \in \mathcal{I}$ and $(E_j(z) \cap Y^c) \in \mathcal{I}$. From property of finite additivity of ideal and De Morgan Rule, we obtained that $(E_j(z) \cap X^c) \cup (E_j(z) \cap Y^c) = (E_j(z) \cap (X \cap Y)^c) \in \mathcal{I}$ and $z \in \mathcal{I}\underline{E}_j(X \cap Y)$. Hence, we have $(\mathcal{I}\underline{E}_j(X) \cap \mathcal{I}\underline{E}_j(Y)) \subseteq \mathcal{I}\underline{E}_j(X \cap Y)$ and the equality is obtained.

(4) The proofs are clear from (2).

(5) Let $z \in \mathcal{I}\underline{E}_j(X^c)$. Then for every $z \in Z$, $(E_j(z) \cap X) \in \mathcal{I}$. Hence, we have $z \notin \mathcal{I}\underline{E}_j(X)$ and $z \in (\mathcal{I}\underline{E}_j(X))^c$. This shows that $\mathcal{I}\underline{E}_j(X^c) \subseteq (\mathcal{I}\overline{E}_j(X))^c$. Of course, conversely can be proved similarly. The other equality is obtained similarly using necassary definitions.

(6) From hypothesis and property of hereditary of ideal, we have $(E_j(z) \cap X^c) \in \mathcal{I}$ for every $z \in Z$ and hence $\mathcal{I}\underline{E}_j(X) = Z$. The other proof is obtained similarly.

□

The converse implications of Theorem 1(2) and the converse inclusions of Theorem 1(4) may not true in generally as shown the next example.

Example 10. *Let $(Z, A, \mathcal{K}_n)$ be an $\mathcal{G}_n$-CAS as in Example 1 ,i.e., $Z = \{z_1, z_2, z_3, z_4\}$, $A = \{(z_1, z_1), (z_1, z_3), (z_1, z_4), (z_2, z_1), (z_2, z_3), (z_3, z_1), (z_3, z_2), (z_3, z_4), (z_4, z_2)\}$ and $\mathcal{I} = \{\emptyset, \{z_2\}, \{z_3\}, \{z_2, z_3\}\}$ be an ideal on Z. In an $\mathcal{IG}_n$-CAS $(Z, A, \mathcal{K}_n, \mathcal{I})$ we take*

(1) $X = \{z_3\}$ and $Y = \{z_4\}$ subsets of Z. For $j = r$, we have $\mathcal{I}\underline{E}_r(X) = \{z_1, z_2\}$ and $\mathcal{I}\underline{E}_r(Y) = \{z_2\}$. Although $\mathcal{I}\underline{E}_r(Y) = \{z_2\} \subseteq \{z_1, z_2\} = \mathcal{I}\underline{E}_r(X)$, but $Y \subseteq X$. Besides, $\mathcal{I}\underline{E}_r(X \cup Y) = Z \neq \{z_1, z_2\} = \mathcal{I}\underline{E}_r(X) \cup \mathcal{I}\underline{E}_r(Y)$.

(2) $X = \{z_1, z_3\}$ and $Y = \{z_2, z_4\}$ subsets of Z. For $j = r$, we have $\mathcal{I}\overline{E}_r(X) = \{z_1, z_3, z_4\}$ and $\mathcal{I}\overline{E}_r(Y) = \{z_1, z_3, z_4\}$. Although $\mathcal{I}\overline{E}_r(X) = \{z_1, z_3, z_4\} \subseteq \{z_1, z_3, z_4\} = \mathcal{I}\overline{E}_r(Y)$, but $X \subseteq Y$. Besides, $\mathcal{I}\overline{E}_r(X \cap Y) = \{z_1, z_3, z_4\} \neq \emptyset = \mathcal{I}\overline{E}_r(X \cap Y)$.

Related two operations introduced in Definition 7, we have the next theorem without proof.

Theorem 11. *Let $(Z, A, \mathcal{K}_n, \mathcal{I})$ be an $\mathcal{IG}_n$-CAS and $X \subseteq Z$. Then, the next properties are hold:*

(1) $\mathcal{I}\underline{E}_u(X) \subseteq \mathcal{I}\underline{E}_r(X) \subseteq \mathcal{I}\underline{E}_i(X)$,

(2) $\mathcal{I}\underline{E}_u(X) \subseteq \mathcal{I}\underline{E}_l(X \subseteq \mathcal{I}\underline{E}_i(X)$,

(3) $\mathcal{I}\overline{E}_i(X) \subseteq \mathcal{I}\overline{E}_r(X \subseteq \mathcal{I}\overline{E}_u(X)$,

(4) $\mathcal{I}\overline{E}_i(X) \subseteq \mathcal{I}\overline{E}_l(X \subseteq \mathcal{I}\overline{E}_u(X)$.

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An improved version of ZLindley distribution with mathematical properties and applications

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An advanced extension ZLindley distribution called the size-biased ZLindley (SBZL) distribution has been developed via a weighted approach. This new modification enhances the flexibility of the standard distribution by refining its functional shape and enabling it to model the most probable form of the hazard rate function effectively. The new model includes various distinct sub-models based on its parameter values. Here we discussed and studied its two variants: the length-biased ZLindley and area-biased ZLindley distributions. Key properties, such as moments, mean residual life function, moment-generating function, and entropy and associated computational features, were described in depth. To estimate the model parameters, four different estimation methods were applied, and a detailed simulation study identified the most effective approach. The practicality and efficiency of the SBZL distribution model were validated using datasets from two distinct fields, where it was found to deliver superior results compared to other competing distributions.

Keywords: Extended distribution, ZLindley model, moments, inference, modeling

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Correlation analysis of agricultural data in Kumluca region

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Correlation analysis is a statistical technique employed to assess the strength and direction of the association between variables. In this study, the 2024 agricultural production data obtained from the Kumluca District Agricultural Directorate were evaluated under three main categories: greenhouse production, fruit production, and field production. For each type of production, the relationships between production quantity, cultivated area, and production value were analyzed. Since the data did not meet the normality assumption, Spearman's rank correlation coefficient was used. The analysis results revealed positive and significant relationships between variables in all production categories. Strong correlations were found particularly in greenhouse and fruit production, while these relationships were moderate in field production. The high correlation between production quantity and production value highlights the impact of agricultural productivity on economic outputs, while the weaker relationships between cultivated area and production value indicate that land size alone does not guarantee economic success.

2020 MSC: 62P05, 62H20, 62G10, 62-06

Keywords: Statistics, correlation analysis, Spearman correlation, agricultural data

Introduction

Agriculture has played a decisive role in the economic, social, and cultural development of societies throughout history. Today, it continues to be of strategic importance in many areas, such as food security, employment, rural development, and the sustainable use of natural resources (*cf.* [1]).

Turkey is one of the countries that can grow a wide variety of agricultural products thanks to its diverse climate types and geographical structure. This diversity makes it possible to meet domestic consumption needs and export agricultural products to international markets. In this respect, the agricultural sector contributes significantly to Turkey's economic growth and foreign trade (*cf.* [2]).

Antalya Province, one of Turkey's important agricultural regions, attracts attention with its favorable climate conditions, fertile soils, and irrigation facilities (*cf.* [3]). The district of Kumluca, located in the southwestern part of Antalya, stands out for its high performance and product diversity in greenhouse production (*cf.* [4]).

Kumluca is divided into three different agricultural production zones thanks to its geographical structure. In the coastal zone (0–300 m), greenhouse cultivation and citrus farming are common. In the transitional zone (300–600 m), pomegranate, olive, grain, and open-field vegetable production takes place. In the high mountainous zone (700 m and above), crops such as cherry, walnut, wheat, and beans are cultivated (*cf.* [4]).

This geographical and climatic diversity enables year-round production in Kumluca and has a decisive impact on important agricultural indicators such as production area, quantity, and economic value.

In studies on agricultural production, the relationships between indicators such as production area, quantity, and economic value are generally evaluated using statistical analyses. Correlation and regression analyses are among the methods frequently used to reveal such relationships (*cf.* [5, 6]).

Studies on regional production data in agriculture generally focus on analyzing production volumes and economic outcomes; however, research comparing these variables across categorical production types (e.g., greenhouse, fruit, and field) is limited. In this context, this study aims to reveal the relationship structure between key variables related to agricultural production in Kumluca district through Spearman's rank correlation analysis and to evaluate the differences between production types.

Materials and Methods

This study uses agricultural production data for 2024 from the Kumluca District Agricultural Directorate. The data has been evaluated in three main categories: greenhouse, fruit, and field production. Products in each category have been analyzed in terms of production area (decares), production quantity (tons), and production value (TRY).

In the categories of greenhouse, fruit, and field production, correlation analysis is one of the statistical methods that can be used to reveal the relationship structure between the variables of production area, production quantity, and production value.

Correlation analysis is a statistical technique applied to identify the strength and direction of the relationship among two or more quantitative variables. Through correlation analysis, it is possible to identify whether there is a linear or ordinal relationship between the variables and to assess how strong this relationship is. The correlation coefficient ranges from -1 to $+1$; positive values indicate a relationship in the same direction, negative values indicate a relationship in the opposite direction, while values close to zero indicate no relationship between the two variables (*cf.* [7, 8]).

In correlation analysis, the structure of the relationship between variables is determined based on the type of data, measurement level, and distribution characteristics. In this context, the most commonly used correlation coefficients include the Pearson correlation coefficient, which is suitable for linear relationships, and the Spearman rank correlation coefficient and Kendall rank correlation coefficient, which are preferred for ordered data or monotonic relationships (*cf.* [7, 9]).

The Spearman correlation coefficient (ρ) is a nonparametric method that measures the strength and direction of the relationship between two variables in a data set based on the ranking of values rather than raw data. This approach is employed to investigate the monotonic relationships between variables. Spearman's correlation can be expressed as the Pearson correlation coefficient calculated using the ranked values of the variables (*cf.* [9, 10]).

When calculating Spearman's rank correlation coefficient (ρ), the x and y variables are first ranked from 1 to N (*cf.* [11]). Here, $x_{i,r}$ is defined as the rank value of the i-th observation of the x variable, and $y_{i,r}$ is defined as the rank value of the i-th observation of the y variable .

During ranking, if two or more observations have the same value, the tied rank method is applied to these observations. In this method, the arithmetic mean of the rank numbers that the tied observations should have is assigned to them. Using the tied rank method, the Spearman rank correlation coefficient is calculated with the following formula (*cf.* [11]):

$$\rho = \frac{\sum_{i=1}^{N} x_{i,r} y_{i,r}}{\sqrt{\sum_{i=1}^{N} x_{i,r}^2 \sum_{i=1}^{N} y_{i,r}^2}}$$

If there are no observations with the same value during ranking, the formula can be further simplified and calculated as follows (*cf.* [11]):

$$\rho = 1 - \frac{6\sum_{i=1}^{N}(x_{i,r-} y_{i,r})^2}{N(N^2-1)}$$

Result

Agricultural production data has been divided into three main categories: greenhouse production, fruit production, and field production. For each category, a data set consisting of the variables production area (decares), production quantity (tons), and production value (TRY) has been defined in the IBM SPSS Statistics 23 program.

Considering the distribution characteristics of the data set, it was deemed appropriate to use Spearman's rank correlation coefficient. It was determined that variables such as production area, production quantity, and production value did not meet the normal distribution assumption in some categories, and that there were outliers in the data set. This situation limits the use of Pearson correlation analysis, which is a parametric method.

Spearman correlation, on the other hand, is a non-parametric method that does not require normality assumptions and performs ranking-based calculations, enabling more reliable analysis of the relationships between physical and economic indicators related to the production process.

Correlation tables showing the relationships between the variables of production area, production quantity, and production value for products in each production category were obtained. Spearman correlation coefficients related to the analysis results are given in the tables below.

	Production Area	Production Quantity	Production Value
Production Area	1		
Production Quantity	0.972**	1	
Production Value	0.962**	0.965**	1
**. Correlation is significant at the 0.01 level (2-tailed).			

Table 1: Greenhouse production correlation coefficients

Table 1; shows that the relationships between variables in the greenhouse production category are strong, positively correlated, and statistically significant. High levels of correlation were detected between production area and production quantity ($\rho = 0.972$, $p < 0.01$), production area and production value ($\rho = 0.962$, $p < 0.01$), and production quantity and production value ($\rho = 0.965$, $p < 0.01$). These results indicate that physical production indicators are closely related to economic output in greenhouse production.

	Production Area	Production Quantity	Production Value
Production Area	1		
Production Quantity	0.826**	1	
Production Value	0.923**	0.937**	1
**. Correlation is significant at the 0.01 level (2-tailed).			

Table 2: Fruit production correlation coefficients

Table 2; shows strong, positive, and statistically significant relationships among the variables in the fruit production category. High levels of correlation were detected between production area and production quantity ($\rho = 0.826$, $p < 0.01$), production area and production value ($\rho = 0.923$, $p < 0.01$), and production quantity and production value ($\rho = 0.937$, $p < 0.01$). These findings indicate that production area and production quantity, which are physical production indicators in fruit production, are directly and strongly related to production value, which is an economic output. In particular, the high correlation between production quantity and production value points to the decisive effect of productivity on economic value.

	Production Area	Production Quantity	Production Value
Production Area	1		
Production Quantity	0.697*	1	
Production Value	0.685*	0.830**	1
*. Correlation is significant at the 0.05 level (2-tailed). **. Correlation is significant at the 0.01 level (2-tailed).			

Table 3: Field production correlation coefficients

Table 3; shows that the variables in the field production category exhibit moderate to high levels of positive correlation and statistically significant relationships. There was a moderate correlation between production area and production quantity ($\rho = 0.697$, $p < 0.05$) and between production area and production value ($\rho = 0.685$, $p < 0.05$), while a stronger correlation was found between production quantity and production value ($\rho = 0.830$, $p < 0.01$). These results once again highlight the decisive role of production quantity in economic output.

Conclusion

The relationships between the variables of production area, production quantity, and production value were examined based on the 2024 agricultural production data for the district of Kumluca. Agricultural production was evaluated under three main categories: greenhouse production, fruit production, and field production. The statistical relationships between the physical production indicators and economic outputs of the products in each category were analyzed using Spearman's rank correlation coefficient.

Correlation analyses revealed positive and significant relationships between variables in all production categories. In the greenhouse production category, the relationships between production area, production quantity, and production value are quite strong, indicating that greenhouse agriculture offers a more controlled and efficient production structure. Similar strong relationships were observed in the fruit production category. In field production, although the correlation levels were lower than in other categories, a high relationship was found between production quantity and production value.

The findings are noteworthy, particularly because the correlation between production quantity and production value is highest in all three categories. This result shows that productivity in agricultural production has a direct and decisive impact on economic value. In contrast, the fact that the relationship between production area and production value is weaker in some categories shows that the size of the cultivated area alone does not guarantee economic success.

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On numbers with four term recurrence and binomial transforms

Abdessadek Saib

In this talk we consider sequences of numbers generated by a four-term linear recurrence relation, presenting partial and weighted sums via various techniques. We then explore their p-binomial transforms (falling, rising), proving each adheres to a four-term recurrence. Finally, we examine iterated p-binomial transforms, formed by t recursive applications of the transform, and establish that all such iterated sequences satisfy a four-term recurrence relation depending only on t.

2020 MSC: 11B37, 11B39, 11B65, 11B83, 11K65, 15Bxx

KEYWORDS: Four-term recurrence, p-binomial transform, recurrence relation, weighted sum

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Existence of non-trivial solutions to a homogeneous fractional differential problem

Anabela S. Silva

In this work, a Lyapunov-type inequality is presented for a class of fractional boundary value problems involving Riemann–Liouville derivatives of orders $\alpha \in (1, 2)$ and $\beta \in (0, \alpha-1)$. To obtain the result, the Green's function of the integral solution to the problem is maximised.

2020 MSC: 26A33, 34A08, 26D10, 34B27

KEYWORDS: Fractional differential equations, Green's function, Lyapunov inequality Riemman-Liouville derivative

Introduction

The classical Lyapunov inequality provides a necessary condition for the existence of nontrivial solutions to second-order boundary value problems. These type of inequalities are powerful tools in the analysis of differential equations and dynamical systems. Applications are known in different areas such as oscillation and disconjugacy, eigenvalue estimation and stability analysis.

This work extends the classical Lyapunov inequality (cf. [3]) to fractional differential equations involving Riemann-Liouville derivatives. The present analysis relies on reformulating the problem as an integral equation through the construction of a Green's function (e.g., [1, 2, 4]). The maximum of this function is computed to establish a necessary condition for the existence of nontrivial solutions.

The paper considers the fractional boundary value problem

$$\begin{cases} (\mathcal{D}^{\alpha}_{a+}x)(t) + (\mathcal{D}^{\beta}_{a+}(qx))(t) = 0, \quad t \in [a, b], \\ x(a) = x(b) = 0, \end{cases} \tag{1}$$

where $\alpha \in (1, 2)$, $\beta \in (0, \alpha - 1)$, and $q \in C([a, b])$ is a real-valued function.

Preliminaries

Let us recall the definitions and some properties of the Riemann-Liouville fractional integral and derivative (cf. [5]).

Definition 1. *Let $\alpha > 0$. The (left) Riemann-Liouville fractional integral of order α for a function $x : [a, b] \to \mathbb{R}$ is defined as:*

$$I^{\alpha}_{a+}x(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} x(s)\, ds,$$

where $\Gamma(\alpha)$ is the Euler Gamma function.

Definition 2. *Let $\alpha > 0$ and $n = \lceil \alpha \rceil$. The (left) Riemann-Liouville fractional derivative of order α is given by:*

$$\mathcal{D}_{a+}^{\alpha} x(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t (t-s)^{n-\alpha-1} x(s)\, ds.$$

Two important properties are presented here:

- **Semigroup Property:** The composition of two fractional integrals satisfies:

$$I_{a+}^{\alpha}(I_{a+}^{\beta} x)(t) = I_{a+}^{\alpha+\beta} x(t).$$

- **Inverse Property:** For sufficiently regular functions,

$$\mathcal{D}_{a+}^{\alpha}(I_{a+}^{\alpha} x)(t) = x(t), \quad \text{and} \quad I_{a+}^{\alpha}(\mathcal{D}_{a+}^{\alpha} x)(t) = x(t) + \sum_{i=1}^{n} c_i (t-a)^{\alpha-i},$$

  where $c_i \in \mathbb{R}$.

Main results

The solution $x(t)$ of the boundary value problem (1) can be expressed as

$$x(t) = \int_a^b G(t,s) q(s) x(s)\, ds,$$

where the Green's function $G(t,s)$ is defined by

$$G(t,s) = \frac{1}{\Gamma(\alpha-\beta)} \begin{cases} \frac{(t-a)^{\alpha-1}(b-s)^{\alpha-\beta-1}}{(b-a)^{\alpha-1}}, & a \leqslant t \leqslant s \leqslant b \\ \frac{(t-a)^{\alpha-1}(b-s)^{\alpha-\beta-1}}{(b-a)^{\alpha-1}} - (t-s)^{\alpha-\beta-1}, & a \leqslant s \leqslant t \leqslant b \end{cases}. \tag{2}$$

Theorem 3. *For all $(t,s) \in [a,b] \times [a,b]$, the Green function satisfies:*

$$\max_{t,s \in [a,b]} |G(t,s)| = \frac{\left(\frac{\alpha-1}{2\alpha-\beta-2}\right)^{\alpha-1} \left(\frac{\alpha-\beta-1}{2\alpha-\beta-2}\right)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} (b-a)^{\alpha-\beta-1}.$$

This result is fundamental to establish the main inequality.

Theorem 4. *Let x be a nontrivial continuous solution of (1). Then*

$$\int_a^b |q(s)| ds \geqslant \frac{\Gamma(\alpha-\beta)(b-a)^{1+\beta-\alpha}}{\left(\frac{\alpha-1}{2\alpha-\beta-2}\right)^{\alpha-1} \left(\frac{\alpha-\beta-1}{2\alpha-\beta-2}\right)^{\alpha-\beta-1}}.$$

Conclusion

A Lyapunov-type inequality is constructed for fractional boundary value problems involving Riemann-Liouville derivatives of different orders. The result enhances existing theory and offers a foundation for further exploration in fractional differential equations.

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Bernoulli infinite series and its effects

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The main purpose of this presentation is to study explicit formula for certain classes of the infinite series involving the family of polylogarithm functions, and the Riemann zeta type functions. By applying integral formulas to some special series and functions, with Bernoulli polynomials and numbers, some formuşlas and relations are given.

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KEYWORDS: Polylogarithmic function, Riemann zeta function, Bernoulli number, Bernoulli polynomial, infinite series

Introduction

Bernoulli's series is known as the series:

$$\sum_{k=1}^{\infty} \frac{1}{k^2}.$$

Generalized Bernoulli series are given as follows:

$$\sum_{k=1}^{\infty} \frac{1}{k^3}$$

$\ddots$

$$\sum_{k=1}^{\infty} \frac{1}{k^m}$$

where m is an positive integer. Since the 1400s, numerous mathematicians have studied this type of series. In particular, despite devoting nearly 20 years to this topic in the 1500s, Swiss mathematician Jacob Bernoulli was unable to find the sum of this series

$$\sum_{k=1}^{\infty} \frac{1}{k^2}.$$

After his death, Euler succeeded in finding the sum of this series

$$\sum_{k=1}^{\infty} \frac{1}{k^2}$$

as

$$\frac{\pi}{6}.$$

Later, German mathematician Riemann would generally defined the following series:

$$\sum_{k=1}^{\infty} \frac{1}{k^s},$$

where s is an complex numbers with real part greater than 1. Today, this function is known as the Riemann zeta function.

The polylogarithm function $\mathbf{Li}_n(z)$ is defined by

$$\mathbf{Li}_n(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^n} \tag{1}$$

where $|z| \leqslant 1$, $n \in \mathbb{Z}^+$ and $z \neq 1$. This function $\mathbf{Li}_n(z)$ has important applications in mathematics, in evaluating special series etc. The function $\mathbf{Li}_n(z)$ is continuous on the closed unit disk $|z| \leqslant 1$ and analytic in its interior (cf. [1], [2], [8], [9]).

Substituting n=2 and z=1 into (1), we have the Bernoulli series:

$$\mathbf{Li}_2(1) = \zeta(2) = \frac{\pi^2}{6}$$

where $\zeta(s)$ denotes the Riemann zeta function (cf. [1]- [9]).

For $n \in \mathbb{N}$, with $n \geqslant 2$,and $|z| < 1$, with the aid of the mathematical induction method, one has

$$\mathbf{Li}_n(z) = \int_0^z \frac{\mathbf{Li}_{n-1}(t)}{t} dt$$

(cf. [1], [2], [8], [9]).

For $a > -1$ and $\lambda \in (0, \infty)$. We now study on the following integral formulas involving the function $\mathbf{Li}_n(z)$:

$$\int_0^1 \int_0^1 \frac{dxdy}{\lambda + xy} = \int_0^1 \frac{\ln(1 + \frac{y}{\lambda})}{y} dy,$$

$$\int_0^1 \frac{\ln(1 + y)dy}{y} = \frac{\pi^2}{12},$$

and

$$\int_0^1 \frac{\ln(1 + \frac{y}{2})dy}{y} = \frac{\pi^2}{12} + \frac{1}{2} \ln^2 2$$

(cf. [1], [2], [8], [9]).

The following formula can be used in the next section: For $n \in \mathbb{N}_0$,

$$\mathbf{Li}_n(z) = -(-1)^n \mathbf{Li}_n(\frac{1}{z}) - \frac{(2\pi i)^n}{n!} B_n \left(\frac{\log z}{2\pi i} \right) \tag{2}$$

(cf. [3]).

Main results

Our aim is to find explicit formula for the following integral in terms of the function $\mathbf{Li}_n(z)$ and other special functions and series involving special polynomials and numbers:

$$\underbrace{\int_0^1 \int_0^1 \cdots \int_0^1}_{k-times} \frac{dx_1 dx_2 \ldots dx_k}{\lambda + x_1 x_2 \ldots x_k} \tag{3}$$

The above integral was studied in [6].

For instance when $\lambda = 1$, one has

$$\int_0^1 \int_0^1 \frac{dxdy}{1+xy} = \frac{\pi^2}{12}$$

(cf. [6]).

$$\int_0^1 \int_0^1 \frac{dxdy}{1-xy} = \frac{\pi^2}{6}$$

(cf. [5]).

$$\int_0^1 \int_0^1 \int_0^1 \frac{dxdydz}{1+xyz} = \frac{3}{8}\zeta(3).$$

By using integral by parts and combining with geometric series, the following formula can be deriven:

$$\int_0^1 \int_0^1 \int_0^1 \frac{dxdydz}{1+xyz} = \frac{3}{8}\zeta(3).$$

By the help of the above integral formulas, and mathematical induction method, it is possible to give explicit formula for (1).

Conclusion

Our future aim is to give relations among (1), (2),

$$\underbrace{\int_0^1 \int_0^1 \int_0^1 \cdots \int_0^1}_{n-times} \frac{dx_1 dx_2 ... dx_{n-1} dx_n}{-\frac{1}{z} + x_1 x_2 ... x_{n-1} x_n},$$

the Bernoulli polynomials, and also othe special series and functions.

Is it possible to give relation between (3) and the following zeta type function:

$$\mathcal{Z}_3(s; a, \lambda) = \frac{2}{(\log a)^s} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^s \lambda^{n+1}},$$

which was studied in detail in (cf. [7])?

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A study on geometry of Bezier-like curves in the three-dimensional real space

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The main purpose of this work is to construct the Frenet frame and calculate curvatures of Bézier-like curves in the three dimensional real space. In addition, we calculate this frame and curvatures at the beginning and ending points.

2020 MSC: 33B10, 53A04, 68U05

Keywords: Bernstein polynomials, Bézier curve, exponential function, Frenet frame, curvature

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The Poisson copula distribution: An alternative framework for modeling count data

Abdullah Zaagan

In this article, we propose a novel two-parameter count probability model derived from the compounding of the Poisson and Copoun distributions. This model exhibits considerable versatility in terms of its probability density and hazard rate functions. To assess the efficacy of the proposed model, key statistical properties including moments, moment-generating function, probability generating function, factorial moments, Rényi entropy, Shannon entropy, and order statistics have been rigorously derived. The parameters of the new model are estimated using the maximum likelihood estimation technique. A comprehensive simulation study is conducted to examine the behavior of the derived estimates. Furthermore, the flexibility and practical applicability of the proposed distribution are demonstrated through its application to three real-world datasets from diverse domains. The results indicate that the new distribution provides a more efficient fit to these datasets compared to existing competing models. As a flexible extension of the Poisson distribution, this model is particularly wellsuited for analyzing over-dispersed count data.

Keywords: Compounding, Copoun distribution, count data analysis, estimation, Poisson distribution

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Applications of differential subordination on certain subclass of meromorphically multivalent functions of order defined by a linear operator

Badriah Alamri

In this paper, we introduce and study a new subclass of meromorphically multivalent functions of order α defined by means of a generalized linear operator. By using the theory of differential subordination, we establish several inclusion relationships, sharp subordination results, and best dominant estimates for functions in this class.

The study begins by establishing the foundational definitions and notations, particularly the concept of subordination between analytic functions and the Hadamard product (or convolution). A linear operator involving hypergeometric functions is defined, generalizing operators previously introduced in the literature by Liu, Srivastava, Aouf, and others. The paper introduces a new function class denoted by $\sum\limits_{p,m}^{n}(A, B, \alpha)$, defined by a differential subordination condition involving the proposed linear operator as follows:

For fixed parameters $A, B(-1 \leqslant B < A \leqslant 1)$ and $0 \leqslant \alpha < p$, we say that a function $f(z) \in \sum\limits_{p,m}$ is in the class $\sum\limits_{p,m}^{n}(A, B, \alpha)$, if it is satisfies the following subordination condition:

$$\frac{1}{\left(1-\frac{\alpha}{p}\right)}\left(-\frac{z^{p+1}\left(D_p^n f(z)\right)'}{p}-\frac{\alpha}{p}\right) \prec \frac{1+Az}{1+Bz}$$

or

$$-\frac{z^{p+1}\left(D_p^n f(z)\right)'}{p} \prec \frac{1+\left[B+(A+B)\left(1-\frac{\alpha}{p}\right)\right]z}{1+Bz},$$

$$(n \in \mathbb{N}_0;\ p \in \mathbb{N};\ 0 \leqslant \alpha < p;\ z \in U).$$

Which equivalent to the following condition:

$$\left|\frac{z^{p+1}\left(D_p^n f(z)\right)'+p}{Bz^{p+1}(D_p^n f(z))+[pB+(A-B)(p-\alpha)]}\right| < 1,\ (z \in U)$$

For convenience, we write $\sum\limits_{p,m}^{n}(1, -1, \alpha) = \sum\limits_{p,m}^{n}(\alpha)$, where $\sum\limits_{p,m}^{n}(\alpha)$ denotes the class of functions in $\sum\limits_{p,m}$ satisfying the following inequality:

$$\Re\left\{-z^{p+1}\left(D_p^n f(z)\right)'\right\} > \alpha,\ \ (0 \leqslant \alpha < p;\ n \in \mathbb{N}_0;\ p \in \mathbb{N};\ z \in U)$$

Several preliminary lemmas are recalled supporting the main results, including classical results related to convex and univalent functions, the hypergeometric

function identities, and properties of convex hulls. The core of the paper presents a sequence of theorems that:

- Provide sufficient conditions under which a function belongs to the proposed class;
- Determine best possible dominant functions;
- Establish inclusion relationships between the new subclass and other known subclasses;
- Demonstrate sharp inequalities and optimal estimates under various constraints on the parameters involved.

Through special cases and corollaries, the paper recovers and improves upon numerous known results in the field, showcasing the generality and strength of the proposed framework. Furthermore, the study includes convolution conditions, Schwarz lemma applications, and parameter-specific refinements that lead to tighter bounds and enhanced understanding of meromorphic function behavior under the new operator.

The work concludes with acknowledgments to prior scholars whose contributions laid the groundwork for this extension, especially the collaborations in differential subordination theory.

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An examination on the concept of proof

Beyza Coskuner

In mathematics, proof is a fundamental tool at the core of the discipline, revealing relationships between concepts and contributing to processes of verification, prediction, and generalization. This process is not limited to algebraic and arithmetic connections but establishes new relationships among different mathematical structures and is considered an inseparable element of teaching reasoning. Although the historical origins of proof cannot be precisely determined, the earliest examples are known to date back to the Babylonians. Figures such as Pythagoras, Eudoxus, and Euclid pioneered the development of proof understanding, while in the 19th century, Fourier, Galois, and Riemann made significant contributions. In the 20th century, Hilbert provided important contributions. The development of proof skills begins at an early age with concepts such as classification and comparison, forming the foundations of logical thinking. With secondary education, the acquisition of abstract thinking enables individuals to develop skills in logical reasoning, mathematical prediction, and constructing proofs.

2020 MSC: 97B10

KEYWORDS: Proof, mathematics education, mathematical thinking, proof and history, concept of proof

The Concept of Proof

Mathematics is expressed as a means of communication for those engaged in the discipline. Mathematics emerges as a result of individuals' cognitive activities, and the results produced are generated through reasoning.

Proof, also known as demonstration, is one of these communication methods in cited [15] (Solow, 2005 as cited in Keçeli, S., Uğurel, I. and Bukova Güzel, E., 2015). Mathematical proofs determine both the foundation and development of mathematics. Almeida (2003) emphasizes that attitudes toward proof are shaped by the process of mathematical knowledge formation, while Harel and Sowder (1998) indicate that students' proof development processes reveal logical structures and ways of defending truth. These studies show that mathematical proof allows individuals to defend truths, persuade others, and contribute to the advancement of mathematics. In this way, mathematics functions as a universal language and enables knowledge sharing.

Proof is central to mathematics and serves multiple functions (Almeida, 2003; Knuth, 2000, 2002; Martin and Harel, 1989; Piatek-Jimenez, 2004; Saeed, 1996; Silver, 1998; Tall, 1995 as cited in Keçeli, S., Uğurel, I., and Bukova Güzel, E., 2015). Mathematical proofs reveal connections between concepts, help make predictions, verify findings, and reach new generalizations (Schabel, 2005). Moreover, they uncover forms of connection different from traditional algebraic and arithmetic types (Barnard and Tall, 1997 as cited in Keçeli, S., Uğurel, I., and Bukova Güzel, E., 2015). Proofs and the process of proving remain central not only to the science of mathematics but also to mathematics education, forming fundamental components in teaching mathematical reasoning (Hanna, 1995; Knuth, 2002).

Proof and Its History

The development of mathematical proof is unclear, and its emergence cannot be precisely determined. In this context, the idea of justifying mathematical propositions can be considered a new concept. Developing methods for how these justifications formed is also an important step. One of the earliest known examples of recorded proof is attributed to the Babylonians. Babylonians, along with the Chinese, were aware of the Pythagorean Theorem. Existing tablets and diagrams show an early form of this proof understanding (Krantz, 2007, pp. 9).

Additionally, Eudoxus initiated the trend of expressing mathematics through theorems. Since Pythagoras lived before Euclid, it can be said that he laid the groundwork for Euclid's studies (Krantz, 2007, pp. 10). In the 19th century, mathematicians such as Fourier, Galois, and Riemann made significant contributions (Krantz, 2007, pp. 6, 17). In the 20th century, Hilbert consolidated the foundations of mathematical proofs. The twenty-three problems he presented guided subsequent studies (Krantz, 2007, pp. 7, 19).

Mathematical Proof and Its Functions

Mathematicians claim that proof is not only a tool for verifying the correctness of a claim but also an element that encourages understanding of the theorem and why it is correct (Hanna, 2000). Proofs help develop algorithms and definitions and systematize knowledge. Additionally, proof has different functions: verification, explanation, discovery, and generalization (Bell, 1976; CadwalladerOlsker, 2011; de Villiers, 1999; Hanna and Jahnke, 1996).

Among the main functions of mathematical proof, verification and explanation are important. Verification is the fundamental process used to show the correctness of a proposition and to ensure acceptance of the theorem. While verification is a prerequisite for proof, the reverse is not true (de Villiers, 1990). The explanation function, on the other hand, goes beyond merely showing correctness and reveals why a proposition is true. In classrooms, this aspect is emphasized because explanation supports conceptual understanding. Although verification ensures the validity of the claim, without explanation, sufficient depth of understanding does not occur (Bell, 1976; Hanna and Jahnke, 1996). The systematization function allows mathematical knowledge to be organized and fundamental concepts to be understood within a small framework (Barendregt and Wiedijk, 2005). The discovery function, although rare, allows the emergence of new mathematical structures and theorems. Historically, in areas such as non-Euclidean geometries, new theorems have been discovered through abstract deductive reasoning; for example, by studying geometric shapes on curved surfaces instead of using the parallel postulate, non-Euclidean geometries were developed.

Mental inquiry is an important factor that increases the persuasiveness of mathematical proofs. Intellectual debates that mathematicians experience during this process can make it enjoyable and fun. Therefore, mathematics educators should encourage and motivate students to engage in such inquiries.

The communication function is seen as a tool. It can provide an opportunity to develop different theorems and definitions. The verification of definitions function plays an important role in ensuring the validity of propositions and increasing confidence in mathematical knowledge. Definitions, axioms, and theorems continuously interact and harmonize in the formation of mathematical knowledge. For example, Euclidean geometry is built on undefined terms, axioms, and theorems; this structure can be said to achieve an axiomatic order through logical deductions and proofs built on axioms and definitions. Similarly, the proof of the Mean Value Theorem taught in analysis courses serves as an effective example in verifying the definitions of the real number system.

Moreover, this function of proof is important for demonstrating different proof methods such as proof by contradiction or induction. The functions of mathematical proofs may vary according to the person performing or reading the proof; in a mathematical paper, the proof of a theorem may only aim to show correctness, while for students, the same proof may serve as an explanation for understanding the fundamental idea. Therefore, the purpose of constructing a mathematical proof may vary depending on the context and the user (Dede and Karakuş, 2015).

Teaching the Structure of Proof and Its Development

The formation of the concept of proof in individuals begins to be observed in the preschool period. These are fundamental concepts, including classification and comparison. This period is considered a transition to logical thinking, making it crucial for internalizing logical relationships. If these foundations are not established, problems may arise in subsequent periods of concrete and abstract thinking (Altıparmak and Öziş, 2005).

With the acquisition of abstract thinking in secondary education, individuals can become skilled in logical reasoning, making mathematical predictions, developing reasons and proofs, and selecting and applying them (NCTM, 2000).

Constructing Mathematical Proof and Its Purpose

It is important for students to answer the questions "Why do we prove?" and "What is its purpose?". In this context, proofs can be heuristic, explanatory, exploratory, or visual.

Heuristic Proof

Heuristic proofs are conjecture- and intuition-based verifications that are considered more practical than formal proofs. They allow students to explore, discover, and learn through observation, making them educationally effective (Hanna, 2000a). For example, Ali and Hüseyin draw different triangles and measure the sum of the interior angles. Both students observe that the sum is 180° for each triangle. During the learning process, they realize that this is not coincidental and reach the assumption: "In any triangle, the sum of the interior angles is 180°." In this way, students prove the interior angle sum of a triangle using an intuitive approach (Reis and Renkl, 2002).

Explanatory Proof

The primary function of proof in mathematics is to show the correctness of propositions. Additionally, proofs are used to explain and ensure understanding. The best proofs not only show correctness but also contribute to understanding the theorem, making them more easily accepted by mathematicians. The explanatory proof approach dates back to ancient Chinese mathematicians; unlike the Greeks, Chinese mathematicians focused more on the explanatory and persuasive aspects of proofs rather than deductive verification. This approach also includes the balanced use and understanding of heuristic reasoning and non-verbal proofs (Hanna, 2000b). For example, in the 3rd century CE, Liu Hui demonstrated the Pythagorean theorem using dissected squares (see Figure 1).

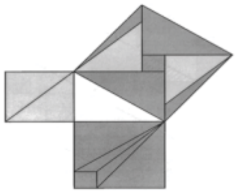

Figure 1: Proof of Pisagor Theorem

Figure 1 Proof of the Pythagorean Theorem Note. Adapted from Nelsen (2000, as cited in (Dede and Karakuş, 2014).

Exploratory Proof

Dynamic software programs have accelerated mathematical discoveries, especially in geometry. Examples include Cabri, GeoGebra, and Geometer's Sketchpad. However, this is not entirely new; exploration has always been an important part of mathematics.

For instance, a student may wish to prove that "the perpendicular bisectors of the three sides of any triangle intersect at a single point." To do this, the student can draw a triangle and its perpendicular bisectors on paper to show the correctness of the theorem. However, dynamic software programs such as Cabri, GeoGebra, or Geometer's Sketchpad provide significant advantages. These programs allow students to grasp the intersection point inside the triangle and observe the triangle's shape by adjusting it on the screen. Thus, it can be continuously observed that the perpendicular bisectors intersect correctly, enhancing the student's intuitive understanding. This helps students understand that the intersection of the three perpendicular bisectors is the circumcenter of the triangle (Hanna, 2002) (see Figure 2).

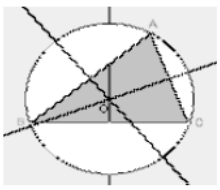

Figure 2: Center of Circumscribed Circle

Visual Proof

Visual representations are also used to verify mathematical propositions and expressions. Graphs and representations can be utilized as fundamental components of mathematical expressions and propositions and are key elements of mathematics curricula (Hanna, 2000b).

Example 1. *Let $n \geqslant 1$, the sum of the first n consecutive integers*

$$1 + 2 + ... + n = \frac{n(n+1)}{2}$$

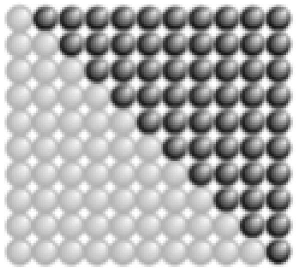

Figure 3: The Sum of Consecutive Integers

can be shown using a visual proof (see Figure 3).

Figure 3. Visual proof of the sum of the first n consecutive integers from 1 to n (Alsina and Nelsen, 2010, p. 120). In Figure 3, the gray circles represent the integers from 1 to n. When this is completed into a rectangle, there are n rows and $n+1$ columns. Accordingly, half of the area of this rectangle gives the sum of the n consecutive numbers from 1 to n, deriving the formula.

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On recent developments in applications of statistics

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Recent advances in statistical methods have significantly expanded the scope of data-driven research across many disciplines. The integration of data science techniques and big data analytics has enabled more accurate modeling, prediction, and interpretation of complex phenomena. These developments not only increase the reproducibility and validity of scientific findings but also enable innovative approaches to evidence-based decision-making processes. Future trends in statistics will focus more on the collaboration between traditional methodologies and new computational tools. It will also emphasize the importance of statistics in overcoming the challenges of large-scale, complex data sets.

2020 MSC: 91G70, 62R07, 94A16

KEYWORDS: Statistical methods, data science, big data analytics, future trends in statistics

Introduction

The growing availability of complex datasets has broadened the use of statistical methods, highlighting not only their potential for scientific advancement but also concerns about validity and reproducibility. Reliable conclusions require not just appropriate statistical techniques, but also rigorous analysis and accurate interpretation [1]. Complex data pose significant challenges for analysis, requiring traditional methods to be adapted or extended in order to extract meaningful insights and draw accurate inferences [2]. In the future of big data analytics, automating statistical analysis is essential due to the volume and complexity of data. Automation supports tasks like data preparation, exploration, replication, and warehouse management [3]. Machine learning and artificial intelligence play a key role in the future of big data analytics by automating tasks like data integration, quality control, and database management. These technologies enhance productivity, reduce errors, and simplify the organization and analysis of large, complex datasets (*cf.* [4, 5]).

Main results

The primary motivation for this presentation is to identify the most appropriate statistical methods and to analyze the application methods and criteria for real-world problems by blending data science and big data analytics. Furthermore, it explores various application areas within the scope of Future Trends in Statistics.

Conclusion

Reliable statistical standards are a fundamental element of scientific integrity. These standards encompass carefully designed studies, transparent data collection, comprehensive summaries, context-appropriate interpretation, and complete reporting. As the volume and complexity of data increase, the future of science will rely more than ever on these standards to ensure validity, reproducibility, and evidence-based innovation. Therefore, the future role of statistics lies not only in preserving existing methodologies but also in developing innovative approaches through integration with artificial intelligence and data science.

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On new representations of the Nuttall function with applications

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In 1972 Albert H. Nuttall [3] introduced the function

$$Q_{\mu,\nu}(a,b) = \mathrm{e}^{-\frac{a^2}{2}} \int_b^\infty t^\mu \, \mathrm{e}^{-\frac{t^2}{2}} \, I_\nu(at) \, \mathrm{d}t,$$

defined in terms of the modified Bessel function of the first kind I_ν, where ν is the order of the function. The Nuttall function is defined for all real values of the parameters a, b, μ, ν, with possible extension to complex arguments a and b *via* analytic continuation. Nuttall originally introduced this function as a generalization of the Marcum Q–function $Q_\nu(a,b)$ [1].

Bearing in mind its great applications, including digital communication performance analysis, outage probability in wireless systems, and capacity evaluation in uncoded MIMO (multiple-input multiple-output) systems, the Nuttall function, along with its special case, the Marcum Q–function, is frequently encountered in both applied and theoretical studies.

The aim of this talk is to present several novel representation formulae for the Nuttall function, some of which offer simplifications of existing results. Furthermore, we provide numerical simulations to compare the computational performance of the new expressions with those already available in the literature (for more details see [2]).

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KEYWORDS: Nuttall function, Marcum Q–function, hypergeometric function, Grünwald–Letnikov fractional derivative, incomplete gamma function

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Generalized Dold and Newton sequences

Dorota Chańko *[1] *and Mateusz Zdunek* [2]

Dold sequences are an important class of integer sequences. They play a significant role in number theory, periodic point theory, combinatorics, topology and dynamical systems. The notion of Dold sequences can be generalized to matrix sequences. The aim of this paper is to present their relationship to another important class of sequences, Newton sequences, and to examine their generalizations.

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Keywords: Dold sequences, number theory, periodic point theory

Introduction

Integer Dold sequences are considered a bridge between number theory, periodic point theory, combinatorics, and dynamics [5]. They play an important role in these fields. For example, the numbers of the fixed points of iterations of a map form a Dold sequence. On the other hand, it is worth pointing out that if we start with a Dold sequence, then, in a natural way, we can construct a mapping on a discrete space such that the number of its periodic points corresponds to the terms of the desired sequence [1].

It is important to mention here the class of other integer sequences, Newton sequences, which are generated as described in Definition 3. Du, Huang and Li in [2] proved that the classes of Dold and Newton sequences coincide.

In [3], G. Graff, J. Gulgowski, and M. Lebiedź generalized the notion of Dold sequences to vector and matrix sequences. J. Gulgowski raised the question of whether the classes of Dold and Newton sequences remain the same after the natural generalization of the latter. The aim of this note is to show that this coincidences does not hold.

Preliminaries

In order to introduce the notion of an integer Dold sequence, we first need to define the Möbius function (1). Similarly, before defining a matrix Dold sequence, we will introduce a generalized Möbius function (6), defined for the partially ordered set (*cf.* Def. 5). This function will then be used to define the Möbius matrix function (7).
All of the following definitions come from either [3] or [4].

Definition 1. *A function $\mu\colon \mathbb{N} \to \{-1,0,1\}$ is called a classical* ***Möbius function*** *if:*

$$\mu(n) = \begin{cases} 1, & \text{when } n = 1, \\ (-1)^k, & \text{when } n \text{ is the product of } k \text{ different prime numbers,} \\ 0, & \text{otherwise.} \end{cases}$$

Definition 2. *An integer sequence* $(x_n)_{n\in\mathbb{N}}$ *is called* ***Dold sequence*** *if the Dold congruences are fullfilled:*

$$\forall_{n\geqslant 1} \sum_{m|n} \mu\left(\frac{n}{m}\right) x_m \equiv 0 \mod n$$

and $\mu\colon \mathbb{N} \to \{-1, 0, 1\}$ *is the classical Möbius function.*

Definition 3. *An integer sequence* $(a_n)_{n\in\mathbb{N}}$ *is called* ***Newton sequence*** *generated by the sequence of integers* $(c_n)_{n\in\mathbb{N}}$ *if:*
$a_n = c_1 a_{n-1} + c_2 a_{n-2} + ... + c_{n-1} a_1 + n c_n$.

Theorem 4. *The classes of Dold sequences and Newton sequences coincide.*

This result was proven in [2] by Du, Huang and Li. The following definitions of poset with assigned weights, Möbius function generalizations and matrix Dold sequence were proposed by G. Graff, J. Gulgowski and M. Lebiedź in [3].

Definition 5. *Let* P *be countable partially ordered set (**poset**) with a partial order "*$\leqslant$*" and let* $w\colon P \to \mathbb{Z}$ *be a function assigning a weight* $w \in \mathbb{Z}$ *to each* $x \in P$ *and satisfying the following conditions:*

- $\forall_{p\in P}$ *the set* $\{q\colon q \leqslant p\}$ *is finite,*
- $\forall_{x,y\in P}\ x < y \implies w(x)|w(y) \wedge w(x) < w(y)$,
- *for all weights* $k \in \mathbb{N}$ *there is at most* N *elements* P *with weight* k*. Such elements are denoted* $k_1, k_2, ..., k_N$.

Definition 6. *A function* $\mu^*\colon P \times P \to \mathbb{Z}$ *is called a generalized* ***Möbius function*** *if:*

$$\mu^*(x,y) = \begin{cases} 1 & \text{if } x = y, \\ -\sum\limits_{x\leqslant z<y} \mu^*(x,z) & \text{if } x < y, \\ 0 & \text{otherwise.} \end{cases}$$

Definition 7. *We define* ***Möbius matrix function*** *as*

$$M^*_{k\to l} = [\mu^*(k_i, l_j)]_{1\leqslant j,\ i\leqslant N}$$

for $k, l \in \mathbb{N}$.

$M^*_{k\to l}(j,i) = \mu^*(k_i, l_j)$, hence the term in j-th row and i-th column is equal to generalized Möbius function for the pair of points k_i and l_j.
It is important to mention here that one dimensional Möbius matrix function with generalized Möbius function defined on $\mathbb{N} \times \mathbb{N}$ with $\mathbb{N}$ ordered by divisibility is equivalent to the classical Möbius function.

Definition 8. *A matrix sequence* $(X_l)_{l\in\mathbb{N}}$, $X_l \in M_{N\times N}(\mathbb{Z})$ *is called a* ***matrix Dold sequence*** *if for all* $l \in \mathbb{N}$ *there exists a matrix* $J_l \in M_{N\times N}(\mathbb{Z})$ *such that*

$$\sum_{k|l} M^*_{k\to l} X_k = l J_l.$$

Moreover, G. Graff and J. Gulgowski proposed the following definition of matrix Newton sequence.

Definition 9. *A matrix sequence* $(A_n)_{n\in\mathbb{N}}$, $A_n \in M_{N\times N}(\mathbb{Z})$ *is called* ***matrix Newton sequence*** *generated by sequence of matrices* $(C_n)_{n\in\mathbb{N}} \in M_{N\times N}(\mathbb{Z})$ *(matrix generating sequence) if:*

$$A_n = C_1 A_{n-1} + C_2 A_{n-2} + ... + C_{n-1} A_1 + n C_n.$$

Matrix Dold and Newton sequences

The main results of our study provide counterexamples to the problem whether the classes of matrix Dold and Newton sequences generally coincide.
In this example $(C_n)_n$ is a 2×2 matrix sequence.
Let $C_1 = I$ and $C_2, C_3, C_4, ... = 0$. $(C_n)_n$ generates matrix Newton sequence $(A_n)_n$:
$A_1 = C_1 = I$,
$A_2 = C_1 A_1 + 2C_2 = I + 0 = I$,
...,
$A_n = C_1 A_{n-1} + C_2 A_{n-2} + ... + C_{n-1} A_1 + nC_n = I \cdot A_{n-1} = I \cdot I = I$,
hence the identity sequence is a Newton sequence.

Let $P = \{1_1, 1_2, 2_1, 2_2\}$ be ordered as shown in the figure 1 below.

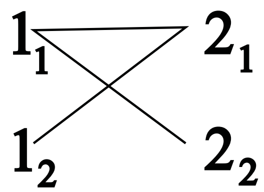

Figure 1: Poset 1

We check whether $(A_n)_n = (I, I, I, ...)$ is a Dold sequence for poset P:
$M^*_{1\to 2} = \begin{bmatrix} \mu^*(1_1, 2_1) & \mu^*(1_2, 2_1) \\ \mu^*(1_1, 2_2) & \mu^*(1_2, 2_2) \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ -1 & 0 \end{bmatrix}$, $M^*_{2\to 2} = I$.
$\sum\limits_{k|2} M^*_{k\to 2} A_k = M^*_{1\to 2} A_1 + M^*_{2\to 2} A_2 = \begin{bmatrix} -1 & -1 \\ -1 & 0 \end{bmatrix} \cdot I +$
$+ I \cdot I = \begin{bmatrix} 0 & -1 \\ -1 & 1 \end{bmatrix}$
and there is no such integer matrix J_2 that $2J_2$ is equal to $\sum\limits_{k|2} M^*_{k\to 2} A_k$. Hence identity matrix sequence, which is a Newton matrix sequence, is not a matrix Dold sequence for given poset P.

Second counterexample involve a partially ordered set for which every vertex in the graph is connected by exactly one edge to the vertex of higher weight.
Let $P = \{1_1, 1_2, 2_1, 2_2, 4_1\}$ be ordered as shown in the figure 2 below.

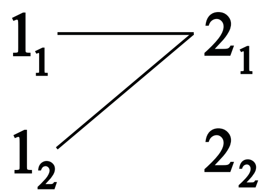

Figure 2: Poset 2

Let $(A_n)_n$ be a 2×2 identity matrix sequence.
$M^*_{1\to 2} = \begin{bmatrix} -1 & 0 \\ -1 & 0 \end{bmatrix}$, $M^*_{2\to 2} = I$.
$\sum\limits_{k|2} M^*_{k\to 2} A_k = M^*_{1\to 2} A_1 + M^*_{2\to 2} A_2 = \begin{bmatrix} -1 & 0 \\ -1 & 0 \end{bmatrix} \cdot I + I \cdot I =$
$= \begin{bmatrix} 0 & 0 \\ -1 & 1 \end{bmatrix}$

There is no such integer matrix J that $2J = \begin{bmatrix} 0 & 0 \\ -1 & 1 \end{bmatrix}$.
As a result, even for relatively simple class of partially ordered sets we can find a Newton matrix sequence which is not a Dold matrix sequence.

Both counterexamples show that Newton matrix sequence which is not a Dold matrix sequence can be found, however they are not counterexamples to the opposite, i.e. whether Dold matrix sequences are generated by some integer matrix sequences in a way given by Definition 9.

Vector Dold and Newton sequences

Definition of vector Dold sequence is similar to the definition of Dold matrix sequence, where $(X_n)_n$ is a vector sequence instead of matrix sequence.

Definition 10. *[3] A vector sequence $(X_n)_{n\in\mathbb{N}} \in \mathbb{Z}^N$ is called a **vector Dold sequence** if for all $n \in \mathbb{N}$ there exists vector $J_n \in \mathbb{Z}^N$ such that*

$$\sum_{k|n} M^*_{k\to n} X_k = nJ_n$$

.

We propose a definition of the vector Newton sequence with a special operation on vectors, denoted by $\bullet$, as explained below.

Definition 11. *A vector sequence $(A_n)_{n\in\mathbb{N}}$, $A_n \in \mathbb{Z}^N$ is called **vector Newton sequence** generated by sequence of vectors $(C_n)_{n\in\mathbb{N}}$, $C_n \in \mathbb{Z}^N$ if:*

$$A_n = C_1 \bullet A_{n-1} + C_2 \bullet A_{n-2} + ... + C_{n-1} \bullet A_1 + nC_n,$$

where $\bullet$ is a multiplication of corresponding vector coordinates, i.e. if
$C = [c_1, c_2, ..., c_N]^T$, $A = [a_1, a_2, ..., a_N]^T$, then
$C \bullet A = [c_1a_1, c_2a_2, ..., c_Na_N]^T$.

Remark 12. *With a Newton sequence defined as above it can be observed that if $(c_n^1), (c_n^2), ..., (c_n^N)$ are integer sequences generating Newton sequences $(a_n^1), (a_n^2), ..., (a_n^N)$ respectively, then vector sequence (C_n), $C_n = [c_n^1, c_n^2, ..., c_n^N]^T$ generates vector Newton sequence (A_n), $A_n = [a_n^1, a_n^2, ..., a_n^N]^T$.*

Theorem 13. *Let P be the disjoint sum of s copies of the poset $(\mathbb{N}, |)$. Then every vector Newton sequence is a vector Dold sequence on poset P.*

Proof. Let n, s be arbitrary fixed natural numbers and let

$$P = \{1_1, 1_2, ..., 1_s, 2_1, 2_2, ..., 2_s, ..., n_1, n_2, ..., n_s\}$$

be ordered as in the figure 3.

For $k \leqslant n$, $M^*_{k\to n} = \begin{bmatrix} \mu(\frac{n}{k}) & & & & \\ & \mu(\frac{n}{k}) & & 0 & \\ & & \ddots & & \\ & 0 & & \mu(\frac{n}{k}) & \\ & & & & \mu(\frac{n}{k}) \end{bmatrix} = \mu(\frac{n}{k})I,$

where I is $s \times s$ identity matrix.
We will show that for poset P ordered as shown in the figure 3, every vector Newton

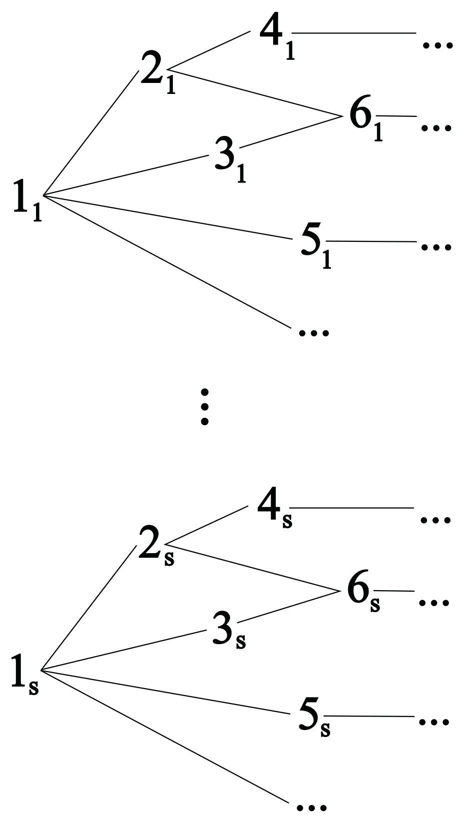

Figure 3: s copies of $(\mathbb{N}, |)$

sequence is a Dold sequence - if $(C_n) = ([c_n^1, c_n^2, ..., c_n^s]^T)$, $((c_n^1), (c_n^2), ..., (c_n^s)$ - integer sequences generating sequences $(a_n^1), (a_n^2), ..., (a_n^s))$ is a generating sequence of (A_n), then

$$\sum_{k|n} M^*_{k\to n} A_k = \sum_{k|n} \mu\left(\frac{n}{k}\right) I A_k = \sum_{k|n} \mu\left(\frac{n}{k}\right) [a_k^1, a_k^2, ..., a_k^s]^T,$$

where $(a_n^1), (a_n^2), ..., (a_n^s)$ are the number Newton sequences, hence the number Dold sequence, thus $\sum\limits_{k|n} \mu(\frac{n}{k}) a_k^i \equiv 0 \mod n$ for $i \in \{1, ..., s\}$. Hence there exists such integer vector J_n, such that

$$\sum_{k|n} M^*_{k\to n} A_k = nJ_n.$$

□

Nevertheless, for more complicated poset conjecture does not hold. Let (A_n) be the vector Newton sequence with terms $A_n = [2^n, 3^n]^T$, which is generated by the vector sequence $(C_n) = ([2,3]^T, [0,0]^T, [0,0]^T, ...)$ by definition 11. Matrix Möbius function $1 \to 2$ for a poset given in the figure 4 above takes the form
$M^*_{1\to 2} = \begin{bmatrix} -1 & 0 \\ -1 & 0 \end{bmatrix}$, a $M^*_{2\to 2} = I$, hence

$$\sum_{k|2} M^*_{k\to 2} = \begin{bmatrix} -1 & 0 \\ -1 & 0 \end{bmatrix} A_1 + IA_2 = [-2,-2]^T + [4,9]^T = [2,7]^T$$

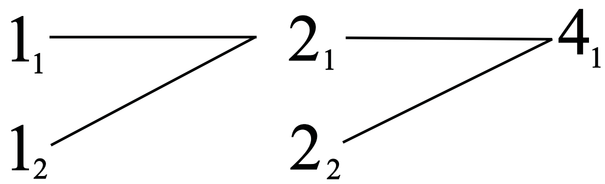

Figure 4: Poset 3

and there is no such integer vector J_2, that $[2,7]^T = 2J_2$.

Conclusion

It was shown that matrix (vector) Newton sequences are not, in general, matrix (vector) Dold sequences. Nevertheless, the question remains whether we can redefine matrix Newton sequences so that they depend on the partially ordered sets appearing in the definition of matrix Dold sequences. The problem could be expanded, and we may raise the question of whether we can find partially ordered sets such that the class of generalized Newton sequences includes the class of generalized Dold sequences, which may be the subject of further research.

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p-adic integral representations of exponential splines and Eulerian polynomials

Damla Gun

The main purpose of this paper is to provide a novel approach to deriving formulas for the p-adic q-integral, including the Volkenborn integral and the p-adic fermionic integral. By applying integral equations and these integral formulas to binomial coefficients, we derive relations related to well-known special numbers such as the Bernoulli and Euler numbers. The aim of this study is to apply the p-adic Volkenborn integral to the spline families for the variable u, and to derive explicit representations in terms of p-adic special numbers. Specifically, we apply the p-adic Volkenborn integral to the exponential spline $\Phi_n(x;u)$ and Eulerian polynomials. We give new formulas in terms of Bernoulli numbers B_n and Euler numbers E_n, and explore their implications in p-adic analysis and analytic number theory.

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KEYWORDS: p-adic Volkenborn integral, Fermionic integral, Exponential spline, B-spline, Eulerian polynomials, Bernoulli numbers, Euler numbers, Combinatorial identities

Introduction

p-adic analysis plays a significant role in the development of modern number theory and combinatorics. In particular, the p-adic Volkenborn integral and its fermionic integral have been widely used to derive identities involving Bernoulli and Euler numbers.

On the other hand, spline functions, such as exponential and B-splines, play an important role in approximation theory and numerical analysis. Recent studies have revealed a relation between spline theory and special numbers.

In this paper, we apply p-adic integral techniques to exponential spline functions and Eulerian polynomials. Moreover, we establish new identities that involve Bernoulli and Euler numbers, and provide p-adic integral representations of polynomials. Let $\mathbb{N} = \{1, 2, 3, ...\}$, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\mathbb{Z}$ denote the set of integers, $\mathbb{C}$ denote the set of complex numbers. This notation frequently appears in the study of special functions and combinatorial identities. It serves as a fundamental tool in establishing recurrence relations and generating functions for various number sequences.

The Apostol Bernoulli numbers and polynomials of order α (real or complex) are defined by means of the following generating functions, respectively:

$$F_{aB_n}(t;\lambda,\alpha) = \left(\frac{t}{\lambda e^t - 1}\right)^{\alpha} = \sum_{n=0}^{\infty} \mathcal{B}_n^{(\alpha)}(\lambda)\frac{t^n}{n!} \tag{1}$$

and

$$F_{aB_P}(t,b;\lambda,\alpha) = F_{aB_n}(t;\lambda,\alpha)\,e^{bt} = \sum_{n=0}^{\infty} \mathcal{B}_n^{(\alpha)}(b;\lambda)\frac{t^n}{n!}, \tag{2}$$

where $|t| < 2\pi$ when $\lambda = 1$; $|\alpha| < |\ln \lambda|$ when $\lambda \neq 1$; $1^{\alpha} := 1$, $\lambda, \alpha \in \mathbb{C}$ and $b \in \mathbb{R}$ (*cf.* [1]- [25]).

Substituting $b = 0$ into (2), we have

$$\mathcal{B}_n^{(\alpha)}(0; \lambda) = \mathcal{B}_n^{(\alpha)}(\lambda).$$

When $\lambda = 1$ in (1) and (2), the Bernoulli numbers and polynomials of order z are given as follows:

$$\mathcal{B}_n^{(\alpha)}(1) = B_n^{(\alpha)} \tag{3}$$

and substituting $\alpha = 1$ into (3), we have

$$\mathcal{B}_n^{(1)}(1) = B_n$$

The Apostol Euler numbers and polynomials of higher order are defined by, respectively,

$$\frac{2^{\alpha}}{(\lambda e^t + 1)^{\alpha}} = \sum_{n=0}^{\infty} \mathcal{E}_n^{(\alpha)}(\lambda) \frac{t^n}{n!} \tag{4}$$

and

$$\frac{2^{\alpha}}{(\lambda e^t + 1)^{\alpha}} e^{xt} = \sum_{n=0}^{\infty} \mathcal{E}_n^{(\alpha)}(x; \lambda) \frac{t^n}{n!} \tag{5}$$

where $|t| < 2\pi$ when $\lambda = 1$; $|\alpha| < |\ln \lambda|$ when $\lambda \neq 1$; $1^{\alpha} := 1$, $\lambda, \alpha \in \mathbb{C}$ and $b \in \mathbb{R}$ (*cf.* [15, 25]).

Substituting $x = 0$ into (5), we have

$$\mathcal{E}_n^{(\alpha)}(0; \lambda) = \mathcal{E}_n^{(\alpha)}(\lambda). \tag{6}$$

When $\lambda = 1$ and $\alpha = 1$ in (6), the Euler numbers as follows:

$$\mathcal{E}_n^{(1)}(1) = E_n. \tag{7}$$

The Eulerian polynomials, which are many combinatorial applications, are defined by

$$\sum_{l=0}^{\infty} l^n z^l = \frac{A_n(z)}{(1-z)^{n+1}} \tag{8}$$

and

$$\sum_{n=0}^{\infty} A_n(z) \frac{t^n}{n!} = \frac{1-z}{1-ze^{t(1-z)}} \tag{9}$$

(*cf.* [3]).

B-spline functions are probably the most applicable one-dimensional polynomial spline functions, where the B-spline, denoted by $M(x) := M(x; x_0, \ldots, x_n)$ with $x_0 < x_1 < \ldots < x_n$, is defined by

$$M(x; x_0, \ldots, x_n) = \sum_{j=0}^{n} \frac{(x_j - x)_+^{n-1}}{\omega'(x_j)}, \quad x \in \mathbb{R},$$

where $\omega(x) = (x - x_0)(x - x_1) \cdots (x - x_n)$ and $x_+ := \max\{0, x\}$ (*cf.* [19]).

The nth order cardinal forward B-spline $N_n := M(x; 0, 1, \ldots, n)$, or simply, the B-spline of order n, can be defined by the recursive convolution formula

$$N_n(x) = (N_{n-1} * N_1)(x) = \int_0^1 N_{n-1}(x-t)\, dt, \quad n \geqslant 2.$$

It is clear that $N_n(x) \in C^{n-2}$ and is a piecewise polynomial of degree $n-1$.

Let n be a non-negative integer. The symbol $\mathcal{S}_n$ denotes the class of splines of order n. An exponential spline $f \in \mathcal{S}_n$ means an element in $\mathcal{S}_n$ satisfying the functional equation

$$f(x+1) = cf(x). \tag{10}$$

Let us recall the notations presented in the central part of Schoenberg's lectures on *Cardinal Spline Interpolation* (CSI) (*cf.* [19]). First, the exponential spline $\phi_n(x; z)$ of degree n to the base z is defined by

$$\phi_n(x; z) := \sum_{j=0}^{\infty} z^j N_{n+1}(x-j), \tag{11}$$

where $N_n(x) := M(x; 0, 1, \ldots, n)$ denotes the cardinal B-spline, or simply, B-spline (*cf.* [19]). Obviously, $N_n(x) \in S_{n-1}$. Therefore, $\phi_n \in S_n$, and it is easy to find that ϕ_n satisfies equation (10); that is,

$$\phi_n(x+1; z) = z\, \phi_n(x; z).$$

As a special consequence of the connection between B-splines and exponential splines, Eulerian polynomials can be expressed in terms of the cardinal forward B-splines. Specifically, the following identity holds (*cf.* Corollary 2.2 in [7]):

$$A_n(z) = n! \sum_{j=1}^{n} N_{n+1}(j) z^j, \tag{12}$$

where $A_n(z)$ denotes the nth Eulerian polynomial, and $B_n(x)$ is the cardinal forward B-spline of order n.

Let $\mathbb{Z}_p$ denote the ring of p-adic integers. The Volkenborn integral (or p-adic bosonic integral) of a function $f : \mathbb{Z}_p \to \mathbb{Q}_p$ is defined by

$$\int_{\mathbb{Z}_p} f(x)\, d\mu_1(x) = \lim_{N\to\infty} \frac{1}{p^N} \sum_{x=0}^{p^N-1} f(x) \tag{13}$$

(*cf.* [12], [18]- [24]).

By using Eq. (13), the Bernoulli numbers B_n is given by the p-adic integral of the function x^n as follows:

$$\int_{\mathbb{Z}_p} x^n\, d\mu_1(x) = B_n \tag{14}$$

(*cf.* [12], [18]- [24]).

By using Eq. (13), we have

$$\int_{\mathbb{Z}_p} x^n\, d\mu_{-1}(x) = E_n, \tag{15}$$

where E_n denotes the nth Euler number and μ_{-1} is the fermionic Volkenborn measure on $\mathbb{Z}_p$ (*cf.* [12], [18]- [24]).

Main results

In this section, we derive several identities by applying the p-adic Volkenborn integral and its fermionic counterpart to exponential spline functions and Eulerian polynomials. These results establish connections between spline-based functions and classical Bernoulli and Euler numbers through p-adic integration.

By applying the p-adic Volkenborn integral to Eq. (11), we obtain:

$$\int_{\mathbb{Z}_p} \phi_n(x;z) d\mu_1(z) = \sum_{j=0}^{\infty} N_{n+1}(x-j) \int_{\mathbb{Z}_p} z^j d\mu_1(z).$$

By using Eq. (14), we arrive at the following result:

Theorem 1. *The p-adic integral of the exponential spline in terms of Bernoulli numbers*

$$\int_{\mathbb{Z}_p} \phi_n(x;z) d\mu_1(z) = \sum_{j=0}^{\infty} N_{n+1}(x-j) B_j. \tag{16}$$

Similarly, applying the fermionic Volkenborn integral to Eq. (11) yields:

$$\int_{\mathbb{Z}_p} \phi_n(x;z) d\mu_{-1}(z) = \sum_{j=0}^{\infty} N_{n+1}(x-j) \int_{\mathbb{Z}_p} z^j d\mu_{-1}(z).$$

By using Eq. (15), we obtain the following theorem:

Theorem 2. *The exponential spline with Euler numbers via fermionic p-adic integration*

$$\int_{\mathbb{Z}_p} \phi_n(x;z) d\mu_{-1}(z) = \sum_{j=0}^{\infty} N_{n+1}(x-j) E_j. \tag{17}$$

Now, we apply the p-adic Volkenborn integral to Eq. (12), we have:

$$\int_{\mathbb{Z}_p} A_n(z)\, d\mu_1(z) = n! \sum_{j=1}^{n} N_{n+1}(j) \int_{\mathbb{Z}_p} z^j\, d\mu_1(z).$$

By using Eq. (14), we arrive at the following theorem:

Theorem 3. *p-adic representation of Eulerian polynomials involving Bernoulli numbers*

$$\int_{\mathbb{Z}_p} A_n(z)\, d\mu_1(z) = n! \sum_{j=1}^{n} N_{n+1}(j) B_j. \tag{18}$$

Applying the fermionic Volkenborn integral to Eq. (12), we have

$$\int_{\mathbb{Z}_p} A_n(z)\, d\mu_{-1}(z) = n! \sum_{j=1}^{n} N_{n+1}(j) \int_{\mathbb{Z}_p} z^j\, d\mu_{-1}(z).$$

By using Eq. (15), we obtain the following theorem:

Theorem 4. *The Eulerian polynomial with Euler numbers under the fermionic p-adic integral*

$$\int_{\mathbb{Z}_p} A_n(z)\, d\mu_{-1}(z) = n! \sum_{j=1}^{n} N_{n+1}(j) E_j. \tag{19}$$

Conclusion

In this paper, we explored the connection between exponential spline functions, Eulerian polynomials, and special numbers via p-adic Volkenborn integrals. By applying both the classical and fermionic versions of the p-adic integral to exponential splines and Eulerian polynomials, we obtained new identities involving Bernoulli and Euler numbers. These results demonstrate how spline theory and p-adic analysis can be unified with special polynomials. The integral representations established in this work not only provide formulas, but also highlight the usefulness of p-adic integration in deriving relationships among special functions and polynomials.

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Integral representation of $B(\mathcal{H})$-valued positive definite functions on convex cones using the Laplace transform

Diana Hunjak

We establish a Nussbaum-type theorem for $B(\mathcal{H})$-valued positive definite functions defined on convex cones of lattice-structured Banach spaces. Unlike Banach lattices, these spaces include important examples such as Sobolev spaces $W^{1,p}(\Omega)$. Using the Berg–Maserick theorem and refining techniques from Glöckner's framework, we prove that every suitably bounded $B(\mathcal{H})$-valued positive definite function on such cones admits a unique integral representation as a Laplace transform of a Radon measure concentrated on a subset of the topological dual.

2020 MSC: 43A35, 44A10, 46A40

Keywords: Positive definite functions, integral representation, Laplace transform, α-boundedness, Banach lattices

Introduction

Integral representations of positive definite functions play a central role in harmonic analysis and representation theory. The classical Hausdorff–Bernstein–Widder theorem characterizes completely monotone functions on the half-line as Laplace transforms of finite positive measures. This result inspired later developments in the study of positive definite functions on cones and semigroups, where the question naturally arises whether every such function can be represented as a Laplace transform and under what conditions. A Nussbaum-type theorem refers to an integral representation of positive definite functions on convex cones via Laplace transforms of measures, extending the classical Hausdorff–Bernstein–Widder theorem.

The study of positive definite functions on convex cones has a long history, with important contributions by Bochner [3], Nussbaum [10], and Neeb [8, 9], among others. Later developments, often motivated by problems in probability theory and mathematical physics, include the works of Dettweiler [4], Hoffmann-Jørgensen and Ressel [6], Ressel and Ricker [11], and Šikić [12].

Berg, Christensen and Ressel [1] developed a general theory showing that exponentially bounded positive definite functions on commutative semigroups admit integral representations via unique Radon measures supported on the set of characters. This integral representation extends the Laplace transform, since the characters are not necessarily exponential. To recover a Laplace-type form, the measure needs to be restricted to a smaller set.

More recently, Glöckner [5] established a broad version of this result for convex cones in vector spaces, extending earlier scalar theories to the setting of $B(\mathcal{H})$-valued positive definite functions ($B(\mathcal{H})$ being the complex algebra of bounded operators on a Hilbert space $\mathcal{H}$). Recall that a function $\phi\colon S \circ S \to B(\mathcal{H})$ on a semigroup with involution $(S, \circ, *)$ is called positive definite if

$$\sum_{j,k=1}^{n} \langle \phi(s_j \circ s_k^*) v_k, v_j \rangle \geqslant 0, \quad \forall\, n \in \mathbb{N}, \{v_1, \ldots, v_n\} \subseteq \mathcal{H}, \{s_1, \ldots, s_n\} \subseteq S.$$

Glöckners analysis also revealed a limitation: if the cone has empty interior, then a *Radon* representing measure on V^* may fail to exist, and no representative measure need exist on the topological dual $(V', \Sigma(V'))$.

The present work [7] contributes to this line of research by establishing a Nussbaum-type theorem for lattice-structured Banach spaces with order units, which are not necessarily Banach lattices, thereby extending the class of spaces admitting integral representations and ensuring the existence of a unique Radon representing measure.

Main theorem

Let $(B, \|\cdot\|, \leqslant)$ be a lattice-structured Banach space (ordered Banach space that is also a vector lattice, but not necessarily a Banach lattice) with a closed generating cone B_+. If B possesses an order unit, then the cone has non-empty interior, which is a crucial condition ensuring the existence of *Radon* representing measures. A $B(\mathcal{H})$-valued positive definite function ϕ on B_+ is called α-bounded if its growth is controlled by an absolute value α on B_+. This condition guarantees that the associated characters of exponential type are continuous, and it is convenient to collect the corresponding continuous linear functionals into the set:

$$C_\alpha := \{\lambda \in B^* : e^{\lambda(x)} \leqslant \alpha(x), \forall\, x \in B_+\} \subseteq B'.$$

Theorem 1. *Let $(B, \|\cdot\|, \leqslant)$ be a lattice-structured Banach space with an order unit, and let α be a locally bounded absolute value on B_+. Suppose $\phi : B_+ \to \mathrm{Herm}^+(\mathcal{H})$ is an α-bounded positive definite function and there exists an order unit u and a sequence of positive real numbers $r_n \downarrow 0$ such that $\phi(r_n u) \to \phi(0)$ in the ultraweak topology. Then there exists a unique Radon $\mathrm{Herm}^+(\mathcal{H})$-valued measure μ on B' supported in $C_\alpha \subset B'$ such that*

$$\phi(x) = \int_{B'} e^{\lambda(x)}\, d\mu(\lambda), \quad \forall\, x \in B_+.$$

Outline of the proof

The proof proceeds by reduction to scalar-valued kernels. Each ϕ induces scalar α-bounded positive definite function $\phi_A(x) = \mathrm{tr}(A\phi(x))$ for $A \in \mathrm{Herm}_1^+(\mathcal{H})$. By the Berg–Maserick theorem (see [2]), each ϕ_A admits a Radon measure representation. These scalar measures are additive in A and therefore combine into a $\mathrm{Herm}^+(\mathcal{H})$-valued measure. The existence of an order unit ensures the non-emptiness of the interior of B_+, which rules out pathological cases without Radon representatives. Finally, uniqueness follows from the concentration of the representing measure on continuous α-bounded characters. Details of the proof and further examples can be found in [7].

Conclusion

Theorem 1 confirms that $B(\mathcal{H})$-valued positive definite functions on lattice-structured Banach spaces with order units always admit Laplace integral representations under mild continuity and boundedness conditions. This extends classical Nussbaum-type theorems to a wider class of spaces, including Sobolev spaces $W^{1,p}(\Omega)$ which are vector lattices but not Banach lattices. The result strengthens the link between harmonic analysis on convex cones and operator-valued representation theory.

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A note on approximation properties for Kantorovich type positive linear operators including Appell type polynomials

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In this study, the approximation properties of Kantorovich-type positive linear operators involving Appell-type polynomials are investigated. Positive linear operators occupy a significant place in classical approximation theory, and their Kantorovich-type variants, which are defined in an integral form, provide effective approximation over a broader class of functions. In this context, the definition of the considered operators is first presented, followed by a discussion of some of their fundamental properties, such as positivity and linearity preservation. The uniform convergence behavior of the operators is analyzed by means of classical Korovkin-type theorems, and the rates of convergence are evaluated in terms of the modulus of continuity and Lipschitz classes. Furthermore, the advantages of incorporating Appell-type polynomials into the definition of these operators are emphasized, and it is shown that certain well-known classical operators can be obtained as special cases of the generalized forms.

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KEYWORDS: Apostol-type polynomials, Korovkin theorem, generating functions, positive linear operators

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On Euler product, Theta functions and Riemann Zeta function with their relations

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The purpose of this presentation is to survey theta functions and their relationship with the Riemann zeta function the Euler product. The relationship between the Jacobi theta function, the Riemann zeta function and the Euler sum will be examined. The connection between Euler's counting functions and the Jacobi theta function is given. Special values of the spectral zeta function will be examined a connection among Bernoulli numbers, the Riemann zeta function and the spectral zeta function is given.

2020 MSC: 40A25, 14K25, 11M26, 11R59, 35K05

KEYWORDS: Euler summation formula, Theta functions, Zeta functions, Riemann zeta function, heat equation

Introduction

Riemann zeta function,

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \tag{1}$$

where $\mathrm{Re}(s) > 1$. The function $\zeta(s)$ is holomorphic at $\mathbb{C}/\{1\}$ (*cf.* [8], [19], [20], [16]).

The multiplication form of the Riemann zeta function with prime numbers is expressed as the Euler product.

$$\zeta(s) = \prod_{p \text{ prime}} \frac{1}{1 - p^{-s}},$$

where $\mathrm{Re}(s) > 1$ (*cf.* [8], [19], [20], [16]).

Jacobi theta function is defined by

$$\theta(z, \tau) = \sum_{n=-\infty}^{\infty} e^{\pi i n^2 \tau + 2\pi i z} \tag{2}$$

(cf. [2], [3], [8], [12]- [14], [19], [20], [16], [22], [23]- [26]).

For $z = 0$, equation (2) deduces to

$$\theta_3(\tau) = \sum_{n=-\infty}^{\infty} e^{\pi i n^2 \tau}$$

(cf. [2], [3], [8], [12]- [14], [19], [20], [16], [22], [23]- [26]).

For $\tau = it$ with $t > 0$ yields

$$\theta_3(it) = \sum_{n=-\infty}^{\infty} e^{-\pi n^2 t}$$

with aid of the Paisson summation formula, one has

$$\theta_3(it) = \frac{1}{\sqrt{t}}\theta_3\left(\frac{i}{t}\right)$$

is the modular transformation property and this transformation shows the behaviour of the theta function in the modular group $SL_2(\mathbb{Z})$ since $\frac{i}{t} = \frac{0t+i}{1t+0}$ (cf. [2], [3], [8], [12]-[14], [19], [20], [16], [22], [23]- [26]). This transformation is important for giving the functional equation of the zeta function. That is

$$\theta_3(it) = 1 + 2\sum_{n=1}^{\infty} e^{-\pi n^2 t}.$$

Thus,

$$\theta_3(it) - 1 = 2\sum_{n=1}^{\infty} e^{-\pi n^2 t}.$$

With the aid the Laplace transform, one has

$$\int_0^{\infty} t^{\frac{s}{2}-1}\left(\theta_3(it) - 1\right)dt = \pi^{\frac{-s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s),$$

where $\Gamma(s)$ denotes the Euler gamma function. After some calculations, we also have

$$\pi^{\frac{-s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \pi^{\frac{-(1-s)}{2}}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s) \tag{3}$$

(*cf.* [20]- [26]).

The theta function can help to explains the analytical properties of the zeta function. So, while the Euler product exmines the distribution of prime numbers, the modular transformation of the theta function provides analytic symmetry, thus showing the relationship between the distribution of prime numbers and Fourier analysis.

By applying the Fourier transform to

$$g(x) = e^{-\pi t x^2}$$

for $t > 0$, we have

$$\widehat{g}(\zeta) = \int_{-\infty}^{\infty} e^{-\pi t x^2} e^{-2\pi i x \zeta} dx = t^{\frac{-1}{2}} e^{\frac{-\pi \zeta^2}{t}}$$

Due to the following Poisson summation formula:

$$\sum_{n=-\infty}^{\infty} g(n) = \sum_{n=-\infty}^{\infty} \widehat{g}(n)$$

and $\widehat{g}(\zeta)$, we have

$$\sum_{n=-\infty}^{\infty} e^{-\pi n^2 t} = t^{\frac{-1}{2}} \sum_{n=-\infty}^{\infty} e^{\frac{-\pi n^2}{t}}$$

For $t > 0$, the above equation, gives us the following well-known result

$$\theta(t) = t^{\frac{-1}{2}}\theta\left(\frac{1}{t}\right),$$

where

$$\theta(t) = \sum_{n=-\infty}^{\infty} e^{-\pi n^2 t}$$

(cf. [3], [8], [12]- [14], [19], [20], [16], [22], [23]- [26]).

Setting

$$\Psi(t) = \sum_{n=1}^{\infty} e^{-\pi n^2 t}.$$

Thus, we have

$$\theta(t) = 1 + 2\Psi(t).$$

Combining the above relation with the functional equation of the following well-known Riemann zeta function and Poisson summation, the modular relation for to Ψ is given by

$$\Psi(t) = t^{\frac{-1}{2}} \Psi\left(\frac{1}{t}\right) + \frac{t^{\frac{-1}{2}} - 1}{2}. \tag{4}$$

Applying the Mellin transform to

$$\theta(t) - 1 = 2\Psi(t),$$

we have the following integral representation for the function $\zeta(s)$:

$$\zeta(s) = \frac{1}{2} \int_{0}^{\infty} t^{\frac{s}{2}-1} (\theta(t) - 1) dt.$$

Hence,

$$\zeta(s) = \int_1^{\infty} (u^{\frac{s}{2}-1} + u^{-\frac{s+1}{2}}) \Psi(u) du + \frac{1}{2} \int_1^{\infty} (u^{-\frac{s+1}{2}} - u^{-\frac{s}{2}-1}) du. \tag{5}$$

(*cf.* [19], [16]). For $\mathrm{Re}(s) > 1$, one also has

$$\zeta(s) = \int_1^{\infty} (u^{\frac{s}{2}-1} + u^{-\frac{s+1}{2}}) \Psi(u) du + \frac{1}{s(s-1)}. \tag{6}$$

For this operator, the following notation can be used:

$u = u(x)$, d: dimension, Δ: Laplacian, t: time, $t > 0$, ∇: gradient(first derivative vector), $\Delta = \mathrm{div}\,\nabla$, $u = u(x)$: scaler function, $x = (x_1, x_2, x_3, ..., x_d)$: space variable (cf. [7]- [16]).

The heat equation,

$$\partial_t u(t, x) = \Delta u(t, x) \tag{7}$$

with

$$u(0, x) = u_0(x)$$

(cf. [1]- [24]), where $u(t, x)$: time and location dependent field, g: start function, $K(t, x, y)$: heat kernel. Thus, the solution is as follows,

$$u(t, .) = e^{-t(-\Delta)} u_0,$$

which is written as the heat semigroup (cf. [5]- [24]).

Heat kernel, M on the surface $K(t, x, y)$,

$$(e^{-t(-\Delta)} g)(x) = \int_M K(t, x, y) g(y) dy \tag{8}$$

where K is the basic solution that transports heat from y to x after time t (*cf.* [11]- [12]- [20]- [16]).

Heat signature,

$$K(t) = Tr(e^{-t(-\Delta)}) = \sum_{n=-\infty}^{\infty} e^{-t\lambda n} \tag{9}$$

The Laplace operator

$$-\Delta u(x) = -\sum_{j=1}^{d} \frac{\partial^2 u(x)}{\partial x_j^2},$$

where λ eigenvalues$\geqslant 0$ and $-\Delta u(x) > 0$. λ are the energy levels of the oscillation or vibration modes in the system. For example, in the vibration of a string, each λ is proportional to the square of the frequency.(cf. [17], [19], [16], [6]).

$\{\lambda n\}$ are the eigenvalues of $-\Delta$.(cf. [13]- [12]- [7]- [24]).

The theta function repeats itself in certain directions. It is periodic with respect to integers:

$$\theta(z+1,\tau) = \theta(z,\tau)$$

It is multiply periodic with respect to the complex period determined by τ:

$$\theta(z+\tau,\tau) = e^{-\pi i\tau - 2\pi i z}\theta(z,\tau)$$

(*cf.* [13]- [23]- [20]- [1]).

The Jacobi theta functions is in a periodic environment,

$$\theta_3(it) = \sum_{n=-\infty}^{\infty} e^{-\pi n^2 t}$$

where $t > 0$ (*cf.* [19], [20], [22], [23]- [26]).

Eigenvalues of $S^{-1} = \mathbb{R}/\mathbb{Z}$ on circle $-\Delta$, for $n \in \mathbb{Z}$,

$$\lambda_n = (2\pi n)^2$$

trace of the heat core,

$$K_{s^1}(t) = \sum_{n=-\infty}^{\infty} e^{-4\pi^2 n^2 t} \tag{10}$$

Substituting $z = 0$ and $\tau = i4\pi t$ into (2), we get

$$K_{s^1}(t) = \theta_3(0, i4\pi t) \tag{11}$$

(*cf.* [19], [20], [22], [23]- [26]).

Using (9), for $\lambda > 0$, we also have

$$\int_0^\infty t^{s-1} e^{-t\lambda} dt = \lambda^{-s}\Gamma(s),$$

By applying the Mellin transform to the following function

$$K(t) = \sum_{\lambda_n > 0} e^{-t\lambda_n}$$

$t > 0$ and using the following known relation:

$$\int_0^\infty t^{s-1} e^{-t\lambda_n} dt = \lambda_n^{-s}\Gamma(s),$$

we also have

$$\int_0^\infty t^{s-1}K(t)dt = \int_0^\infty t^{s-1}\sum_{\lambda_n>0} e^{-t\lambda_n}dt.$$

(*cf.* [7]- [24]- [15]- [13]).

If R is large enough, it can be replaced with the help of Fubini's theorem.

$$\int_0^\infty t^{s-1}K(t)dt = \sum_{\lambda_n>0}\int_0^\infty t^{s-1}e^{-t\lambda_n}dt.$$

Therefore

$$\sum_{\lambda_n>0}\lambda_n^{-s}\Gamma(s) = \Gamma(s)\sum_{\lambda_n>0}\lambda_n^{-s} \tag{12}$$

(cf. [1]- [26]).

The spectral zeta function depend on -Δ is given by $\zeta_{-\Delta}^{S^1}(s)$. (cf. [7]).

Hence we can also write,

$$\zeta_{-\Delta}(s) = \frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}K(t)dt. \tag{13}$$

Some Results

In this section, some comments and results will be given on the heat equation and heat kernel in physics with the help of the spectral zeta function. Heat signature $K_{S^1}(t)$ is defined as follows:

$$K_{s^1}(t) = \sum_{n=-\infty}^{\infty} e^{-4\pi^2n^2t}$$

where $S^1 = \{z \in \mathbb{C},\ |z| = 1\}$ is the unit circle in the complex plane.

Due to equation (11), the Jacobi theta function, the heat kernel $K_{s^1}(t)$ is obtained so the theta function reduces to the heat kernel. By applying again values on S^1 we get the following result to (13):

$$\lambda_n = (2\pi n)^2$$

where , $n \in \mathbb{Z}$, $n \neq 0$,

$$\zeta_{-\Delta}(s) = \frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}K_{S^1}(t)dt,$$

and

$$\zeta_{-\Delta}^{S^1}(s) = \frac{1}{\Gamma(s)}\sum_{n\neq 0}\int_0^\infty t^{s-1}e^{-4\pi^2n^2t}dt. \tag{14}$$

By the help of these transformations yields,

$$\begin{aligned}\zeta_{-\Delta}^{S^1}(s) &= \frac{1}{\Gamma(s)}\sum_{n=-\infty,n\neq 0}^{\infty}(4\pi^2n^2)^{-s}\Gamma(s)\\ &= \sum_{n=-\infty,n\neq 0}^{\infty}(4\pi^2n^2)^{-s}.\end{aligned}$$

using symmetry property in the equation yields,

$$\zeta_{-\Delta}^{S^1}(s) = 2(4\pi^2)^{-s}\sum_{n=1}^{\infty}n^{-2s},$$

with the aid of zeta function the above equation that is $\zeta_{-\Delta}^{S^1}(s)$ can be written in terms of the Riemann zeta function:

$$\zeta_{-\Delta}^{S^1}(s) = 2(2\pi)^{-2s}\zeta(2s). \tag{15}$$

Substituting $s = v$, where $v \in \mathbb{N}$ into (15), and using the following well-known formula

$$\zeta(2v) = (-1)^{v+1}\frac{(2\pi)^{2v}}{2(2v)!}B_{2v}$$

(cf. [8], [16]), where B_v denotes the Bernoulli numbers, defined by means of the following generating function:

$$\frac{t}{e^t - 1} = \sum_{v=0}^{\infty}\frac{B_v}{v!}t^v$$

($|t| < 2\pi$), we have the following result:

$$\zeta_{-\Delta}^{S^1}(v) = (-1)^{v+1}\frac{1}{(2v)!}B_{2v}.$$

Some values of the function $\zeta_{-\Delta}^{S^1}(v)$ are given by

$$\begin{aligned}
\zeta_{-\Delta}^{S^1}(1) &= \frac{1}{12} \\
\zeta_{-\Delta}^{S^1}(2) &= -\frac{1}{4!}\left(-\frac{1}{30}\right) = \frac{1}{720}
\end{aligned}$$

and so on.

Substituting $s = -v$, where $v \in \mathbb{N}$ into (15), and using the following well-known formula

$$\zeta(-v) = -\frac{1}{v+1}B_v$$

(cf. [8], [16]), we have the following result:

$$\zeta_{-\Delta}^{S^1}(-v) = 2(2\pi)^{2v}\zeta(-2v).$$

Thus, we get

$$\zeta_{-\Delta}^{S^1}(-v) = -\frac{2(2\pi)^{2v}}{2v+1}B_{2v}.$$

Some values of the function $\zeta_{-\Delta}^{S^1}(-v)$ are given by

$$\begin{aligned}
\zeta_{-\Delta}^{S^1}(-1) &= -\frac{4\pi^2}{9} \\
\zeta_{-\Delta}^{S^1}(-2) &= -\frac{2(2\pi)^4}{5}\left(-\frac{1}{30}\right) = \frac{16\pi^4}{75}
\end{aligned}$$

and so on.

Conclusion

The main objective of this study is to survey the properties of the Jacobi theta function and zeta type function involving the Riemann zeta function. We also gave the connection between the Jacobi theta function and the heat kernel. We also give the relationship between the Mellin transform of the heat signature and the zeta function.

Motivation of our future work is to study on the Epstein zeta functions and theta functions in quadratic forms and related areas.

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Applications of Derangement numbers in real-world problems involving DNA sequences

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In this work study and survey some well-known applications of derangement numbers related to their generating function and also DNA sequences, combinatorial analysis, and related fields.

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KEYWORDS: Alignment errors, bioinformatics, DNA Sequencing, derangement Numbers, permutation modeling

Introduction

To motivation of these work is to study on the known the DNA sequencing, which is fundamental to biology, however errors come from misalignments or incorrect nucleotide placement remain a significant challenge. In order to give a mathematical perspective on these errors, this work introduces a new use of derangements. That is, in theory of combinatorics, a derangement is a permutation of a set of n elements, in which none of the elements occupy their original positions. In order to formulate this, we need a formula for the numbers derangement with their generating function, which has been many applications in applied sciences.

In this work d_n denotes the number of derangements of n elements. Explicit formula for the numbers d_n can be given by means of the following equation:

$$d_n = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}$$

(*cf.* [1]- [2]- [3]- [6]- [7]- [8]).

The numbers d_n is given by means of the following generating function:

$$\sum_{n=0}^{\infty} D_n \frac{t^n}{n!} = \frac{e^{-t}}{1-t}$$

(*cf.* [1]- [2]- [3]- [6]- [7]- [8]).

The following formula is a recurrence relation for the numbers d_n.

$$d_n = (n-1)(d_{n-1} + d_{n-2}),$$

with

$$d_0 = 1, \quad d_1 = 0$$

(*cf.* [3]- [6]). By applying derivatie operator to the generating function for the numbers d_n, this forrmula can be proved.

Simillarly, $d_{n,k}$ denotes the number of permutations of n elements having exactly k fixed points.

Explicit formula for the numbers $d_{n,k}$ can be given by means of the following equation:

$$d_{n,k} = \frac{n!}{k!}\sum_{j=0}^{n-k}\frac{(-1)^j}{j!}$$

(*cf.* [3]).

The numbers $d_{n,k}$ is given by means of the following generating function:

$$D_k(t) = \sum_{n=k}^{\infty} d_{n,k}\frac{t^n}{n!} = \frac{t^k e^{-t}}{k!(1-t)}$$

(*cf.* [4]).

It is easy to find the function $D_k(t)$ with the aid of the definition $d_{n,k}$. That is

$$D_k(t) = \sum_{n=k}^{\infty}\frac{n!}{k!}\left(\sum_{j=0}^{n-k}\frac{(-1)^j}{j!}\right)\frac{t^n}{n!} = \frac{1}{k!}\sum_{n=k}^{\infty} t^n \sum_{j=0}^{n-k}\frac{(-1)^j}{j!}.$$

On the right-hand side (RHS) of the above equation to change variable $m = n - k$ yields

$$D_k(t) = \frac{t^k}{k!}\sum_{m=0}^{\infty} t^m \sum_{j=0}^{m}\frac{(-1)^j}{j!}.$$

Reevaluating the inner sums in the above equation with the aid of convolution relation, we have

$$\sum_{m=0}^{\infty}\sum_{j=0}^{m}\frac{(-1)^j t^m}{j!} = \sum_{j=0}^{\infty}\frac{(-1)^j}{j!}\sum_{m=j}^{\infty} t^m = \sum_{j=0}^{\infty}\frac{(-1)^j t^j}{j!}\sum_{f=0}^{\infty} t^f = \frac{e^{-t}}{1-t}.$$

From a probabilistic perspective, the ratio $d_n/n!$ represents the probability that a randomly selected permutation contains no fixed points. As n grows large, this probability converges to:

$$\lim_{n\to\infty}\frac{d_n}{n!} = \frac{1}{e} \approx 0.3679.$$

Derangements are particularly useful for modeling scenarios in which complete positional mismatches occur. They provide a structured mathematical framework to quantify disorder and randomness in sequences. For example, in biological contexts such as DNA sequencing or protein alignment, derangements can model the probability of all nucleotides or amino acids being misaligned (*cf.* [9]- [10]). Consequently, the integration of combinatorial models, including derangements, with biological data not only deepens our understanding of sequencing reliability but also opens new research perspectives in personalized medicine, evolutionary biology, and genetic engineering (*cf.* [11]- [12]).

Further Observations and Remarks

A DNA sequence can be mathematically modeled as a permutation in which each nucleotide occupies a specific position. Let

$$S = (A_1, A_2, \ldots, A_n),$$

where elements of the sequence may repeat and $A_j \in \{A, T, G, C\}$ for $j \in \{1, 2, \ldots, n\}$. Sequencing errors including misplacement of nucleotides are expressed by the following permutation:

$$\sigma : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\},$$

where $\sigma(j)$ denotes the new position of the nucleotide originally at position j. If no nucleotide remains in its original position, σ constitutes a *derangement*. This gives us a combinatorial framework for modeling complete positional mismatches. For instance, we give the following for the above transformation a DNA sequence undergoing complete misplacement (derangement) can be visually represented as follows. Green boxes indicate nucleotides in their original positions, red boxes indicate displaced nucleotides, and arrows illustrate the movement from original to new positions.

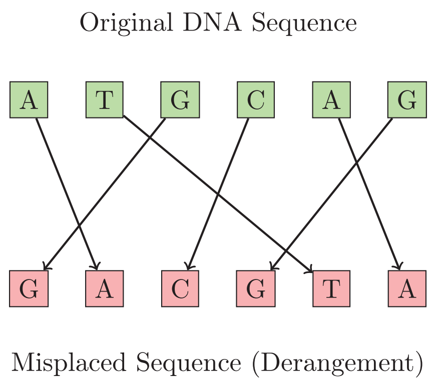

Figure 1: Schematic representation of a DNA sequence and its complete misplacement (derangement).

From above the graf, we take $n = 6$. Thus, the probability that all nucleotides in a DNA sequence are displaced from their original positions converges to a constant value as the sequence length increases. This asymptotic behavior is formally expressed as

$$\lim_{n\to\infty} \frac{d_n}{n!} = \frac{1}{e} \approx 0.3679.$$

In other words, regardless of the sequence length, the likelihood of a complete positional mismatch stabilizes around 36.79%.

Example 1. *Consider the case $n = 6$. The number of derangements is*

$$d_6 = 265,$$

while the total number of permutations is

$$6! = 720,$$

which gives the probability of complete misplacement as

$$\frac{d_6}{6!} = \frac{265}{720} \approx 0.368.$$

This value is remarkably close to the theoretical limit $\frac{1}{e} \approx 0.3679$, confirming that even for relatively small sequence lengths, the probability of total misalignment approaches the asymptotic constant.

We can also consider cases in which only a subset of nucleotides remains in their correct positions, leading to the concept of restricted derangements.

Example 2. *We now take into account a DNA sequence of length $n = 6$, where exactly $k = 2$ nucleotides remain in their original positions, while the remaining $n - k = 4$ nucleotides are given. The number of such permutations is given by means of the following relation:*

$$d_{n,k} = \frac{n!}{k!} \sum_{j=0}^{n-k} \frac{(-1)^j}{j!}.$$

Substituting $n = 6$ and $k = 2$ in to the above equation yields

$$d_{6,2} = \frac{6!}{2!} \sum_{j=0}^{4} \frac{(-1)^j}{j!}.$$

Combining the above equation with

$$\sum_{j=0}^{4} \frac{(-1)^j}{j!} = \frac{3}{8},$$

we get

$$d_{6,2} = \frac{720}{2} \cdot \frac{3}{8} = 135.$$

This result shows that there are 135 permutations in which exactly two nucleotide occupy their correct positions, while the remaining four are misplaced. By the probabilistic comments, the likelihood of this event is given by means of the following formulation:

$$P(\textit{exactly 2 fixed nucleotide}) = \frac{d_{6,2}}{6!} = \frac{135}{720} \approx 0.1875.$$

Conclusion

Here, we gave some the application of derangement numbers model to ordering errors in DNA sequences. Since representing nucleotide arrangements as permutations, we gave the probability of complete and partial positional misalignments by using combinatorial and probabilistic methods. This method showed that the probability of total misplacement converges to following numbers:

$$\frac{1}{e}$$

which gives us the sequence length increases, and partial misplacements with their evaluation by the restricted derangements.

The future aim of this is to study and investigate applications of the Derangement numbers not only to Real-World Problems Involving DNA Sequences, but also other areas.

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Note on a certain family of combinatorial polynomials of complex order

Ezgi Polat [*1] *and Yilmaz Simsek* [2]

The purpose of this work is to study certain family of the Peters-type combinatorial numbers and polynomials of complex-order. We also study and survey two parametric kind of the polynomials. We give some identities for these polynomials with the help of their generating functions. Moreover, we also give some remarks and observations on these polynomials.

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Keywords: Combinatorial numbers and polynomials, generating function

Introduction

In recent years, there has been a growing interest in the study of number theory, special numbers and polynomials and their generating function and mathematical analysis through their rich algebraic structures and wide applications in various branches of mathematics.

Utilizing combinatorial polynomials, such as the Stirling polinomials, the Bernoulli polynomials, and the Euler-type polynomials, new generalized of the polynomials and the numbers have been introduced and investigated incorporating additional parameters or orders to reflect more intricate structures or comprehensive applications. Khan et al. [4, 5] defined the 2-variable Simsek polynomials a new family of special numbers and polynomials. These polynomials exhibit interesting relationships with the Stirling numbers of both kinds, Apostol-Bernoulli numbers, the q-Euler numbers, the q-Changhee numbers, the Daehee numbers, the Cauchy numbers and the Hurwitz-Lerch zeta functions [17, 21].

The purpose of this paper is to contribute to this line of research by extending the known results on Simsek numbers and polynomials to the setting of complex order.

Throughout this paper, we use the notation

$$\mathbb{N}_0 = \{0\} \cup \mathbb{N}$$

and we denoted by $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{C}$ the set of natural numbers, the set of integers and the set of complex numbers, respectively.

We assume that

$$\binom{z}{n} = \frac{z(z-1)\dots(z-n+1)}{n!},$$

where $z \in \mathbb{C}$ and $n \in \mathbb{N}$.

The Peters-type combinatorial numbers and polynomials of complex order are defined by means of the following generating functions, respectively:

$$F(u, \lambda; z) = \frac{2^z}{(\lambda^2 u + \lambda - 1)^z} = \sum_{n=0}^{\infty} Y_n^{(z)}(\lambda) \frac{u^n}{n!}, \tag{1}$$

$$G(u,x,\lambda;z) = (1+\lambda u)^x F(u,\lambda;z) = \sum_{n=0}^{\infty} Y_n^{(z)}(x;\lambda)\frac{u^n}{n!}, \tag{2}$$

where $z \in \mathbb{C}$ (*cf.* [16]).

When z is an integer numbers, $Y_n^{(z)}(x;\lambda)$ and $Y_n^{(z)}(x;\lambda)$ reduce to work of Kucukoglu et al (*cf.* [8]- [14], [21]). Substituting $k=1$ into (1) and (2), one obtains

$$Y_n^{(1)}(\lambda) = Y_n(\lambda)$$

and

$$Y_n^{(1)}(x;\lambda) = Y_n(x;\lambda),$$

(*cf.* [21]).

For the polynomials $Y_n(x;\lambda)$, Khan et al. [4, 5] called the Simsek polynomials. After Khan et al.,for the polynomials $Y_n(x;\lambda)$ and the numbers $Y_n(\lambda)$, Kucukoglu [8]- [14] called the Peters-type Simsek polynomials and numbers of the first kind. After that Kilar [6], Gun [2] etc. have studied these numbers and polynomials.

The Stirling numbers of the first kind $S_1(n,k)$ can be defined by means of the following generating functions:

$$\frac{(\log(1+u))^k}{k!} = \sum_{n=0}^{\infty} S_1(n,k)\frac{u^n}{n!}, \tag{3}$$

wehere we noted that $S_1(n,k)=0$ if $k>n$ (*cf.* [1, 21]).

Two Tarametric Kinds of the Polynomials $Y_n^{(z)}(x;\lambda)$

In this section we give not only new relations related to the Stirling numbers of the first kind but also we survey two parametric kinds of the Peters-type combinatorial polynomials $Y_n^{(z)}(x;\lambda)$ and give some new identities for these kinds.

Let $z \in \mathbb{C}$, $k,t \in \mathbb{C}$ and set $x = k+it$. We define two parametric types of the polynomials $Y_n^{(z)}(x;\lambda)$ as follows. Considering Eq-(2), we have

$$G(u,k+it,\lambda;z) \quad = \quad F(u,\lambda;z)(1+\lambda u)^{k+it}.$$

That is,

$$G(u,k+it,\lambda;z) \quad = \quad G(u,k,\lambda;z)e^{it\ln(1+\lambda u)}. \tag{4}$$

By applying the following Euler identity

$$e^{wi} = \cos w + i\sin w$$

to the above equation, we have

$$G(u,k+it,\lambda;z) = G(u,k,\lambda;z)\left[\cos\left(t\ln(1+\lambda u)\right) + i\sin\left(t\ln(1+\lambda u)\right)\right]. \tag{5}$$

By using Eq-(5), we derive the following generating functions:

$$\mathcal{G}_c = G(u,k,\lambda;z)\cos\left(t\ln(1+\lambda u)\right) = \sum_{n=0}^{\infty} Y_n^{(C;z)}(k;\lambda)\frac{u^n}{n!} \tag{6}$$

and

$$\mathcal{G}_s = G(u,k,\lambda;z)\sin\left(t\ln(1+\lambda u)\right) = \sum_{n=0}^{\infty} Y_n^{(S;z)}(k;\lambda)\frac{u^n}{n!}. \tag{7}$$

Theorem 1. *Let $n \in \mathbb{N}$. Then*

$$Y_n^{(C;z)}(k;\lambda) = \sum_{d=0}^{n}\sum_{m=0}^{d}\binom{n}{d}Y_{n-d}^{(z)}(k;\lambda)\,(-1)^m\,t^{2m}\lambda^d S_1(d,2m). \tag{8}$$

Proof. Considering Eq-(6), using the series expansion of the cos function, then we have

$$\sum_{n=0}^{\infty}Y_n^{(C;z)}(k;\lambda)\frac{u^n}{n!} \quad = \quad \sum_{n=0}^{\infty}Y_n^{(z)}(k;\lambda)\frac{u^n}{n!}\sum_{m=0}^{\infty}(-1)^m\ln(1+\lambda u)^{2m}\frac{t^{2m}}{2m!}.$$

Combined with the above equation and the left side of Eq-(3), we give

$$\sum_{n=0}^{\infty}Y_n^{(C;z)}(k;\lambda)\frac{u^n}{n!} = \sum_{n=0}^{\infty}Y_n^{(z)}(k;\lambda)\frac{u^n}{n!}\sum_{m=0}^{\infty}(-1)^m\sum_{n=0}^{\infty}S_1(n,2m)\frac{\lambda^n u^n}{n!}t^{2m}. \tag{9}$$

After some calculations, we give:

$$\sum_{n=0}^{\infty}Y_n^{(C;z)}(k;\lambda)\frac{u^n}{n!} \quad = \quad \sum_{n=0}^{\infty}\sum_{d=0}^{n}\binom{n}{d}Y_{n-d}^{(z)}(k;\lambda)\sum_{m=0}^{d}(-1)^m\,t^{2m}\lambda^d S_1(d,2m)\frac{u^n}{n!}.$$

Comparing the coefficients of $\frac{u^n}{n!}$ on both sides of the above equation, we obtain desired result.

□

Theorem 2. *Let $n \in \mathbb{N}$. Then*

$$Y_n^{(S;z)}(k;\lambda) = \sum_{d=0}^{n}\sum_{m=0}^{d}\binom{n}{d}Y_{n-d}^{(z)}(k;\lambda)\,(-1)^m\,t^{2m+1}\lambda^d S_1(d,2m+1). \tag{10}$$

Proof. Similarly considering Eq-(7) using the series expansion of the sin function, then we have

$$\sum_{n=0}^{\infty}Y_n^{(S;z)}(k;\lambda)\frac{u^n}{n!} \quad = \quad \sum_{n=0}^{\infty}Y_n^{(z)}(k;\lambda)\frac{u^n}{n!}\sum_{m=0}^{\infty}(-1)^m\,\frac{\ln(1+\lambda u)^{2m+1}t^{2m+1}}{(2m+1)!}.$$

Combined with the above equation and the left side of Eq-(3), we give

$$\sum_{n=0}^{\infty}Y_n^{(S;z)}(k;\lambda)\frac{u^n}{n!} = \sum_{n=0}^{\infty}Y_n^{(z)}(k;\lambda)\frac{u^n}{n!}\sum_{m=0}^{\infty}(-1)^m\sum_{n=0}^{\infty}S_1(n,2m+1)\frac{\lambda^n u^n}{n!}t^{2m+1}. \tag{11}$$

After some calculations, we give:

$$\sum_{n=0}^{\infty}Y_n^{(S;z)}(k;\lambda)\frac{u^n}{n!} \quad = \quad \sum_{n=0}^{\infty}\sum_{d=0}^{n}\binom{n}{d}Y_{n-d}^{(z)}(k;\lambda)\sum_{m=0}^{d}(-1)^m\,t^{2m+1}\lambda^d S_1(d,2m+1)\frac{u^n}{n!}.$$

Comparing the coefficients of $\frac{u^n}{n!}$ on both sides of the above equation, we arrive at desired result. □

By using Eq-(8), some values of the polynomials $Y_n^{(C;z)}(k;\lambda)$ are given by

$$\begin{aligned}
Y_0^{(C;z)}(k;\lambda) &= Y_0^{(z)}(k;\lambda), \\
Y_1^{(C;z)}(k;\lambda) &= Y_1^{(z)}(k;\lambda), \\
Y_2^{(C;z)}(k;\lambda) &= Y_2^{(z)}(k;\lambda) - t^2\lambda^2 Y_0^{(z)}(k;\lambda), \\
Y_3^{(C;z)}(k;\lambda) &= Y_3^{(z)}(k;\lambda) - t^2\lambda^2 3Y_1^{(z)}(k;\lambda) - t^2\lambda^3 3Y_0^{(z)}(k;\lambda), \\
Y_4^{(C;z)}(k;\lambda) &= Y_4^{(z)}(k;\lambda) - t^2\lambda^2 6Y_2^{(z)}(k;\lambda) - t^2\lambda^3 12Y_1^{(z)}(k;\lambda) \\
&\quad -t^2\lambda^4 11Y_0^{(z)}(k;\lambda) + t^4\lambda^4 Y_0^{(z)}(k;\lambda),
\end{aligned} \tag{12}$$

and so on.

Using Eq-(10), some values of the polynomials $Y_n^{(S;z)}(k;\lambda)$ are given by

$$\begin{aligned}
Y_0^{(S;z)}(k;\lambda) &= 0, \qquad (13)\\
Y_1^{(S;z)}(k;\lambda) &= t\lambda Y_0^{(z)}(k;\lambda) - t^3\lambda Y_0^{(z)}(k;\lambda),\\
Y_2^{(S;z)}(k;\lambda) &= t\lambda 2Y_1^{(z)}(k;\lambda) + t\lambda^2 Y_0^{(z)}(k;\lambda),\\
Y_3^{(S;z)}(k;\lambda) &= t\lambda 3Y_2^{(z)}(k;\lambda) + t\lambda^2 3Y_1^{(z)}(k;\lambda) + t\lambda^3 2Y_0^{(z)}(k;\lambda) - t^3\lambda^3 Y_0^{(z)}(k;\lambda),\\
Y_4^{(S;z)}(k;\lambda) &= t\lambda 4Y_3^{(z)}(k;\lambda) + t\lambda^2 6Y_2^{(z)}(k;\lambda) + t\lambda^3 8Y_1^{(z)}(k;\lambda)\\
&\quad -t^3\lambda^3 4Y_1^{(z)}(k;\lambda) + t\lambda^4 6Y_0^{(z)}(k;\lambda) - t^3\lambda^4 6Y_0^{(z)}(k;\lambda),
\end{aligned}$$

and so on.

Partial Derivative Formulas for the $G(u, k + it, \lambda; z)$

In this section we give some new identities related to the partial derivative formulas for the function $G(u, x, \lambda; z)$.

Considering Eq-(4), let $x = k + it$. Then we have

$$G(u, k + it, \lambda; z) = (1 + \lambda u)^{k+it} F(u, \lambda; z)$$

That is,

$$G(u, k + it, \lambda; z) = (1 + \lambda u)^{it} G(u, k, \lambda; z)$$

Let us take the partial derivative of the function $G(u, x, \lambda; z)$ with respect to k and t. We have the following equations:

$$\frac{\partial}{\partial k} G(u, k + it, \lambda; z) = \ln(1 + \lambda u) G(u, k + it, \lambda; z), \qquad (14)$$

and

$$\frac{\partial}{\partial t} G(u, k + it, \lambda; z) = i \ln(1 + \lambda u) G(u, k + it, \lambda; z) \qquad (15)$$

By using Eq-(14), we have

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{\partial}{\partial k} \left\{ Y_n^{(z)}(k + it; \lambda) \right\} \frac{u^n}{n!} &= \sum_{n=0}^{\infty} (-1)^n \frac{\lambda^{n+1} u^{n+1}}{n+1} \sum_{n=0}^{\infty} Y_n^{(z)}(k + it; \lambda) \frac{u^n}{n!}\\
&= \sum_{n=1}^{\infty} \sum_{k=0}^{n-1} (-1)^k \binom{n}{k+1} Y_{n-k-1}^{(z)}(k + it; \lambda) \frac{\lambda^{k+1} k! u^n}{n!}.
\end{aligned}$$

Comparing the coefficients of $\frac{u^n}{n!}$ on both sides of the above equation, we arrive at

$$\frac{\partial}{\partial k} \left\{ Y_n^{(z)}(k + it; \lambda) \right\} = \sum_{k=0}^{n-1} (-1)^k \binom{n}{k+1} Y_{n-k-1}^{(z)}(k + it; \lambda) \lambda^{k+1} k!.$$

By using Eq-(15), we have

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{\partial}{\partial t} \left\{ Y_n^{(z)}(k + it; \lambda) \right\} \frac{u^n}{n!} &= i \sum_{n=0}^{\infty} (-1)^n \frac{\lambda^{n+1} u^{n+1}}{n+1} \sum_{n=0}^{\infty} Y_n^{(z)}(k + it; \lambda) \frac{u^n}{n!}\\
&= i \sum_{n=1}^{\infty} \sum_{k=0}^{n-1} (-1)^k \binom{n}{k+1} Y_{n-k-1}^{(z)}(k + it; \lambda) \frac{\lambda^{k+1} k! u^n}{n!}.
\end{aligned}$$

Comparing the coefficients of $\frac{u^n}{n!}$ on both sides of the above equation, we arrive at

$$\frac{\partial}{\partial t}\left\{Y_n^{(z)}(k+it;\lambda)\right\}=\sum_{k=0}^{n-1}(-1)^k\, i\binom{n}{k+1}Y_{n-k-1}^{(z)}(k+it;\lambda)\lambda^{k+1}k!.$$

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A note on the Nörlund sum and their applications

Elif Sukruoglu *[1] *and Yilmaz Simsek* [2]

The goal of this article is to derive the relationship among the Hurwitz zeta function, the Bernoulli numbers and polynomials, the Laplace and Stirling numbers, and the Nörlund sum. By applying the integral representation of the Laplace numbers to the functional equation of the Nörlund sum, we also derive a new explicit formula for Bernoulli numbers in terms of the Laplace numbers. Furthermore, with the aid of this formula, a relationship among the Hurwitz zeta function and finite sum involving the Laplace numbers and the Stirling numbers of the second kinds given.

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KEYWORDS: Nörlund sum, Laplace transform, generating function, Stirling numbers, Bernoulli polynomials and numbers, Hurwitz zeta function

Introduction

Special numbers and polynomials have long held a central place in analysis and applied mathematics, particularly in the study of differential equations, integral transforms, and summation techniques. In recent years, considerable attention has been paid to the study of the Nörlund sum in the context of integral transforms and operator theory. The Laplace transform and the inverse Laplace transform which are closely related to the Nörlund sum and its functional equation, are indispensable tools in applied mathematics, engineering, physics, and statistics. They also make them suitable for studying generating functions and identities related to special polynomials. Similarly, Bernoulli numbers and polynomials play a significant role in the theory of the Nörlund sum, and also in analytic number theory, combinatorics, and special functions. These numbers and polynomials are deeply connected with the Riemann zeta function and the Hurwitz zeta function have been thoroughly studied, and their structural properties have been revealed. Such connections not only enrich the theory of special functions, but also provide powerful tools for applications in pure and applied mathematics and other fields (*cf.* [1, 2, 3, 6, 8, 10]).

Another important analytical technique involves the Nörlund sum, which is also known as the Nörlund sum operator, a generalization of classical sum methods useful for extending the concept of convergence and assigning finite values to divergent series. This operator has been used in various settings to describe and analyze sequences of special numbers and families of polynomials, providing new insights into their structures.

Motivation of this study focuses on both the Nörlund sum method and the interaction between special numbers and polynomials, and the analytical frameworks provided by Laplace transforms, the Bernoulli numbers and polynomials, the Stirling numbers, and the Nörlund sum. By utilizing these, we explore the discovery of new identities and new connections among these topics.

Some standard notations of this study are given as follows: The sets of natural numbers, integers, real numbers, and complex numbers are denoted by $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ respectively, and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

The Nörlund principal solution formula is defined by:

$$F(x;w) = \overset{x}{\underset{a}{\boldsymbol{S}}} K(z) \underset{w}{\Delta} z = \int_a^{\infty} K(t)dt - w \sum_{j=0}^{\infty} K(x+jw) \tag{1}$$

where $\overset{x}{\underset{a}{\boldsymbol{S}}} K(z) \underset{w}{\Delta} z$ is so-called *summing* $K(z)$ *from* a *to* x and w is so-called span of the sum (*cf.* [4, 5, 9, 11, 12]).

Now, we give some generating functions for certain family of special numbers and polynomials and notations. They will serve as a main tool for our results in the next sections.

Let $k \in \mathbb{N}_0$. The Stirling numbers of the first kind and the Stirling numbers of the second kind are defined by respectively

$$\frac{(\log(1+t))^k}{k!} = \sum_{n=0}^{\infty} S_1(n,k) \frac{t^n}{n!}, \tag{2}$$

or

$$x^{(k)} = \sum_{j=0}^{k} S_1(k,j) x^j, \tag{3}$$

and

$$\frac{\left(e^j - 1\right)^k}{k!} = \sum_{j=0}^{\infty} S_2(j,k) \frac{t^j}{j!},$$

or

$$x^j = \sum_{k=0}^{j} S_2(j,k) x^{(k)} \tag{4}$$

where $x^{(k)} = x(x-1)\ldots(x-k+1)$ (*cf.* [2, 6, 8, 10]).

The Laplace numbers, which are defined by means of the following generating functions

$$\frac{t}{\ln(1+t)} = \sum_{n=0}^{\infty} L_n \frac{t^n}{n!},$$

and

$$L_n = \int_0^1 \binom{x}{n} dx, \tag{5}$$

(*cf.* [4, 6, 7, 11, 12]).

The Bernoulli polynomials and numbers are defined by respectively

$$\frac{te^{tx}}{e^t - 1} = \sum_{n=0}^{\infty} \frac{B_n(x)}{n!} t^n,$$

and

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} t^n,$$

with $B_0(x) = 1$ and $|t| < 2\pi$ (*cf.* [6, 8, 10]).

The Hurwitz zeta function is defined by

$$\zeta(s,a)=\sum_{n=0}^{\infty}\frac{1}{(n+a)^s},$$

where $s=\alpha+i\beta$ and $\alpha>1$ (*cf.* [2, 6, 8, 10]). Bernoulli numbers and polynomials appear in many areas of number theory and analysis. They are closely related to the Riemann zeta function through the identity

$$\zeta\left(1-n\right)=-\frac{B_n}{n}, \tag{6}$$

where $n\geqslant 1$ (*cf.* [2, 6, 10]). A deeper generalization is provided by the Hurwitz zeta function $\zeta(s,a)$, whose evaluation at negative integers yields Bernoulli polynomials via

$$\zeta\left(-n,\alpha\right)=-\frac{B_{n+1}\left(\alpha\right)}{n+1},$$

where $n\geqslant 0$ (*cf.* [2, 6, 10]). The interaction between Bernoulli numbers and polynomials and the Hurwitz zeta function forms an important part of modern analytic number theory.

Identities for certain special numbers and polynomials derive from the Nörlund sum and Laplace numbers

In this section, we give novel identities by using the Nörlund sum involving Laplace numbers. We also derive the relationship between these formulas and some special numbers. Furthermore, we derive the relationship among the Stirling numbers of the second kind, the Laplace numbers, and the Hurwitz zeta function.

Substituting

$$K(z)=\lambda^z\binom{z}{v},$$

into equation (1), we fond a new functional equation which was given in [11]. Putting $v=0$ in the related equation, we have the following reduced formula for $|\lambda^w|<1$ and $n\in\mathbb{N}$.

$$\sum_{n=1}^{\infty}\left(\lambda^w-1\right)^{n-1}L_n=\frac{1}{\ln\lambda^w}-\frac{1}{\lambda^w-1}. \tag{7}$$

In order to prove the next theorem, we need the above formula:

Theorem 1. *Let $m\in\mathbb{N}$ with $m\geqslant 1$. Then we have*

$$\sum_{n=0}^{m-1}n!L_{n+1}S_2\left(m+1,n\right)=-\frac{B_m}{m} \tag{8}$$

where L_n Laplace numbers and B_m Bernoulli numbers.

Proof. Substituting

$$\lambda^w=e^t$$

into equation (7), we get

$$\sum_{n=1}^{\infty}\left(e^t-1\right)^{n-1}L_n=\frac{1}{\ln e^t}-\frac{1}{e^t-1}.$$

Therefore

$$\sum_{m=0}^{\infty}\sum_{n=1}^{\infty} w^m(n-1)!L_nS_2\left(m,n-1\right)\frac{t^m}{m!} = \frac{1}{tw} - \frac{1}{tw}\sum_{m=0}^{\infty} B_m\frac{t^m}{m!}w^m.$$

After some elementary calculations on both sides of the above equation, we get

$$\sum_{m=0}^{\infty} m\sum_{n=1}^{\infty} w^m(n-1)!L_nS_2\left(m-1,n-1\right)\frac{t^m}{m!} = 1 - \sum_{m=0}^{\infty} B_m\frac{t^m}{m!}w^m.$$

Comparing the coefficients $\frac{t^m}{m!}$ on both sides, the desired result is obtained. □

By putting equation (6) in equation (8), the following corollary:

Corollary 2. *The following relation holds true:*

$$\zeta(1-m) = \sum_{n=0}^{m-1} n!L_{n+1}S_2\left(m+1,n\right).$$

Theorem 3. *Let $m \in \mathbb{N}$ with $m \geqslant 1$. Then we have*

$$B_m = m\sum_{n=0}^{m-1}\sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k+1}k^{m+1}L_{n+1}. \tag{9}$$

Combining equation (8) and equation (9), after some calculations the following corollary:

Corollary 4. *The following relation holds true:*

$$\sum_{k=0}^{m}\frac{(-1)^k k!}{k+1}S_2\left(m,k\right) + \sum_{k=0}^{m-1} k!L_{k+1}S_2\left(m+1,k\right) = 0.$$

After some calculations in the above equation, we get the following result:

$$\frac{(-1)^{m+1}m!}{m+1} = \sum_{k=0}^{m-1} k!\left(\frac{(-1)^k S_2\left(m,k\right)}{k+1} + L_{k+1}S_2\left(m+1,k\right)\right).$$

Conclusion

In this study, we derived novel identities using the relationship between the Laplace numbers and the Nörlund sum. We also derived the relationship between Laplace numbers and special numbers involving the Bernoulli numbers, the Laplace numbers and the Stirling numbers using functional equation for the Nörlund sum. By assigning some special values, we derived the relationship between the Hurwitz zeta function and a finite sum involving the Laplace numbers and the Stirling numbers.

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On ideal semitopological groups

Florion Cela [*1] *and Ledia Subashi* [2]

We introduce and investigate *ideal semitopological groups*, obtained by requiring left and right translations to be I–continuous on an ideal topological space (G, τ, I). The notion strictly generalizes classical semitopological groups: we give an explicit example of an ideal semitopological group that is not a (classical) semitopological group. We establish basic translation invariance facts: images of open (resp. closed) sets under translations are I–open (resp. I–closed), and unions AO, OA with $O \in \tau$ are I–open. Under the additional hypothesis that (G, τ) is submaximal, the entire I–neighbourhood system at any point is generated by translating the I–neighbourhood system at the identity, recovering the classical picture; in particular, ideal semitopological groups are I–homogeneous. We also study homomorphisms: for $f : G_1 \to G_2$ with G_1 ideal semitopological and G_2 semitopological, I–continuity at the identity implies I–continuity at every point. Finally, we present a counterexample showing that I–continuity at the identity does not in general propagate when the codomain is also an ideal semitopological group.

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KEYWORDS: Ideal semitopological group, ideal topological space, I-open set, I-continuity, submaximal space, translations, I-neighbourhood, homomorphisms

Introduction

Topological groups form a central interface between algebra and topology. A well–known weakening is the *semitopological group*, where only left and right translations are required to be continuous. In parallel, the theory of *ideal topological spaces* replaces classical openness/closure by their ideal–refinements; see the foundational works [4, 5, 6]. The synthesis of these directions in the setting of groups has led to the study of *ideal topological groups* [7, 8].

In this paper we develop the intermediate notion of an *ideal semitopological group*: a group $(G, \cdot)$ endowed with an ideal topological structure (G, τ, I) such that for every $a \in G$ the translations $L_a(x) = ax$ and $R_a(x) = xa$ are I–continuous. This replaces usual continuity by I–continuity while retaining the translation–based control that is characteristic of (semi)topological groups; compare the classical translation picture (e.g. [2, Ch. 1]).

Our contributions are as follows. First, we exhibit a concrete example where (G, τ, I) is an ideal semitopological group but (G, τ) fails to be a classical semitopological group, showing the generalization is nontrivial. Next, we prove standard translation facts in the I–setting: translations send open sets to I–open sets and closed sets to I–closed sets, and for $O \in \tau$ and $A \subseteq G$ the sets AO and OA are I–open. Under submaximality of the ambient topology, we recover the classical neighbourhood transport by translations: the I–neighbourhood filter at any $g \in G$ equals the left/right translate of the I–neighbourhood filter at the identity. As a corollary, ideal semitopological groups are I–homogeneous. Finally, for homomorphisms $f : G_1 \to G_2$ with G_1 ideal semitopological and G_2 semitopological, we show that I–continuity at the identity propagates to every point; we also give a counterexample demonstrating that such propagation can fail when G_2 is itself an ideal semitopological group.

The paper is self–contained: Section 2 recalls the needed facts on ideals and I–openness; Section 2 contains the main results outlined above. For standard background on (semi)topological groups we refer to [2], and for the ideal–theoretic framework to [6].

Ideal semitopological group

Definition 1. [1] *Let X be a set. A nonempty family $I \subseteq \mathcal{P}(X)$ is an* ideal *on X if:*

1. *If $A \in I$ and $B \subseteq A$, then $B \in I$.*
2. *If $A, B \in I$, then $A \cup B \in I$.*

The ideal I is proper *if $X \notin I$ (equivalently, $I \neq \mathcal{P}(X)$).*

Definition 2. [1] *Let (G, τ) be a topological space and let $I \subseteq \mathcal{P}(G)$ be a proper ideal. For $A \subseteq G$ define*

$$A^* = \{x \in G : \ \forall U \in \tau \ (x \in U \Rightarrow U \cap A \notin I)\}.$$

*A set $U \subseteq G$ is called I–*open *if $U \subseteq (U^*)^\circ$. The triple (G, τ, I) is then called an* ideal topological space*.*

Definition 3. [1] *Let (G, τ) be a topological space and let $I \subseteq \mathcal{P}(G)$ be a proper ideal. For $A \subseteq G$ set*

$$A^* = \{x \in G : \ \forall U \in \tau \ (x \in U \Rightarrow U \cap A \notin I)\}.$$

*A subset $U \subseteq G$ is called I–*open *if $U \subseteq (U^*)^\circ$. A map $f : (G, \tau, I) \to (G, \tau, I)$ is called I–*continuous *if $f^{-1}(V)$ is I–open for every open V.*

Remark 1. *In any ideal topological space (X, τ, I), the intersection of an open set with an I-open set is again I-open; that is, if $O \in \tau$ and U is I-open, then $O \cap U$ is I-open.*
Intuition. *Since $U \subseteq (U^*)^\circ$, around each $x \in U$ there is an open neighbourhood $N \subseteq U^*$. Intersecting with O keeps us inside $(O \cap U)^*$ (because any open neighbourhood inside O still meets U in a set not belonging to I), hence $O \cap U \subseteq ((O \cap U)^*)^\circ$.*
In general, the intersection of I-open sets need not be I-open.

Submaximal spaces and I–openness. We will frequently use the classical notion of submaximality.

Definition 4. *[1] A topological space X is* submaximal *if it satisfies one of the following equivalent conditions:*

(i) Every subset of X is locally closed (i.e., an intersection of an open set and a closed set).

(ii) Every dense subset of X is open.

(iii) Every preopen subset of X is open.

Remark 2. *Observe that if (X, τ, I) is an ideal topological space, then every I–open set is preopen in X [6]. And if X is submaximal, then every I–open set in X is open.*

Definition 5 ([2])**.** *A* semitopological group *is a triple $(G, \cdot, \tau)$ where $(G, \cdot)$ is a group and τ is a topology on G such that:*

1. *for every $a \in G$, the left and right translations*

$$L_a(x) = a \cdot x, \qquad R_a(x) = x \cdot a$$

are τ–continuous (equivalently, the multiplication map $m : G \times G \to G$, $m(x, y) = x \cdot y$, is separately continuous*); and*

2. *the inversion map $i : G \to G$, $i(x) = x^{-1}$, is τ–continuous.*

Remark 3. *Conventions vary in the literature: some authors call a group with (1) alone a semitopological group, reserving the term* quasitopological group *for the case when (2) also holds. In this paper we adopt the definition above, which includes continuity of inversion; see, e.g., [2, Ch. 1].*

Proposition 1 ([2])**.** *Let $(G, \cdot, \tau)$ be a semitopological group (left/right translations and inversion are τ–continuous). For every $a \in G$, the maps $L_a(x) = a \cdot x$, $R_a(x) = x \cdot a$, and $i(x) = x^{-1}$ are homeomorphisms of (G, τ).*

Corollary 1 ([2])**.** *Every semitopological group $(G, \cdot, \tau)$ is homogeneous: for all $a, b \in G$ there exists a homeomorphism $f : G \to G$ with $f(a) = b$ (e.g. $f = R_{a^{-1}b}$ or $f = L_{ba^{-1}}$).*

Proposition 2 ([2])**.** *If $V(x)$ denotes the family of neighbourhoods of x in a semitopological group, then $V(g) = g\,V(e) = V(e)\,g$ for every $g \in G$.*

Proposition 3 ([2])**.** *For every $A \subseteq G$ and $a \in G$, $\mathrm{Int}(aA) = a\,\mathrm{Int}(A)$, $\overline{aA} = a\,\overline{A}$, $\mathrm{Int}(Aa) = \mathrm{Int}(A)\,a$, and $\overline{Aa} = \overline{A}\,a$.*

Proposition 4 ([2])**.** *Let $f : (G_1, \tau_1) \to (G_2, \tau_2)$ be a group homomorphism between semitopological groups. Then f is continuous at e_1 iff f is continuous at every $g \in G_1$.*

Main Result

Definition 6. *Let $(G, \cdot)$ be a group. A triple (G, τ, I) is called an* ideal semitopological group *if:*

1. *(G, τ, I) is an* ideal topological space *(as above), and*

2. *for every $a \in G$, the* translations *$L_a(x) = a \cdot x$ and $R_a(x) = x \cdot a$ are I–*continuous.

Example 1. *Let $G = (\mathbb{Z}, +)$, let $T := \mathbb{Z}\backslash\{0\}$, and set*

$$\tau = \{\varnothing,\, T,\, \mathbb{Z}\}, \qquad I = \mathrm{Fin}(\mathbb{Z}).$$

(1) Not semitopological. *For $a \neq 0$,*

$$L_a^{-1}(T) = \{x \in \mathbb{Z} : a + x \neq 0\} = \mathbb{Z}\backslash\{-a\},$$

which is not in τ (the only cofinite open set in τ is T). Hence L_a is not τ–continuous, so $(\mathbb{Z}, \tau)$ is not semitopological.

(2) Ideal semitopological. *We must show $L_a^{-1}(O)$ is I–open for each $a \in \mathbb{Z}$ and $O \in \tau$ (similarly for R_a). Trivially true for $O = \varnothing, \mathbb{Z}$. For $O = T$ put $U := L_a^{-1}(T) = \mathbb{Z}\backslash\{-a\}$ (cofinite). Fix $x \in U$. Every open nbhd of x in τ is either T (if $x \neq 0$) or $\mathbb{Z}$ (if $x = 0$); in both cases $V \cap U$ is infinite, hence $V \cap U \notin I =$ Fin. Thus $x \in U^*$ for all x, so $U^* = \mathbb{Z}$ and $U \subseteq (U^*)^\circ = \mathbb{Z}$. Therefore U is I–open and all translations are I–continuous.*

So $(\mathbb{Z}, \tau, I)$ is an ideal semitopological group, but $(\mathbb{Z}, \tau)$ is not semitopological group.

Proposition 5. *Let (G, τ, I) be an ideal semitopological group. Then every open set $O \in \tau$ is I–open.*

Proof. Take $a = e$. Since $L_e = \mathrm{id}_G$ is I–continuous, for any open $O \in \tau$ we have $L_e^{-1}(O) = O$ I–open by the definition of I–continuity. Hence all open sets are I–open. □

Definition 7. *Let (G, τ, I) be an ideal semitopological group and let e denote the identity of G. For each $x \in G$ define*

$$\mathbb{L}_I(x) := \{\, L_{x^{-1}}^{-1}(O) : \ O \in \tau,\ e \in O \,\}, \qquad \mathbb{R}_I(x) := \{\, R_{x^{-1}}^{-1}(O) : \ O \in \tau,\ e \in O \,\}.$$

Every element of these families is an I-open neighbourhood of x, because $L_{x^{-1}}$ and $R_{x^{-1}}$ are I-continuous and, whenever $e \in O$, we have $x \in L_{x^{-1}}^{-1}(O)$ and $x \in R_{x^{-1}}^{-1}(O)$.

Proposition 6. *Let (G, τ, I) be an ideal semitopological group and let $a \in G$. Then:*

(a) If $O \in \tau$ is open, then $L_a(O)$ and $R_a(O)$ are I-open.

(b) If $F \subseteq G$ is closed, then $L_a(F)$ and $R_a(F)$ are I-closed.

(c) If $O \in \tau$ and $A \subseteq G$, then $AO = \bigcup_{a \in A} aO$ and $OA = \bigcup_{a \in A} Oa$ are I-open.

Proof. (a) Since every translation is I–continuous and $L_a^{-1} = L_{a^{-1}}$, for open O we have

$$L_a(O) = L_{a^{-1}}^{-1}(O) \text{ is } I\text{–open.}$$

The argument for $R_a(O) = R_{a^{-1}}^{-1}(O)$ is identical.

(b) If F is closed, then $F^c \in \tau$, so by part (a) $L_a(F^c)$ and $R_a(F^c)$ are I–open. Because translations are bijections, $(L_a(F))^c = L_a(F^c)$ and $(R_a(F))^c = R_a(F^c)$, hence $L_a(F)$ and $R_a(F)$ are I–closed.

(c) Write $AO = \bigcup_{a \in A} aO = \bigcup_{a \in A} L_a(O)$ and $OA = \bigcup_{a \in A} Oa = \bigcup_{a \in A} R_a(O)$. Each $L_a(O)$ and $R_a(O)$ is I–open by (a); arbitrary unions of I–open sets are I–open, hence the claim. □

Remark 4. *In the theory of (classical) semitopological groups, the neighbourhood system at any $x \in G$ is obtained by translating the neighbourhood system at the identity e via the maps L_x and R_x. The proposition above shows only one direction of this inclusion for the I–setting. In particular, when G is* submaximal, *the next proposition shows that the neighbourhood system at e actually generates the entire I–neighbourhood system at every point of G.*

Proposition 7. *Let (G, τ, I) be an ideal semitopological group and assume (G, τ) is submaximal. For every $a \in G$:*

(a) If O is I–open, then $L_a(O)$ and $R_a(O)$ are I–open.

(b) If F is I–closed, then $L_a(F)$ and $R_a(F)$ are I–closed.

(c) If O is I–open and $A \subseteq G$, then

$$AO = \bigcup_{x \in A} xO \quad \text{and} \quad OA = \bigcup_{x \in A} Ox$$

are I–open.

Proof. Since (G,τ) is submaximal, every I–open set is preopen, hence open. Therefore, by the previous proposition (images of open sets under translations are I–open), we obtain (a). For (b), use complements and the bijectivity of translations. For (c), write $AO = \bigcup_{x\in A} L_x(O)$ and $OA = \bigcup_{x\in A} R_x(O)$ and use (a) together with stability of I–open sets under arbitrary unions. $\square$

Theorem 1. *Let (G,τ,I) be an ideal semitopological group with (G,τ) submaximal. Denote by $V_I(x)$ the family of all I-neighbourhoods of $x \in G$. Then for every $g \in G$,*

$$V_I(g) \;=\; g\,V_I(e) \;=\; V_I(e)\,g,$$

where $g\,V_I(e) := \{\, gU : U \in V_I(e)\,\}$ and $V_I(e)\,g := \{\, Ug : U \in V_I(e)\,\}$.

Proof. *($\mathbb{L}_I(g) \subseteq V_I(g)$).* Let $U \in V_I(e)$. Since (G,τ) is submaximal, every I-open set is open; hence $U \in \tau$. Because $L_g^{-1} = L_{g^{-1}}$ is I-continuous,

$$gU = L_g(U) = (L_{g^{-1}})^{-1}(U)$$

is I-open, and $g \in gU$. Thus $gU \in V_I(g)$.

($V_I(g) \subseteq \mathbb{R}_I(g)$). Let $C \in V_I(g)$. Again submaximality gives $C \in \tau$. Since $R_g^{-1} = R_{g^{-1}}$ is I-continuous, $R_g^{-1}(C)$ is an I-open neighbourhood of e. As R_g is a bijection,

$$C = R_g\big(R_g^{-1}(C)\big) \in V_I(e)\,g.$$

The symmetric argument with L_g^{-1} yields $V_I(g) \subseteq gV_I(e)$. Combining the inclusions gives $V_I(g) = \mathbb{R}_I(g) = \mathbb{L}_I(g)$. $\square$

Proposition 8. *Let (G,τ,I) be an ideal semitopological group. Then G is I-homogeneous: for all $a,b \in G$ there exists an I-homeomorphism $f : G \to G$ with $f(a) = b$.*

Proof. Set $c := a^{-1}b$ and take $f := R_c$, where $R_c(x) = xc$. Then $f(a) = ac = a(a^{-1}b) = b$. The map R_c is a bijection with inverse $R_{c^{-1}}$. By definition of ideal semitopological group, both R_c and $R_{c^{-1}}$ are I-continuous; hence R_c is an I-homeomorphism. (Equivalently, one may use $f := L_{ba^{-1}}$.) $\square$

Remark 5. *Within each family, compositions remain I-homeomorphisms: $L_a \circ L_b = L_{ab}$ and $R_a \circ R_b = R_{ba}$, with inverses $L_{(ab)^{-1}}$ and $R_{(ba)^{-1}}$ again I-continuous. However, in general the class of all I-homeomorphisms need not be closed under composition (e.g. mixed compositions such as $L_a \circ R_b$ need not be I-continuous). In the submaximal case, where I-open = open, the usual closure under composition is recovered.*

Proposition 9. *Let $f : (G_1,\tau_1,I_1) \to (G_2,\tau_2)$ be a* group homomorphism*, where (G_1,τ_1,I_1) is an ideal semitopological group and (G_2,τ_2) is a semitopological group. Denote the identities by $e_1 \in G_1$ and $e_2 \in G_2$. Then*

$$f \text{ is } I_1\text{–continuous at } e_1 \quad\Longleftrightarrow\quad f \text{ is } I_1\text{–continuous at every } g \in G_1.$$

Proof. One direction is evident; we prove the converse. Let $O \in V(f(g))$, so $O = f(g)\,O_{e_2}$ for some open neighbourhood $O_{e_2} \ni e_2$ (since $L_{f(g)}$ is a homeomorphism). As f is (usual) continuous at the identity e_1, there exists an open neighbourhood $O_e \ni e_1$ such that $f(O_e) \subseteq O_{e_2}$. The set O_e is open and contains e_1; hence, by the preceding proposition, gO_e is I_1–open. Moreover,

$$f(gO_e) = f(g)\,f(O_e) \subseteq f(g)\,O_{e_2} = O.$$

Thus $gO_e \subseteq f^{-1}(O)$ with gO_e I_1–open, which shows that f is I_1–continuous at g. $\square$

Remark 6. *The above statement is not true in general in the case when G_2 is an ideal semitopological group.*

Example 7. *Let $G_1 = (\mathbb{Z}, +)$ with the* indiscrete *topology $\tau_1 = \{\varnothing, \mathbb{Z}\}$ and let*

$$I_1 = \{A \subseteq \mathbb{Z} :\ 0 \notin A\}$$

(the ideal of all subsets avoiding 0). Then (G_1, τ_1, I_1) is an ideal semitopological group: indeed, for $U \subseteq \mathbb{Z}$ one computes

$$U^* = \begin{cases} \mathbb{Z}, & \text{if } 0 \in U, \\ \varnothing, & \text{if } 0 \notin U, \end{cases}$$

so the I_1–open sets are exactly those containing 0 (and $\varnothing$); since the only open sets are $\varnothing, \mathbb{Z}$ and translations preimage these to $\varnothing, \mathbb{Z}$, all left/right translations are I_1–continuous.

Let $G_2 = (\mathbb{Z}, +)$ with $\tau_2 = \{\varnothing, T, \mathbb{Z}\}$ where $T = \mathbb{Z}\backslash\{0\}$, and let $I_2 = \mathrm{Fin}(\mathbb{Z})$. As shown earlier, (G_2, τ_2, I_2) is an ideal semitopological group (translations are I_2–continuous).

Consider the identity homomorphism $f : G_1 \to G_2$, $f(n) = n$.

The only open neighbourhood of $f(0) = 0$ in (G_2, τ_2) is $\mathbb{Z}$, and $f^{-1}(\mathbb{Z}) = \mathbb{Z}$ is I_1–open (it contains 0). Hence f is I_1–continuous at $e = 0$.

Take $g = 1$. The set $T = \mathbb{Z}\backslash\{0\}$ is an open neighbourhood of $f(1) = 1$ in (G_2, τ_2), and $f^{-1}(T) = T$. But in (G_1, τ_1, I_1) every I_1–open neighbourhood of 1 must contain 0 (since the I_1–open sets are precisely those containing 0), so no I_1–open neighbourhood of 1 can be contained in T. Therefore f is not *I_1–continuous at $g = 1$.*

Consequently, f is I–continuous at the identity but not I–continuous at all points, with both G_1 and G_2 ideal semitopological groups.

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Time series regression analysis: An application on midwife numbers in Turkey between 1928 and 2020

Fusun Yalcin *[1] *and Yagmur Karaca Ulkutanir* [2]

Time series represent a potent methodology for the analysis of time-dependent data within fields such as mathematics, statistics, and econometrics. In this study, regression models were evaluated using data on midwife counts from Turkey between 1928 and 2020, and the most suitable model was identified. The objective of the present study is to formulate forward-looking predictions based on historical data and to furnish a scientific framework to guide policy-makers in the health sector. The findings demonstrate that the cubic regression model provides the most suitable representation of the data set. The prediction of midwife numbers for 2021 and 2022 was conducted utilising the selected model.

2020 MSC: 62-06, 62M10, 62J05

KEYWORDS: Time series, regression analysis, cubic regression, health statistics, forecasting

Introduction

Time series regression analysis is a powerful statistical method for analyzing data points collected at regular intervals over time (*cf.* [1, 2]). This technique is extensively utilized across a wide range of disciplines in both the social and natural sciences, including mathematics, statistics, econometrics, engineering, biology, and geology (*cf.* [3, 4]). A fundamental feature of time series analysis is its ability to forecast future outcomes by studying past data trends. This adaptable structure has made it a frequently employed methodology for researchers in various scientific fields.Time series regression analysis is a statistical method that examines the relationship between a dependent variable and one or more independent variables, with observations collected in a sequential order over time. (*cf.* [4]). This method is particularly valuable for understanding how a variable changes over time and for making informed predictions about its future behavior. The core principle involves identifying patterns, trends, and seasonal variations within a dataset to extrapolate future values.

Materials and Methods

The dataset used in the study consists of annual midwife numbers from 1928 to 2020 obtained from the Turkish Statistical Institute (TUIK) (*cf.* [5]). SPSS and EViews software were used to analyze the data. Simple linear regression, quadratic regression, cubic regression, logarithmic regression, logistic regression, and other alternative regression models were tested in the study. The cubic model was selected as the most suitable model to explain the data set, and the analysis was continued using this model.

The cubic regression model that best explains the increasing trend in the data set is as follows:

$$y_i = \beta_0 + \beta_1 x_i + \beta_2 x_i^2 + \beta_3 x_i^3 + \varepsilon$$

In this equation, y represents the dependent variable, and x_i represents the independent variables. β_0 is the regression constant, while β_1, β_2 and β_3 are coefficients indicating the effect of the independent variables on y. ε is the error term (*cf.* [6]).

Statistical suitability criteria such as root mean square error (RMSE), mean absolute error (MAE), Akaike information criterion (AIC), and Schwarz information criterion (SIC) values were considered for model selection.

In addition, the stationarity of the data set was tested using the Augmented Dickey-Fuller (ADF) test, seasonal effects were examined, and necessary transformations were performed (*cf.* [2]).

One of the most critical points in time series analysis is determining whether the series is stationary. Since misleading results can be obtained in non-stationary series, methods such as differencing or logarithmic transformation are used to make the series stationary. In this study, special importance was given to the stationarity analysis due to the long-term nature of the midwife data.

Result

The analyses revealed that the cubic regression model had the lowest RMSE and MAE values (Table 1 and Figure 1). This result indicates that the cubic regression model best explains the increasing trend in the dataset (Table 1 and Figure 1). The parameters obtained from the model were found to be statistically significant and showed significance at the $p < 0.01$ level. Midwife numbers for 2021 and 2022 were estimated using the cubic regression model. The estimated values indicate that the upward trend in the number of midwives in Turkey is continuing.

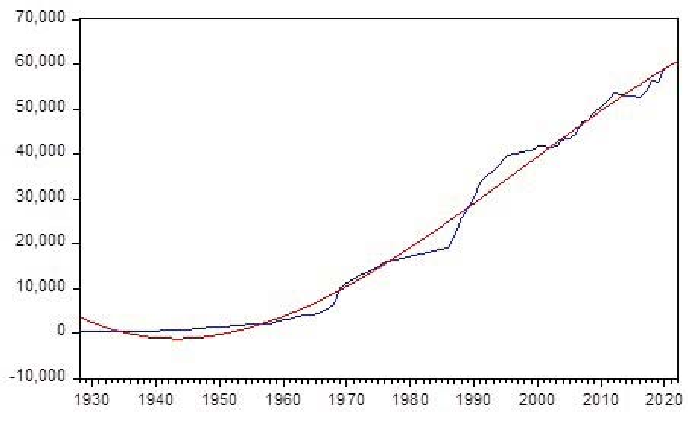

Figure 1: Estimation of the number of midwives in Turkey using a cubic regression model (*cf.* [7])

Model	RMSE	MAE	AIC	SIC
Simple Linear Regression Model	6028.26	5390.66	20.29	20.34
First-Difference Regression Model	3343.20	2597.44	16.33	16.39
Exponential Regression Model	17602.42	8818.57	0.96	1.01
Quadratic Regression Model	2814.26	2096.61	18.79	18.87
Logistic Regression Model	17582.12	8811.10	0.96	1.01
Cubic Regression Model	2173.51	1714.69	18.29	18.40
Logarithmic Regression Model	13089.61	11660.19	21.84	21.89

Table 1: Time series regression models (*cf.* [7])

In model comparisons, while simple linear regression and quadratic regression models are explanatory to a certain extent, it has been determined that the most appropriate model in terms of error terms is cubic regression. Exponential and logistic regression models have been found to be unsuitable for the data set. This situation once again highlights that model selection in time series should be based not only on theoretical but also on empirical criteria.

Variable	Coefficient	Standard Error	t-Statistic	Prob.
a	3686.340	885.619	4.162	0.0001
b	-658.122	83.818	-7.852	0.0000
c	24.515	2.124	11.544	0.0000
d	-0.118			
R^2	0.988			
AIC	18.292			
SIC	18.401			
HQ	18.336			
F-statistic	2399.701	Durbin-Watson statistic		0.144
Prob (F-statistic)	0.000000			

Table 2: Cubic regression model estimate for the number of midwives in Turkey (*cf.* [7])

Table 2 shows the estimated values for the cubic regression model. According to these estimated values, a= 3686.340, b= -658.122, c= 24.515, and d= -0.118 were found, and these coefficients were found to be statistically significant at the .01 level since the probability values were p $<$ 0.01. Table 3 shows the estimated values obtained for the cubic regression model.

2021	59834.8469587145
2022	59834.8469587145

Table 3: Predicted values for the cubic regression model (*cf.* [7])

Conclusion

This study involved performing a time series regression analysis using data on the number of midwives in Turkey between 1928 and 2020. Data were obtained from the Turkish Statistical Institute (TUIK) and seven different regression models were applied. The performance of these models was then evaluated using statistical measures such as root mean square error (RMSE), mean absolute error (MAE), the Akaike information criterion (AIC) and the Schwarz information criterion (SIC). Comparisons revealed that the cubic regression model best represented the data. This model was found to be statistically significant, and the number of midwives in 2021 and 2022 was estimated. These results suggest that the upward trend in the number of midwives in Turkey is continuing.

In conclusion, time series regression analysis is an effective method for forecasting health data. Long-term analysis of midwife numbers in Turkey provides a scientific basis for human resource planning in the health sector. This study demonstrates the power and versatility of time series models in the social and natural sciences.

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Remarks on the family of antisymmetric T_0-quasi-metric functions

Filiz Yıldız

In the previous studies [1], it had been shown that the function A_X described via the difference values of two quasi-pseudo-metrics is a T_0-quasi-metric on the family of all quasi-pseudo-metrics defined on a set X. Thus, the asymmetry value of a T_0-quasi-metric d was introduced with the notation $A_d = A_X(d, d^{-1})$ as a new approach to the asymmetry degree of d.

On the other hand, the notion of antisymmetric T_0-quasi-metric function which is some sense opposite to the metric function appeared in [2] and some various aspects of antisymmetric functions are discussed in [3].

Hence, it would be natural to ask whether the family of antisymmetric T_0-quasi-metrics may be closed, dense and complete according to the symmetrization metric A_X^s of A_X, in the T_0-quasi-metric space of all quasi-pseudo-metrics over X. Following that in addition to proving A_X is not an antisymmetric T_0-quasi-metric function, some other asymmetric and topological properties of it will be observed in this study.

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Some integrability properties of modified Martínez Alonso–Shabat equation

Hynek Baran

We study the modified Martínez Alonso–Shabat equation

$$u_y u_{xz} + \alpha u_x u_{ty} - (u_z + \alpha u_t) u_{xy} = 0$$

and present its recursion operator and an infinite commuting hierarchy of full-fledged nonlocal symmetries. To date, such hierarchies have been found only for very few integrable systems in more than three independent variables.

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Some remarks on cancellation law in semigroups of convex sets

Hubert Przybycień

Cancellation law is a main property in a structures like semigroups, abstract cones, rings etc. which allow to embeded it naturally into some richer structures like groups, vector spaces fields etc.. In this context a canellation law was considered by many authors especially for semigroups of convex sets ([2] , [3] , [4], [5] , [6] , [7] [8], [9] , [10]). In this paper we present discuss and also give a some new remarks concerning the cancellation law in a some algebraical structures.

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Keywords: Cancellation law , semigroups, convex sets

Introduction

Cancellation law plays an important role in many areas of mathematics and was studied by many authors ([2] , [3] , [4], [5] , [6] , [7] [8], [9] , [10]).. Namely if we had a commutative semigroup with one binary operation say $(S, +)$ and not every element in S is invertible then the equation $x+b = b+c$ with the unknown x may posses more that one solution but if the semigroup satisies the cancellation law then it guarantees that the equation x + b = b +c has exactly one solution x = c. This observation is crucial and allows us to introduce an equivalence relation in the cartesian product $S \times S$ in the following way

$$(a, b) \sim (c, d) \leftrightarrow a + d = b + c.$$

The quotient set $S \times S/_{\sim}$ with a suitable defined operation has a structure of group. Moreover the semigroup S can be embedded into ithe quotitnt group. This kind of constructions of embedding of a structure into a richer structure was used by many authors in many areas of mathematics even classical construction of the set of integers is made in this way. In this article we present discuss and also give a some new remarks concerning the cancellation law.

Cancellation law in noncommutative case

Let $(S, +)$ be a semigroup i.e. a nonempty set S with a binary operation $+ : S \times S$ which is a associative.
We say that S satisfies a *left cancellation law* if for any $a, b, c \in S$ the equality $b + a = b + c$ implies $a = c$.
Analogously w can define property of satisfaying right cancellation law by a semigroup $(S, +)$.

Example 1. *A set $I(\mathbb{N})$ of all injections $f : \mathbb{N} \to \mathbb{N}$ with the binary operation $\circ$ defined as superposition of functions is a semigroup satisfies the left cancellation law but this semigroup does not satisfy the right calcellation law.*
Indeed if for some $f, g, h \in I(\mathbb{N})$ we have

$$f \circ g = f \circ h$$

then by definition of superposition we have

$$f(g(n)) = f(h(n))$$

for all $n \in \mathbb{N}$ and by the injectivity of f w obtain that $h = g$.
This semigroup does not satisfies the right cancellation law since for

$$g(n) = \begin{cases} 2n \text{ if } n \text{ is even} \\ n+2 \text{ if } n \text{ is odd} \end{cases}$$

$$h(n) = \begin{cases} 2n \text{ if } n \text{ is even} \\ n^3 \text{ if } n \text{ is odd} \end{cases}$$

and

$$f(n) = 2n$$

we have

$$g \circ f = h \circ f$$

but $h \neq g$.

If $(S, +)$ is a semigroup which satisfies a cancellation law then we can introduce an relation R in $S \times S$ as follow:

$$aRb \Leftrightarrow a + b = b + a$$

Obviously the ralation is reflexive and symmetric,
To see its transivity let us assume that for some $a, b, c \in S$ we have

$$aRb \wedge bRc$$

then

$$a + b = b + a \wedge b + c = c + b$$

hence

$$a + b + c + b = b + a + b + c$$

now using the equalities $b + c = c + b$ and $b + a = a + b$ we get

$$a + c + b + b = a + b + b + c$$

thus by left cancellation law

$$c + b + b = b + b + c$$

using the equlity $c + b = b + c$ we obtain

$$b + c + b = b + b + c$$

and thus by left cancellation law

$$c + b = b + c$$

and therefore aRb.
Let $(S,+)$ be a semigroup then another relation R_1 may be introduced in S in the following way :

$$aR_1b \Leftrightarrow c+a=c+b \text{ for some } c \in S \text{ such that } cRa \text{ and } cRb$$

It is easy to see that the relation is reflexive and symmetric. To prove the the transivity let

$$aR_1b \wedge bR_1d$$

hence

$$c_1+a=c_1+b$$

$$c_2+b=c_2+d$$

for some $c_1, c_2 \in S$ such that

$$c_1Ra \wedge c_1Rb \wedge c_2Rb \wedge c_2Rd$$

If we additinally assume that the relation R is transitive then it follows that

$$x+y=y+x$$

for $x, y \in \{a, b, d, c_1, c_2\}$ hence by adding the equalities

$$c_1+a=c_1+b$$

$$c_2+b=c_2+d$$

we get

$$c_2+b+c_1+a=c_2+d+c_1+b$$

thus

$$(c_2+b+c_1)+a=(c_2+b+c_1)+d$$

and

$$(c_2+b+c_1)+a=a+(c_2+b+c_1) \wedge (c_2+b+c_1)+d=(d+c_2+b)+c_1.$$

So if we denote $s=c_2+b+c_1$ then we have that

$$s+a=s+d$$

and

$$sRa \wedge sRb$$

therefore aR_1d therefore the relation R_1 is transitive.
Let us denote by S/R_1 the quotient set on which divide S the equivalence reelation R_1 and let us denore by

$$[a]_{R_1}=\{x \in S : aR_1x\}$$

if we define

$$[a]_{R_1} \oplus [b]_{R_1}=[a+b]_{R_1}$$

then for any $a_1 \in [a]_{R_1}$ and $b_1 \in [b]_{R_1}$ we have

$$c+a_1=c+a$$

where cRa_1 and cRa and

$$c_1+b_1=c_1+b$$

where c_1Rb_1 and c_1Rb.
So if we additionaly assume thar $a, b \in S$ commute then the binary operation $\oplus$ is well defined for elements $[a]_{R_1}, [b]_{R_1} \in S/R_1$. Moreover if $a, c \in S$ commute then

$$[a]_{R_1} \subset [a+c]_{R_1}$$

and if additionaly $a, b, c \in S$ commute and

$$[a+c]_{R_1} \subset [b+c]_{R_1}$$

then for some $c_1, c_2 \in S$ such that aRc_1, aRx, bRx, c_2Rb, we have

$$a + c + c_1 = x + c_1$$

$$x + c_2 = b + c + c_2$$

by adding the last two equalities we obtain

$$a + c + c_1 + c_2 = x + c_1 + c_2 = b + c + c_1 + c_2$$

and

$$a + c + c_1 + c_2 = c + c_1 + c_2 + a$$

and

$$a + c + c_1 + c_2 = c + c_1 + c_2 + a$$

hence

$$aR_1b$$

and therefore

$$[a]_{R_1} = [b]_{R_1}.$$

Commutative semigroups

Now we assume that the semigroup $(S, +)$ is a commutative semigroup i.e

$$a + b = b + a$$

for all $a, b \in S$.

In this case the quotient set $S/_{R_1}$ with operation $\oplus$ becomes a semigroup with cancellation law i.e

$$[a]_{R_1} \oplus [c]_{R_1} = [b]_{R_1} \oplus [c]_{R_1}$$

implies

$$[a]_{R_1} = [b]_{R_1}$$

for $[a]_{R_1}, [b]_{R_1}, [c]_{R_1} \in S/_{R_1}$

Example 2. *Let $\mathcal{I}(\mathbb{R})$ be the set of all bounded intervals of the real line with addition defined by*

$$A + B = \{x + y : x \in A \wedge y \in B\}$$

for $A, B \in \mathcal{I}(\mathbb{R})$. Then obviously for $a, b \in \mathbb{R}$, $a < b$ we have

$$[a, b] + (0, 1) = (a, b + 1) = [a, b) + (0, 1) = (a, b] + (0, 1) = (a, b) + (0, 1)$$

hence

$$[a, b]R_1[a, b)R_1(a, b]R_1(a, b).$$

An easy observarion shows that if $a < b$ then

$$[(a,b)]_{R_1} = \{[a,b],[a,b),(a,b],(a,b)\}$$

and

$$[[a,a]]_{R_1} = \{\{a\}\}$$

and the quotient semigroup $\mathcal{I}(\mathbb{R})_{R_1}$ may be consider as a semigroup of intervals with forgoten ends which with the addition $\oplus$ becomes a commutative semigroup with cancellation law while the semigroup $\mathcal{I}(\mathbb{R})$ with addition $+$ ia a commutative semigroup which does not satisfy a cancellation law.

If $(S,+)$ is a commutative semigroup which satisfies the cancellation law then we can introduce an relation $\sim$ in the cartesian product $S^2 = S \times S$ as follow

$$(a,b) \sim (c,d) \Leftrightarrow a + d = b + c.$$

Obviously the relation $\sim$ is reflexive and symmetric. To see that it is also transitive let $(a,b),(c,d),(e,f) \in S^2$ be such that $(a,b) \sim (c,d)$ and $(c,d) \sim (e,f)$ then

$$a + d = b + c$$

$$c + f = d + e$$

hence

$$f + c + a + d = e + d + b + c$$

thus

$$f + a + c + d = e + b + c + d$$

and by cancellation law we obtain that

$$a + f = b + e$$

which shows that $\sim$ is transitive.
If in the quotient set $S^2/_\sim$ we define an addition $+$ as follow:

$$[(a,b)]_\sim + [(c,d)]_\sim = [(a+c,b+d)]_\sim$$

It is easy to see that if $(a_1,b_1) \sim (a,b)$ and $(c_1,d_1) \sim (c,d)$ then

$$(a_1 + c_1, b_1 + d_1) \sim (a + c, b + d)$$

since

$$a_1 + c_1 + b + d = b_1 + a + c_1 + d = b_1 + a + d_1 + c = a + c + b_1 + d_1$$

therefore the addition in $S^2/_\sim$ is well defined.
Moreover for any $a,b,c,x,y \in S$ the relation

$$(a + c, b + c) \sim (x,y)$$

is equivalent to the relation

$$(a,b) \sim (x,y)$$

thus for any $a,b,c \in S$ we have

$$[(a,b)]_\sim + [(c,c)]_\sim = [(a,b)]_\sim$$

and the addition in the quotient set posses a neutral element (zero) which is the class $0_\sim = [(c,c)]_\sim$.
Additionally for any $[(a,b)]_\sim \in S^2/_\sim$ we have

$$[(a,b)]_\sim + [(b,a)_\sim = 0_\sim$$

From the above properties it follows the following theorem:

Theorem 3. *Let S be a commutative semigroup and let the relation $\sim$ be defined as follows*

$$(a,b) \sim (c,d) \Leftrightarrow a+d = b+c$$

for $(a,b),(c,d) \in S^2$. Then the quotient set $S^2/_\sim$ with addition defined by

$$[(a,b)]_\sim + [(c,d)]_\sim = [(a+c, b+d)]_\sim$$

becomes a commutative group.
Moreover if there exists an element $c \in S$ such that

$$c + c = c$$

then the function $i : S \to S^2/_\sim$ defined by

$$i(a) = [(a,c)]_\sim$$

is an injection which satisfies the equality

$$i(a+b) = i(a) + i(b)$$

for all $a, b \in S$.

Example 4. *Let $S = \mathbb{N} \cup \{0\} = \{0, 1, ...\}$ be the set of natural numbers with usual addition.*
Then the set $S^2/_\sim$ can be interpreted as a set of integers. More precisely an integer $x \in \mathbb{Z}$ is represented by a class

$$[(x,0)]_\sim$$

when x is positive and

$$[(0,|x|)]_\sim$$

when x is not positive.

The above theorem shows that the cancellation law is essential in the possibility of embedding a semigroup into a group. In the next part we concentrare our attention on the conditions that guarantee the cancellation law to be satisfied. Some semigroups in natural way can be equiped with some richer structure. Here we discuss some variants or such situation.

Definition 5. *Let S be a commutative semigroup. We say that an element $a \in S$ is divisible by 2 if there exists exactly one $x \in S$ such that $x + x = a$. Such an element x will be denoted by $\frac{1}{2}a$.*

Obiviously if every element of a semigroup S is divisible by 2 then we can define in a natural way a multiplication of elements of a semigroup S by the *positive dyadic numbers.* More precisely if

$$D_2^+ = \left\{ \frac{m}{2^k} : m \in \mathbb{N}, k \in \mathbb{Z} \right\}$$

is the set of positive dyadic numbers and $\frac{m}{2^k}$ and $a \in S$ then by setting

$$a_1 = \frac{1}{2}a$$

and

$$a_j = \frac{1}{2}a_{j-1}$$

for $j > 1$. We can define an element

$$\frac{1}{2^t}a$$

as

$$a_t$$

for every natural number t. Hence we can define an element

$$\frac{m}{2^k}a$$

as a

$$\underbrace{\frac{1}{2^k}a + ... + \frac{1}{2^k}a}_{m-\text{times}}$$

So if in a semigroup S every element is divisible by 2 then the semigroup has natural structure of multiplaying by dyadic numbers.
Analogously for any natural number l we can cosider the property of divisibilyty of an element a by a natural number l as a existence of an exactly one element $x \in S$ such that

$$a = \underbrace{x + ... + x}_{l-\text{times}}$$

If we denote this element by $\frac{1}{l}a$. and by

$$D_l^+ = \left\{\frac{m}{l^k} : m \in \mathbb{N}, k \in \mathbb{Z}\right\}$$

then in a similar way we can define a multiplication of elements of a semigroup S by the numbers from D_l.

Example 6. *Let S be a semigroup of a positive real numbers with the binary operation*

$$a + b = ab$$

where ab denotes the usual product or real numbers.
Then the element

$$\frac{1}{2}a$$

is eqal to the arythmetic square root of a usually denoted by

$$\sqrt{a}$$

and the element

$$\frac{m}{l}a$$

is equal to

$$\sqrt[l]{a^m}.$$

Example 7. *Let S be a commutative semigroup in which every element is divisible by l then the set $2^S \backslash \{\varnothing\}$ of all nonempty subsets of S with addition $+$ defined by*

$$A + B = \{x + y : x \in A \wedge y \in B\}$$

is a commutative semigroup in which not every element is divisible by 2,
For example if S is a semigroup from the Example 6 then there is no $A \in 2^S \backslash \{\varnothing\}$ such that

$$A + A = \{1, 4\}.$$

Example 8. *Let X be a vector space and $\mathcal{C}(X)$ be a family of nonepty convex subsets of X. Then $\mathcal{C}(X)$ with addition defined by*

$$A + B = \{x + y : x \in A \wedge y \in B\}$$

for $A, B \in \mathcal{C}(X)$ is a commutative semigroup in which every element is divisible by any natural number l. To see this let $A \in \mathcal{C}(X)$. We have

$$A = \underbrace{\frac{1}{l}A + ... + \frac{1}{l}A}_{l-times}$$

if for some $B \in \mathcal{C}(X)$ the equality

$$A = \underbrace{B + ... + B}_{l-times}$$

holds true then from the convexity of the set B we obtain that

$$A = lB$$

therefore $B = \frac{1}{l}A$, what shows the uniqueness of the set B. The semigroup does not satisfy the cancellation law since

$$X + A = A$$

for any $A \in \mathcal{C}(X)$.

The limit operator

Let S be a a commutative semigroup and let let $S^{\mathbb{N}}$ be a set of all countable sequences which terms belonging to S. Then $S^{\mathbb{N}}$ is also a commutative semigroup with addition defined by

$$(a_n) + (b_n) = (a_n + b_n)$$

for $(a_n), (b_n) \in S^{\mathbb{N}}$. Moreover if every element of S is divisible by natural number l then this property posses also all elements of $S^{\mathbb{N}}$.

Definition 9. *Let S be a commutative semigroup. An operator $L : Z \to S$ where$Z \subset S^{\mathbb{N}}$ is called a limit operator on S if it satisfies the following conditions:*

i) for any $(a_n), (b_n) \in Z$ we have

$$L((a_n) + (b_n)) = L((a_n)) + L((b_n)),$$

ii) if $a_n = a$ for all $n \in \mathbb{N}$ then $(a_n) \in Z$ and

$$L((a_n)) = a,$$

iii) if $L((a_n)) = a$ and n_k is increasing sequence of natural numbers then $(a_{n_k}) \in Z$ and

$$L((a_{n_k})) = a.$$

It is clear that if on the commutative semigroup S is defined a translation invariant metric $d : S \times S \to \mathbb{R}$ i.e

$$d(x + v, y + v) = d(x, y$$

then since

$$d(a_n + b_n, a + b) \leqslant d(a_n b_n, a + b_n) + d(a + b_n, a + b) =$$
$$= d(a_n, a) + d(b_n, b)$$

the usual limit defined by metric d detrmines an limit operator L_d defined on the subsemigroup of all convergent sequences with terms belonging to S.

Example 10. *Let S a set of all open intervals (a, ∞) where $a \in \mathbb{R}$ then S with addition defined by*

$$(a, \infty) + (b, \infty) = (a + b, \infty)$$

is a commutative semigroup. If Z denotes a subset of $S^{\mathbb{N}}$ consists of all seqences of the form

$$((a_n, \infty))$$

where (a_n) is a convergent sequence of real numbers. Then an operation $L : Z \to S$ defined by

$$L((a_n \infty)) = \left(\lim_{n\to\infty} a_n, \infty\right)$$

is a limit operator on S in the sense of definition 2.

Now we are going to prove some version of cancellation law for suitable elements of a semigroup. We start with some definitions.

Definition 11. *Let S be a commutative semigroup whose every element is divisible by 2. We say that an element $a \in S$ is dyadic convex if for every dyadic numbers $s, t \in D_2$ such that $s + t = 1$ we have*

$$sa + ta = a.$$

Observe that if S is a commutative semigroup whose every element is divisible by 2. Then for $a, b \in S$ we have

$$\frac{1}{2}(a + b) = \frac{1}{2}a + \frac{1}{2}b$$

and hence

$$\alpha(a + b) = \alpha a + \alpha b$$

for any dyadic number α.
Observe that from the above fact it follows that the subset S_{conv} of convex elements of an semigroup whose every element is divisible by 2, forms a subsemigroup of S.

Definition 12. *Let S be a commutative semigroup with zero whose every element is divisible by 2, an let us assume that there exists a limit operator L defined on S. We say that an element $a \in S$ is L-bounded if*

$$L((2^{-n}a)) = 0.$$

Since

$$L(s(a_n)) = sL((a_n))$$

for every $s \in D_2$ therefore the set S_{bd} of L-bounded elements of an commutative semigroup with zero whose every element is divisible by 2 and in which exists a limit operator L defined on it, forms itself a subsemigroup of S.
These fact implies that if S is an commutative semigroup with zero whose every element is divisible by 2 and in which exists a limit operator L defined on it then the subset S_{bdc} of all L- bounded and convex elements of S is a semigroup.
Now we prove the cancellation law for a bounded and convex elements of a suitable semigroup.
More precisely we prove the followng theorem 13 which will be called a cancellation law:

Theorem 13. *Let S be an commutative semigroup with zero whose every element is divisible by 2 and in which exists a limit operator L defined on it. Then the semigroup S_{bdc} of all L-bounded and convex elements of S satisfies the cancellation law. i.e for all $a, b, c \in S_{bdc}$ the equality*

$$a + b = b + c$$

implies that

$$a = c.$$

Proof. Let $a, b, c \in S$ and assume that

$$a + b = b + c$$

. Then

$$2a + b = a + a + b = a + b + c = b + c + c = b + 2c$$

and

$$4a + b = 2a + 2a + b = 2a + b + 2c = b + 2c + 2c = b + 4c$$

and similary for every natural number n we can prove that

$$2^n a + b = b + 2^n c$$

multplying the last equality by 2^{-n} we obtain

$$a + 2^{-n}b = c + 2^{-n}b.$$

Hence we have a following equality

$$(a + 2^{-n}b) = (c + 2^{-n}b),$$

therefore

$$L((a + 2^{-n}b)) = L((c + 2^{-n}b)).$$

But

$$L((a + 2^{-n}b)) = L((a)) + L((2^{-n}b)) = a + 0$$

and

$$L((c + 2^{-n}b)) = L((c)) + L((2^{-n}b)) = c + 0$$

therefore $a = c$ and the proof is complete.

Example 14. *Let X be a normed vector space and let $\mathcal{B}(X)$ be family of all nonempty bounded closed and convex subsets of X. If we define a binary operation $\dot{+} : \mathcal{B}(X) \times \mathcal{B}(X) \to \mathcal{B}(X)$ as*

$$A\dot{+}B = \overline{A + B}$$

where

$$A + B = \{x + y : x \in A \wedge y \in B\}$$

and

$$\overline{A + B}$$

denotes the topological closure of the set $A + B$. Then $(\mathcal{B}(X), \dot{+})$ is a commutative semigroup with $0 = 0_{\mathcal{B}(X)} = \{0_X\}$in which every element is divisible by 2.
Let $d : \mathcal{B}(X) \times \mathcal{B}(X) \to \mathcal{B}(X) \to \mathbb{R}$ be any translation invariant and positively dyadic homogoenus metric i.e

$$d(A\dot{+}C, B\dot{+}C) = d(A, B)$$

$$d(\alpha A, \alpha B) = \alpha d(A, B)$$

for $A, B, C \in \mathcal{B}(X)$ and $\alpha \in D_2$.
Let Z be the subset of $\mathcal{B}(X)^{\mathbb{N}}$ consist of all convergent sequances in the metric space $(\mathcal{B}(X), d)$ if we define an operator $L : Z \to \mathcal{B}(X)$ as

$$L((A_n)) = A_0$$

where A_0 is the limit of the sequence (A_n) in the metric space $(\mathcal{B}(X), d)$. Then a such defined limit operator L satisfy the conditions of definition 9. Moreover from the dyadic homogenity we have

$$d(0, 2^{-n}A) = 2^{-n}d(0, A)$$

for any $A \in \mathcal{B}(X)$. Therefore any element of $\mathcal{B}(X)$ is L–bounded and from the theorem 13 we obtain that the semigroup $(\mathcal{B}(X), \dot{+})$ satisfies the cancellation law.
Since the Hausdorff metric d_H defined as

$$d_H(A, B) = \inf\{t > 0 : A \subset B + tU \wedge B \subset A + tU\}$$

where U is the unit ball in the normed space X is well defined metric for $A, B \in \mathcal{B}(X)$ and it is a positively homogenous. Thus from theorem 13 we obtaina following property:
If A, B, C are closed bounded and convex sets and

$$\overline{A + C} = \overline{B + C}$$

then $A = C$.

Now from theorem 3 and theorem 13 we get the following:

Corollary 15. *Let S be an commutative semigroup with zero whose every element is divisible by 2 and in which exists a limit operator L defined on it. If S_{bdc} is a set of all L-bounded and convex elements of S and $S^2_{bdc}/_\sim$ is a set of all equivalence classes of a relation $\sim$ defined by*

$$(a, b) \sim (c, d) \Leftrightarrow a + d = b + c$$

for $(a, b), (c, d) \in S_{bdc} \times S_{bdc}$, then *i)* *the set* $S^2_{bdc}/_\sim$ *with addition defined by*

$$[(a,b)]_\sim + [(c,d)]_\sim = [(a+c, b+d)]_\sim$$

forms a commutative group,

ii) *the semigroup* S_{bdc} *can be embedded into a group* $S^2_{bdc}/_\sim$ *by an injective map* $i : S_{bdc} \to S^2_{bdc}/_\sim$ *defined by*

$$i(a) = [(a,0)]_\sim$$

for $a \in S_{bdc}$,

iii) *the function* i *satisfies the following equlity*

$$i(a+b) = i(a) + i(b)$$

for all $a, b \in S$.

Remark 16. *From collorary 15 and example 14 it follows that the semigroup* $(\mathcal{B}(X), \dot{+})$ *of all nonempty closed bounded and convex subsets of a normed space* X *can ebedded into a group which construction is described in theorem 3, this group is called the Minkowski Rådstroem-Hörmander group*

Abstract convex cones

Our consideration which we made before shows that a suitable semigroup which satisfy a cancellation law can be embeded into a group. Therefore it makes sense to consider some other algebraic structures and expect that it can be a "part" of some richer structure. This leads us to consider structures like *abstract convex cones.*

Definition 17. *We say that a system* $(V, +, \cdot)$ *is an abstract convex cone if* $+ : V \times V \to V$ *and* $\cdot : [0, \infty) \times V \to V$ *are two binary maps such that:*

a) $(V, +)$ *is a commutative semigroup*

b) $\alpha \cdot (a+b) = \alpha \cdot a + \alpha \cdot b$ *for all* $a, b \in V$ *and* $\alpha \in [0, \infty)$.

c) $1 \cdot a = a$ *for all* $a \in V$.

d) $\alpha \cdot (\beta \cdot a) = (\alpha\beta) \cdot a$ *for all all* $a \in V$ *and* $\alpha, \beta \in [0, \infty)$.

e) $(\alpha + \beta)a = \alpha a + \beta a$ *for* $a \in V$ *and* $\alpha, \beta \in [0, \infty)$.

If V is an abstract convex cone and there exists a a neutral element $0 \in V$ such that

$$0 + a = a$$

for all $a \in V$ then we will cal it an *abstract convex cone with zero.*
It is clear that if V is an abstract convex with cone with zero which satisfy the cancellation law then

$$0a = 0$$

since by $e)$ in definition 17 we have

$$0a = 0a + 0a$$

Now let V be an bstract convex with cone with zero which satisfy the cancellation law. Similary like for semigroups we can introduce an equivalence relation $\sim$ defined between elements of $V^2 = V \times V$ as follow

$$(a, b) \sim (c, d) \Leftrightarrow a + d = b + c$$

where $(a, b), (c, d) \in V^2$.
It can be shown that the quotient set

$$V^2/_\sim = \{[a, b]/_\sim : a, b \in V\}$$

where

$$[a, b]/_\sim = \{(c, d) \in V^2 : (c, d) \sim (a, b)\}$$

becomes a real vector space with addition defined by

$$[a, b]/_\sim \oplus [c, d]/_\sim = [a + c, b + d]/_\sim.$$

and scalar multiplication defined by

$$\lambda \odot [a, b]/_\sim = \begin{cases} [\lambda \cdot a, \lambda \cdot b]/_\sim \text{ if } \lambda \geqslant 0 \\ [(-\lambda) \cdot b, (-\lambda) \cdot a]/_\sim \text{ if } \lambda < 0. \end{cases}$$

and then we can prove the following embeding theorem

Theorem 18. *Let $(V, +, \cdot)$ be a an abstract convex cone with zero which satisfies the cancellation law then V can be isomorphically embedded into a vector space $(V^2/_\sim, \oplus, \odot)$ by the cannonical cone homomorphism $i : V \to V^2/_\sim$ defined by*

$$i(x) = [x, 0]/_\sim.$$

Example 19. *If we cosider a semigroup $(\mathcal{B}(X), \dot{+})$ of all nonempty bounded closed and convex subsets of X, which was defined in Example 14 then as was shown this semigroup satisfiy a cancellation law.*

If we additionaly define a multiplication by a nonegative real numbers by

$$\alpha \cdot A = \{\alpha \cdot a : a \in a\}$$

for $A \in \mathcal{B}(X)$ and $\alpha \in [0, \infty)$

Then the structure $(\mathcal{B}(X), \dot{+}, \cdot)$ is an abstract convex cone with zero which satisfy a cancellation law and by theorem 18 *it can be embedded isomorphically into a linear space $\tilde{X} = (\mathcal{B}(X)_\sim, \oplus, \odot)$.*

The space $\tilde{X}$ which is ddefined above is called a Minkowski- Rådstroem-Hörmander space over the space X.

Example 20. *The semigroup $(\mathcal{B}(X), \dot{+})$ of all nonempty bounded closed and convex subsets of X can be considered in more genaral case when X is a topological vector space.*
As was shown by R.Urbański in [9] the following order cancellation law holds true:
If X is a topological vector space and $A, B, C \subset X$ are nonempty sets suchh that B i bounded and C is closed and convex then the following implication holds true

$$A \dot{+} B \subset B \dot{+} C \Rightarrow A \subset C.$$

Obviously these form of ordered cancellation law implies a cancellation law in $\mathcal{B}(X)$ since the condition

$$A\dot{+}B = B\dot{+}C$$

implies that

$$A\dot{+}B \subset B\dot{+}C$$

and

$$B\dot{+}C \subset A\dot{+}B$$

Moreover if we introduce a inclusion in $\mathcal{B}(X)$ as an order relation then the system $(\mathcal{B}(X), \dot{+}, \subset)$ becomes a partialy ordered semigroup in which the order is compatibile with the addition i.e

$$A \subset B \Rightarrow A\dot{+}C \subset B\dot{+}C,$$

for $A, B, C \in \mathcal{B}(X)$. Moreover If we additionaly define a multiplication by a nonegative real numbers by

$$\alpha \cdot A = \{\alpha \cdot a : a \in a\}$$

for $A \in \mathcal{B}(X)$ and $\alpha \in [0, \infty)$
then the system $(\mathcal{B}(X), \dot{+}, \cdot, \subset)$ becomes a partially ordered convex cone with zero which satisfy the ordered cancellation law and the addition and scalar multiplication by nonegative reals are compatibile wit the order structure.

Thus from theorem 18 and the above consideration we have the following embedding theorem:

Theorem 21. *Let X be a real topological vector space. Then the partially ordered convex cone $(\mathcal{B}(X), \dot{+}, \cdot, \subset)$ can be isomorphically embedded (as a cone) into a vector lattice $(\tilde{X}, \oplus, \odot, \leqslant_*)$. Where*

i) $\tilde{X} = \{[A, B]/_\sim : A, B \in \mathcal{B}(X)\}$ *and the equivalence relation $\sim$ is defined on the set $\mathcal{B}(X) \times \mathcal{B}(X)$ by*

$$(A, B) \sim (C, D) \Leftrightarrow A\dot{+}D = B\dot{+}C,$$

ii)

$$[A, B]/_\sim, \oplus [C, D]/_\sim, = [A\dot{+}C, B\dot{+}D]/_\sim$$

for $[A, B]/_\sim, [C, D]/_\sim \in \tilde{X}$.

iii)

$$\lambda \odot [A, B]/_\sim = \begin{cases} [\lambda \cdot A, \lambda \cdot B]/_\sim \ \textit{if } \lambda \geqslant 0 \\ [(-\lambda) \cdot B, (-\lambda) \cdot A]/_\sim \ \textit{if } \lambda < 0, \end{cases}$$

for $\lambda \in \mathbb{R}, [A, B]/_\sim \tilde{X}$,

iv) The order $\leqslant_*$ is defined by $[A, B]/_\sim \leqslant_* [C, D]/_\sim \Leftrightarrow B\dot{+}C \subset A\dot{+}D$.

Moreover homomorphism $i : \mathcal{B}(X) \to \tilde{X}$ defined by

$$i(A) = [A, \{0\}]$$

preserves the order structures.

The space $\tilde{X}$ is called a Minkowski - Rådströem - Hörmander vector lattice over the vector space X.

Remark 22. *It is worth to noticed that if we have also a topological structure defined on the semigroup* $(\mathcal{B}(X), \dot{+})$ *or partially ordered semigroup*

$$(\mathcal{B}(X), \dot{+}, \subset)$$

or abstract convex cone

$$(\mathcal{B}(X), \dot{+}, \cdot)$$

or partially ordered convex cone

$$(\mathcal{B}(X), \dot{+}, \cdot, \subset)$$

then under some conditions we can also embeded it continously and isomprphically respectively into a topological group, partially ordered topological group, topological vector space or topological vector lattice.

Some generalizations of cancellation law for a families of sets which inludes a class of convex sets

We'll show that assuming some additional condition the cancellation law cancellation law due in form proved by to R. Urbanski can be generalized into a wider class of subsets of a topological vector space. We'll mention it in two cases: the first case is concerned on k-convex sets and the second case is related to a class of p-convex sets.

Definition 23. *Let X be a topological vector space and let $A \subset X$ and $k \in \mathbb{R}_+$. We say that the set A is k- convex if*

$$\underbrace{A + ... + A}_{n-times} \subset knA$$

for every $n \in \mathbb{N}$.

Remark 24. *The class of sets which are k- convex for some $k \in \mathbb{R}_+$ are wider than the class of convex sets. More precisely every k-convex set is* 1*-convex since for a convex set A we have inclusion*

$$\underbrace{A + ... + A}_{n-times} \subset nA$$

for every number $n \in \mathbb{N}$.

From the other hand if X is a topological vector space and $A \subset X$ be a set such that $0 \in \mathrm{Int}(A)$ and the convex envelope of the set A is bounded then A is k-convex for some $k \in \mathbb{R}_+$.
Indeed. Since A is a nieighbouhood of zero and $\mathrm{conv}(A)$ is bounded hence

$$\mathrm{conv}(A) \subset kA$$

for some $k \in \mathbb{R}_+$.
Therefore for every $n \in \mathbb{N}$ we have

$$\frac{1}{n}(\underbrace{A + ... + A}_{n-\text{times}}) \subset \frac{1}{n}(\underbrace{\mathrm{conv}(A) + ... + \mathrm{conv}(A)}_{n-\text{times}}) = \mathrm{conv}(A) \subset kA.$$

Now we present a some generalization of cancellation law which was proved in [6]:

Theorem 25. *Let X be a topological vector space and let $A, B, C \subset X$ if the set B is bounded and the set C is k- convex then the following implication:*

$$A + B \subset B + C \Rightarrow A \subset k\overline{C}.$$

Remark 26. *Observe that since*

$$(-1,1) + (-1,1) \subset (-1,1) + \left(\left(-\frac{1}{2}, \frac{1}{2}\right) \cup \{-1,1\}\right)$$

thus w can not replace the k by 1.

Since convex sets are 1- convex and for closed set C we have $\overline{C} = C$ thefore from theorem 25 we obtain as a corollary a a following version of cancellation law:

If X is a topological vector space and $A, B, C \subset X$ are nonempty sets such that B i bounded and C is closed and convex then the following implication holds true

$$A + B \subset B + C \Rightarrow A \subset C.$$

which is due to R. Urbański.
It have to be noticed that this vesion of cancellation law is a start point to prove embedding theorem for the cone $\mathcal{B}(X)$.
The above result can be also generalized to family of p- convex sets where $0 < p \leqslant 1$. We start with the following definition

Definition 27. *Let A be a vector space and let $A \subset X$. We say that the set A is p-convex $0 < p \leqslant 1$ if for any positive real numbers s, t such that $s^p + t^p = 1$ and any elements x, y of the set A the element $sx + ty$ also belong to the set A.*

Example 28. *A classical example of p-convex sets ale unit balls in the Frechet spaces*

$$\ell^p = \left\{ (x_n) \in \mathbb{R}^{\mathbb{N}} : \sum_{j=1}^{\infty} |x_j|^p < \infty \right\}$$

with metric defined by

$$d_p((x_n),(y_n)) = \sum_{j=1}^{\infty} |x_j - y_j|^p$$

or

$$L^p(0,1) = \left\{ x \in \mathbb{R}^{(0,1)} : x \textit{ is Lebesgue measurable and } \int_{(0,1)} |x(t)|^p dx < \infty \right\}$$

with the metric defined by

$$d_p(x,y) = \int_{(0,1)} |x(t) - y(t)|^p dt.$$

Observe that the spaces ℓ^p and $L^((0,1))$ are not locally convex spaces and it can posses a some "pathological" properties such as no existence of convex set with nonempty topological interior or existence of bounded sets whose convex envelope is not bounded. So in these spaces from the topological or metric point of view is more suitable to consider $p-$ convex sets than convex sets.

As was shown by R. Urbański in the paper [10] the cancellation law in the classical form i.e.

$$A + B \subset B + C \Rightarrow A \subset C.$$

does not hold true for p convex set.

Since we have a following example:

Example 29. *Let*

$$B = \{(x,y) \in \mathbb{R}^2 : |x| + |y| \leqslant 1\}$$
$$A = \{(x,y) \in \mathbb{R}^2 : |x|^p + |y|^p \leqslant 1\}$$

then A and B are compact and p =convex sets for any $0 < p \leqslant 1$. Moreover

$$B + B \subset B + A$$

but obviously

$$B \not\subset A,$$

unless $p < 1$.

If we analyze the original proof of order cancellation law given by R. Urbański then obviously the following version of it can be deduced:
If X is a topological vector space and $A, B, C \subset X$ are nonempty sets such that B i bounded and C is closed and convex then the following implication holds true

$$A + B \subset B + C \Rightarrow A \subset \overline{\text{conv}}(C).$$

In the paper [4] there was considered a class of sets which satisfy a so called p- condition but from the below consideration it seemst that it can be generalized to some class of $p-$ convex sets. Before we formulate it we need to define some notions..

Definition 30. *Let X be a vector space and $A \subset X$. We say that the set A satisfies the p-condition if A is a p-covex set and $n^{-1+\frac{1}{p}} A \subset A$ for every natural number $n \in \mathbb{N}$.*

Remark 31. *In the paper [4] it was shown a cancellation law in the folllowing form: If X is a topological vector space and $A, B, C \subset X$ are nonempty sets such that B i bounded and C is closed and satisfies $p-$ condition then the following implication holds true*

$$A + B \subset B + C \Rightarrow A \subset C.$$

There was an open question to the author, of existence A a set whicch satisfy a p-condition and which is not convex? Now is known that the answer is no since following consideration can be done:
If the set satisfies the p- condition then for any $x, y \in C$ we have

$$2^{-1+\frac{1}{p}} x, 2^{-1+\frac{1}{p}} y \in C$$

and since C is p - convex we have

$$\frac{1}{2}x + \frac{1}{2}y = 2^{-\frac{1}{p}}(2^{-1+\frac{1}{p}} x) + 2^{-\frac{1}{p}}(2^{-1+\frac{1}{p}} x) \in C$$

thus since C is closed therefore C must be convex.
If we look at the proof of the cancellation law we can reformulate to the other form. Namely:
If X is a topological vector space and $A, B, C \subset X$ are nonempty sets such that B i bounded and C is closed and $p-$ convex then the following implication holds true

$$A + B \subset B + C \Rightarrow A \subset \overline{\bigcup_{n=1}^{\infty} n^{-1+\frac{1}{p}} C}.$$

Question. There is an open question of a existence of a semigroup which is a subset of the set of p– convex sets and which consists of not only convex sets and in which the cancellation holds true ?

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Local and global duality in Clifford algebras category

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In this work, using the conceptual framework of the local to global principle, we explore duality in graded algebras. Starting with the categorification of graded algebras, we review the tensor support theory associated with the spectrum of the category. Applying the local–global principle, the sub algebras, associated to principal ideals of the spectrum of the graded algebra, are used to review the corresponding global and local duality, in Clifford algebra category.

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Introduction

The aim of this work is to explore the duality concept in graded algebras from the local to global principle approach [5]. Following Lawere's work [11, Grassmann Dialectics and Category Theory], Mac Lane's 1963 book of Homology [13], and other authors [1, 15, 20], we started discussing about the process of categorification of a graded algebra.

The notion of localization in category theory is like an extension of the concept of a local ring. Thereby, it is possible to write a free module of a ring in terms of a direct sum of an index set and a subset of maximal ideals of the module. This set is one of the basis of the free module. [21, Wang and Kim, 2016]. Employing the tensor triangulated categories approach, we explored the local to global principle in a stable monoidal category model [18, 5, 3].

The support theory provides a geometric approach for the study of some algebraic structures. We considered its application to the study of triangulated categories for constructing a local cohomology of functors related to a central ring of operators [4]. The localization of a category in a class of morphism is a procedure that allows to invert all the morphisms [16, 2].

The support theory of triangulated categories is applied to a discrete triangulated category. Applied to a small tensor triangulated category (that is a monoidal category), by taking the module action [19] of a subcategory, permits to construct the analog of the local-global principle [5, 6, 4, Benson etal, 2011]. This process is applied to a categorified Clifford algebra.

Categorification of Clifford algebras

The most familiar Clifford Geometric Algebra is the Geometric Algebra (GA) of an n-dimensional Euclidean vector space. Geometric algebras are usually defined in terms of a quadratic form, or in terms of their isomorphic coordinate matrix algebra representations. They can also be defined as an ideal in the machinery of tensor algebras. Category theory, in turn, provides a powerful abstract framework that generalizes the concepts of vector category to tensor category through the idea of a monoidal category.

A universal arrow in the category of vector spaces over a field $\mathbf{Vct}_{\mathcal{F}}$, sends any object X in the **Set** category to the vector space category, $\mathbf{Vct}_{\mathcal{F}}$ like an object V_X with basis X.

In the $\mathbf{Vct}_{\mathcal{F}}$ category of vector spaces over a field, a contravariant functor $\bar{D}$, called a dual functor, is defined from $\mathbf{Vct}_{\mathcal{F}}$ to $\mathbf{Vct}_{\mathcal{F}}$. Let V, W be vectors over a field $\mathcal{F}$. The object function of $\bar{D}$ is $\bar{D} : V \to \bar{D}(V) = \mathbf{Vct}_{\mathcal{F}}(V, \mathcal{F})$ and the arrow function from $h : V \to W$ to $\bar{D}h : \bar{D}W \to \bar{D}V$ is $(\bar{D}h) : f = fh$, with $f : W \to \mathcal{F}$.

$$\begin{tikzcd} \mathcal{F} \arrow[d, equal] & V \arrow[l, "f"'] \arrow[d, "h:V\to W"'] \\ \mathcal{F} & W \arrow[l, "f"'] \end{tikzcd} \qquad \begin{tikzcd} \bar{D}V = \mathbf{Vct}_{\mathcal{F}}(V, \mathcal{F}) \arrow[r, "(\bar{D}h):f"'] & \mathcal{F} \arrow[d, equal] \\ \bar{D}W = \mathbf{Vct}_{\mathcal{F}}(W, \mathcal{F}) \arrow[u, "\bar{D}h:\bar{D}W\to\bar{D}V"'] \arrow[r, "fh"'] & \mathcal{F} \end{tikzcd} \tag{1}$$

For the contravariant functor, a bijection can be defined. Let $g : V \to \bar{D}W$ be function φ:

$$\varphi = \varphi_{V,W} : \mathbf{Vct}_{\mathcal{F}}(V, \mathbf{Vct}_{\mathcal{F}}(W, \mathcal{F})) \to \mathbf{Vct}_{\mathcal{F}}(W, \mathbf{Vct}_{\mathcal{F}}(V, \mathcal{F})) \tag{2}$$

such that, for all $v \in V$ and $w \in W$, $[(\varphi g)w]v = (gv)w$, φ is a bijection, since $\varphi_{W,V}\varphi_{V,W}$ is the identity.

This bijection can become an adjunction by defining two covariant functors between the $\mathbf{Vct}_{\mathcal{F}}$ category and its opposite $\mathbf{Vct}_{\mathcal{F}}^{op}$.

$$\begin{tikzcd} \mathbf{Vct}_{\mathcal{F}} \arrow[r, bend left, "D^{op}"] & \mathbf{Vct}_{\mathcal{F}}^{op} \arrow[l, bend left, "D"] \end{tikzcd}$$

with object functions

$$D : \mathrm{Obj}(\mathbf{Vct}_{\mathcal{F}}^{op}) \to \mathrm{Obj}(\mathbf{Vct}_{\mathcal{F}}), \quad D^{op} : \mathrm{Obj}(\mathbf{Vct}_{\mathcal{F}}) \to \mathrm{Obj}(\mathbf{Vct}_{\mathcal{F}}^{op})$$

and the arrows function between $h^{op} : W \to V$ and $h : V \to W$:

$$Dh^{op} = \bar{D}h : \bar{D}W \to \bar{D}V, \quad D^{op}h = (\bar{D}h)^{op} : \bar{D}V \to \bar{D}W.$$

Then, the bijection φ can be written as

$$\mathbf{Vct}_{\mathcal{F}}^{op}(D^{op}W, V) \cong \mathbf{Vct}_{\mathcal{F}}(W, DV) \tag{3}$$

Adjunction 3 is natural in V and W [10, Mac Lane, p. 88]. The unit of the adjunction is the map $\eta_W = \kappa_W : W \to DD^{op}W$, and the counit is an arrow $\epsilon_V : D^{op}DV \to V$ in $\mathbf{Vct}_{\mathcal{F}}^{op}$ which turns out to be $\epsilon_V = (\kappa_V)^{op}$ for the same arrow κ

Clifford Graded Modules

Given a $\mathcal{F}$-module Λ^m, if Λ^{m_1} and Λ^{m_2} are submodules of Λ^m, suppose that

$$\Lambda^{m_1} \oplus \Lambda^{m_2} = \Lambda^m$$

we say that Λ^{m_1} and Λ^{m_2} are complementary direct summands and the direct sum spams the module Λ^m

For complementary direct summands of a module, there is a split epimorphism. Let $f : \Lambda^m \to \Lambda^n$ and $f' : \Lambda^n \to \Lambda^m$ be a homomorphism such that:

$$ff' = 1_{\Lambda^n}$$

then f is an epimorphism, f' is a monomorphism, and

$$\Lambda^m = \mathrm{Ker}(f) \oplus \mathrm{Im}(f')$$

we say that f is a split epimorphism and we write:

$$\Lambda^m - \overset{f}{\oplus}\!\!\longrightarrow \Lambda^n \longrightarrow 0$$

f, is called idempotent, if $f^2 = f$. Note that $1 - f$ is an idempotent too. A sub module $\Lambda^k \leqslant \Lambda^m$ is a direct summand of Λ^m if and only if $\Lambda^k = \mathrm{Im}(e)$ for some idempotent endomorphisms e of Λ^m.

A finite group G, acting over V_X, a homomorphism $V_X \to G(\Lambda(V))$, gives a graded representation of a module of V called the space of multivectors. It is a direct sum of graded modules of a tensor algebra

$$\Lambda(V) = \bigoplus \Lambda(V)^n$$

If $\{e_1, e_2, \cdots, e_n\}$ is a basis of V, then a decomposible multivector of degree k can be written like a monomial:

$$X = e_1 \wedge e_2 \wedge \cdots \wedge e_k$$

a complimentary base [7, 14, 22] for V can be written by:

$$\begin{aligned} X^* &= \{e_1^*, e_2^*, \cdots e_n^*\} \quad \text{with} \\ e_i^* &= e_1 \wedge e_2 \wedge \cdots \wedge e_n \quad \text{(no vector } e_i \text{ in the product)} \end{aligned}$$

The base X^* spans the dual space V^*. Then V is self dual.

A graded sub algebra is a graded module of the graded algebra with the same identity and closure in the product. The quotient algebra of a graded algebra Λ and a graded two sided ideal J generate a sub graded algebra with a product determined with the projection:

$$\Lambda \to \Lambda/J$$

Clifford Algebra in the Space of Multivectors

A Clifford algebra is constructed in the space of multivectors, like a universal object of quadratic algebras $\mathcal{C}l(Q)$, a graded associative algebra over $\mathcal{F}$ generated by a quadratic module, a vector space V [9, 8, 12, 17, Helmstetter and Micali, 2008, Lounesto, 2001, Porteus, 2009], and a Clifford map $V \to \mathcal{C}l(Q)$ such that, to a graded algebra map $V \to \Lambda(V)$, $v \mapsto \phi_v$, with

$$(\phi_v)^2 = Q(v) \cdot 1_{\Lambda(V)} \quad \text{for all } v \in V$$

corresponds an algebra homomorphism $\mathcal{C}l(Q) \to \Lambda(V)$

$$\begin{tikzcd} V \arrow[r, "\gamma"] \arrow[dr, "\phi"'] & \mathcal{C}l(Q) \arrow[d, "\psi"] \\ & \Lambda(V) \end{tikzcd}$$

The Clifford graded algebra has, together with the quadratic form Q, two homomorphisms of degree 0:

$$\pi = \pi_\Lambda : \Lambda \otimes \Lambda \to \Lambda$$
$$I = I_\Lambda : \mathcal{F} \to \Lambda$$

The $\mathcal{F}$-algebra Λ, together with homomorphisms π and I, form a monad i. e. a triplet $< \Lambda, \otimes, \mathcal{F} >$

- Λ is a category,
- equipped with a product $\otimes : \Lambda \times \Lambda \to \Lambda$ that is associative:

$$\otimes(1 \times \otimes) = \otimes(\otimes \times 1) : \Lambda \times \Lambda \times \Lambda \to \Lambda$$

- and it has an object $\mathcal{F}$ that is a left and right unit for the product:

$$\otimes(\mathcal{F} \times 1) = id_\Lambda = \otimes(1 \times \mathcal{F})$$

Clifford Product

The Clifford algebra is a vector space with a bilinear map:

$$[,]_{\mathcal{C}l} : (V \otimes V) \to V$$

called Clifford product. The Clifford product has a skewsymmetric part, called the exterior product, and it is written $u \wedge v$, that is a quotient module of the tensor in the graded algebra Λ

$$\Lambda^2(V) = V \otimes V / I_2$$

I_2 is the space spanned by $\{u \otimes v + v \otimes u | u, v \in V\}$

and a symmetric part, written $u \lrcorner v$, that is a quotient module of the tensor product in a graded algebra Λ

$$Sym^2(V) = V \otimes V / J_2$$

J_2 is the space spanned by $\{u \otimes v - v \otimes u | u, v \in V\}$.

The product $u \lrcorner v$ is called the left contraction and it is the dual of the exterior product, because of the symmetric bilinear form:

$$< u, v > = \frac{1}{2}[Q(u+v) - Q(u) - Q(v)]$$

$$\langle u \lrcorner v, w \rangle = \langle v, u \wedge w \rangle \tag{4}$$

For two elements $a, b \in V$, the Clifford product is:

$$[a, b]_{\mathcal{C}l} = ab = a \lrcorner b + a \wedge b.$$

Duality in the Clifford Monad

The monad has a bifunctor:

$$\_ \otimes \_ : \Lambda \times \Lambda \to \Lambda \otimes \Lambda$$

a covariant functor and a contravariant functor acting on the module category over Λ:

$$\mathrm{Hom}_\Lambda(\_, \Lambda^n) \quad \text{and} \quad \mathrm{Hom}_\Lambda(\Lambda^m, \_) \quad \text{with} \quad \Lambda^m, \Lambda^n \in \mathrm{Obj}_\Lambda$$

and a tensor homset adjunction:

$$\mathrm{Hom}_\Lambda(\Lambda^p \otimes \Lambda^m, \Lambda^n) \cong \mathrm{Hom}_\Lambda(\Lambda^p, \mathrm{Hom}_\Lambda(\Lambda^m, \Lambda^n))$$

with the corresponding unit and counit:

$$\eta_{\Lambda^m,\Lambda^n} : \Lambda^n \to \mathrm{Hom}_\Lambda(\Lambda^n, \Lambda^m \otimes \Lambda^n) \quad \epsilon_{\Lambda^m,\Lambda^n} : \mathrm{Hom}\,\Lambda(\Lambda^m, \Lambda^n) \otimes \Lambda^m \to \Lambda^n.$$

The dual for an object $\Lambda^m \in \mathrm{Hom}_\Lambda$ is:

$$(\Lambda^m)^* = \mathrm{Hom}_\Lambda(\Lambda^m, 1),$$

that is associated to an evaluation map:

$$(\Lambda^m)^* \otimes \Lambda^n \to \mathrm{Hom}_\Lambda(\Lambda^m, \Lambda^n)$$

In case that the evaluation map is an isomorphism for all $\Lambda^m, \Lambda^n \in \mathrm{Hom}_\Lambda$, we say that Λ is rigid.

Duality in a Clifford Module

For a graded module the adjunction associated to the monad, have two covariant functors (with the same object function) :

$$\Lambda^*(\iota_{e_n^*}\Lambda^p, \Lambda^m) \cong \Lambda(\Lambda^p, \epsilon_{e_n}\Lambda^m)$$

e_n is an idempotent, a principal ideal that is an element of the base X of the vector space V, $\iota_{e_n^*}$ is the covariant opposite functor of the covariant functor ϵ_{e_n}

$$\iota_{e_n^*} = (\epsilon_{e_n})^{op}$$

The functor ϵ_{e_n} is characterized by the inclusion map of Λ^m in Λ^p and the $\iota_{e_n^*}$ functor by the projection map of Λ^p in Λ^m

Then we can define a creation operator acting on a multivector $\overset{m}{Y} \in \Lambda^m$:

$$\epsilon_{e_n} \overset{m}{Y} \equiv e_n \wedge \overset{m}{Y}$$

and an annihilation operator:

$$\iota_{e_n^*} \overset{p}{Z} \equiv e_n^* \lrcorner \overset{p}{Z}$$

The action of the operator $\iota_{e_n^*}$ is the dual of the action of ϵ_{e_n}, they have the properties:

$$\iota_{e_m^*}\epsilon_{e_n} = \delta_{mn}$$
$$\iota_{e_1^*}\epsilon_{e_1} + \iota_{e_2^*}\epsilon_{e_2} + \cdots + \iota_{e_N^*}\epsilon_{e_N} = 1$$

with N the dimension of V.

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Asymptotically sharp nonlinear Hausdorff–Young inequalities in the discrete SU(1,1) setting

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We study nonlinear variants of the classical Fourier transform taking values in the group SU(1,1). In particular, we study discrete models of the nonlinear Fourier transform and the associated Hausdorff–Young type inequalities, with emphasis on the behavior of sharp constants. Our results highlight cases where improved bounds can be obtained, especially for sufficiently small sequences or for sequences far from extremizers of the linear theory.

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KEYWORDS: Fourier analysis, nonlinear Fourier transform, sharp constant, trigonometric polynomial

Introduction

This paper studies a *discrete model of the nonlinear Fourier transform (NFT)*, introduced by Tao and Thiele in [15]. The discrete NFT connects a finitely supported sequence of complex numbers $F = (F_n)_{n\in\mathbb{Z}}$ in the unit disk, with only finitely many F_n being nonzero, with SU(1,1)-valued trigonometric products.

The motivation comes from the *linear Fourier transform*, where inequalities such as Parseval's identity and the Hausdorff–Young inequality play a fundamental role. In many applications in analysis and physics, nonlinear analogues of the Fourier transform appear naturally, for instance in scattering theory. A central question is how much of the classical Fourier analysis carries over to these nonlinear settings, and whether sharp constants analogous to the linear case can still be obtained.

We start by defining the coefficients

$$A_n := \frac{1}{(1-|F_n|^2)^{1/2}}, \qquad B_n := \frac{F_n}{(1-|F_n|^2)^{1/2}},$$

which satisfy $A_n^2 - |B_n|^2 = 1$. For each $t \in \mathbb{T} = \mathbb{R}/\mathbb{Z} \equiv [0,1\rangle$, one constructs the SU(1, 1)-*valued trigonometric (or Fourier) product with coefficients* F

$$\begin{bmatrix} a(t) & b(t) \\ \overline{b(t)} & \overline{a(t)} \end{bmatrix} = \prod_{n\in\mathbb{Z}} \begin{bmatrix} A_n & B_n e^{2\pi i n t} \\ \overline{B_n} e^{-2\pi i n t} & A_n \end{bmatrix},$$

which lies in the matrix group SU(1,1). This assignment $F \mapsto (a,b)$ is called the *(discrete) nonlinear Fourier transform*. For small sequences F, the function $b(t)$ is close to the linear Fourier transform

$$\hat{F}(t) = \sum_{n\in\mathbb{Z}} F_n e^{2\pi i n t}.$$

Within this framework, two fundamental inequalities are well established and serve as key tools. The first is the *nonlinear Parseval identity*:

$$\big\|(\log|a(t)|^2)^{1/2}\big\|_{\mathrm{L}^2_t(\mathbb{T})} = \big\|(\log|A_n|^2)^{1/2}\big\|_{\ell^2_n(\mathbb{Z})}. \tag{1}$$

which serves as a nonlinear analogue of the classical Parseval formula.

The second is the *nonlinear Hausdorff–Young inequality*:

$$\big\|(\log|a(t)|^2)^{1/2}\big\|_{L^q_t(\mathbb{T})} \leqslant C_p \big\|(\log|A_n|^2)^{1/2}\big\|_{\ell^p_n(\mathbb{Z})}, \tag{2}$$

$1 \leqslant p < 2$, $\frac{1}{p} + \frac{1}{q} = 1$. A central open problem is whether the constants C_p can be chosen uniformly in p, and whether the *sharp version* with constant 1 actually holds.

Main results

The main contributions of this work are presented in the following two theorems (see [8]).

Theorem 1. *If a sequence F satisfies $\|F\|_{\ell^1(\mathbb{Z})} \leqslant \frac{1}{2}$, then*

$$\big\|(\log|a(t)|^2)^{1/2}\big\|_{L^q_t(\mathbb{T})} \leqslant \big(1 + 3\|F\|_{\ell^1(\mathbb{Z})}\big) \big\|(\log|A_n|^2)^{1/2}\big\|_{\ell^p_n(\mathbb{Z})}.$$

This inequality is *asymptotically sharp* as $\|F\|_{\ell^1(\mathbb{Z})} \to 0$, since the multiplicative factor approaches 1.

Theorem 2. *There exist $\alpha, \delta > 0$, depending on $1 < p < 2$, such that if a sequence of coefficients F is not identically zero and satisfies*

$$\|F\|_{\ell^1(\mathbb{Z})} \leqslant \delta \left(1 - \frac{\|F\|_{\ell^\infty(\mathbb{Z})}}{\|F\|_{\ell^p(\mathbb{Z})}}\right)^{\alpha}, \tag{3}$$

then the associated $SU(1,1)$-valued trigonometric product satisfies inequality (2) with sharp constant 1.

Sequences that satisfy (3) are considered far from linear Hausdorff–Young extremizers (with respect to their ℓ^1-norm).

Theorem 1 shows that the nonlinear inequality behaves optimally for small sequences, while Theorem 2 establishes sharpness for sequences far from extremizers. These results provide some indication that the conjectured sharp inequality might extend more generally. Details of the proofs can be found in [8]. The analysis is also connected to work in the continuous setting (see [7]), particularly the Dirac scattering transform, although the discrete case involves additional technical challenges.

Conclusion

This paper provides partial progress toward the conjectured sharp nonlinear Hausdorff–Young inequality in the discrete SU(1,1) framework. Two situations are considered where asymptotically sharp or sharp inequalities can be verified: for small sequences and for sequences far from extremizers in the linear theory. These findings contribute to a better understanding of nonlinear Fourier analysis in the discrete case.

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Hardy type inequalities and Abel-Gontscharoff's interpolating polynomial

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As is well known, ever since its publication in 1925 by G. Hardy, the inequality that now bears his name has been studied, generalized, and extended by many mathematicians. In this talk Hardy-type inequalities are extended to higher-order convex functions. Namely, as we know, under certain conditions, a function can be approximated using Abel–Gontscharoff polynomials. By means of this interpolation, we first present the difference that arises from a Hardy-type inequality involving the general Hardy operator and determined its lower bound. Then, using Čebyšev inequality and the general Grüss inequality, we calculated the upper bound of the observed difference.

These results about Hardy inequality and Abel–Gontscharoff's interpolating polynomial are published in [2].

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Introduction

G. H. Hardy [3] stated and proved that the inequality

$$\int_0^\infty \left(\frac{1}{x} \int_0^x f(t)\, dt \right)^p dx \leqslant \left(\frac{p}{p-1} \right)^p \int_0^\infty f^p(x)\, dx,\ p > 1, \tag{1}$$

holds for all f non-negative functions such that $f \in L^p(\mathbb{R}_+)$ and $\mathbb{R}_+ = (0, \infty)$. The constant $\left(\frac{p}{p-1}\right)^p$ is sharp.

Let $(\Sigma_1, \Omega_1, \mu_1)$ and $(\Sigma_2, \Omega_2, \mu_2)$ be measure spaces with positive σ-finite measures. Let $U(f, k)$ denote the class of functions $g : \Omega_1 \to \mathbb{R}$ with the representation

$$g(x) = \int_{\Omega_2} k(x, t) f(t) d\mu_2(t),$$

and A_k be an integral operator defined by

$$A_k f(x) := \frac{g(x)}{K(x)} = \frac{1}{K(x)} \int_{\Omega_2} k(x, t) f(t) d\mu_2(t), \tag{2}$$

where $k : \Omega_1 \times \Omega_2 \to \mathbb{R}$ is measurable and non-negative kernel, $f : \Omega_2 \to \mathbb{R}$ is measurable function and

$$0 < K(x) := \int_{\Omega_2} k(x, t) d\mu_2(t), \qquad x \in \Omega_1. \tag{3}$$

The following result was given in [4] (see also [5]).

Theorem 1. *Let u be a weight function, $k(x,y) \geqslant 0$. Assume that $\frac{k(x,y)}{K(x)}u(x)$ is locally integrable on Ω_1 for each fixed $y \in \Omega_2$. Define v by*

$$v(y) := \int_{\Omega_1} \frac{k(x,y)}{K(x)} u(x) d\mu_1(x) < \infty. \tag{4}$$

If ϕ is a convex function on the interval $I \subseteq \mathbb{R}$, then the inequality

$$\int_{\Omega_1} \phi(A_k f(x)) u(x) d\mu_1(x) \leqslant \int_{\Omega_2} \phi(f(y)) v(y) d\mu_2(y) \tag{5}$$

holds for all measurable functions $f : \Omega_2 \to \mathbb{R}$, such that $Imf \subseteq I$, where A_k is defined by (2) - (3).

Recently there were many papers involving generalizations of Hardy's inequality and interpolating polynomials.

An important role in the paper will be played by Abel-Gontscharoff interpolation. The Abel-Gontscharoff interpolation for two points and the remainder in the integral form is given in the following theorem (for more details see [1])."

Theorem 2. *Let $n, m \in \mathbb{N}$, $n \geqslant 2$, $0 \leqslant m \leqslant n-1$ and $\phi \in C^n([\alpha,\beta])$. Then*

$$\phi(u) = Q_{n-1}(\alpha,\beta,\phi,u) + R(\phi,u),$$

where Q_{n-1} is the Abel-Gontscharoff interpolating polynomial for two-points of degree $n-1$, i.e.

$$\begin{aligned} Q_{n-1}(\alpha,\beta,\phi,u) &= \sum_{s=0}^{m} \frac{(u-\alpha)^s}{s!} \phi^{(s)}(\alpha) \\ &+ \sum_{r=0}^{n-m-2} \left[\sum_{s=0}^{r} \frac{(u-\alpha)^{m+1+s} (\alpha-\beta)^{r-s}}{(m+1+s)!\,(r-s)!} \right] \phi^{(m+1+r)}(\beta) \end{aligned}$$

and the remainder is given by

$$R(\phi,u) = \int_{\alpha}^{\beta} G_{mn}(u,t) \phi^{(n)}(t) dt,$$

where $G_{mn}(u,t)$ is Green's function defined by

$$G_{mn}(u,t) = \frac{1}{(n-1)!} \begin{cases} \displaystyle\sum_{s=0}^{m} \binom{n-1}{s} (u-\alpha)^s (\alpha-t)^{n-s-1}, & \alpha \leqslant t \leqslant u; \\ \displaystyle -\sum_{s=m+1}^{n-1} \binom{n-1}{s} (u-\alpha)^s (\alpha-t)^{n-s-1}, & u \leqslant t \leqslant \beta. \end{cases} \tag{6}$$

Generalizations of Hardy's inequality

Here we present some results that are published in [2]. Our first result is an identity related to generalized Hardy's inequality. We apply interpolation by the Abel-Gontscharoff polynomial and get the following result.

Theorem 3. *Let $(\Sigma_1, \Omega_1, \mu_1)$ and $(\Sigma_2, \Omega_2, \mu_2)$ be measure spaces with positive σ-finite measures. Let $u : \Omega_1 \to \mathbb{R}$, be a weight function and v is defined by (4). Let $A_k f(x), K(x)$ be defined by (2) and (3) respectively, for a measurable function $f : \Omega_2 \to [\alpha, \beta]$ and let $n, m \in \mathbb{N}$, $n \geqslant 2$, $0 \leqslant m \leqslant n-1$, $\phi \in C^n([\alpha, \beta])$ and G_{mn} be defined by (6). Then*

$$
\begin{aligned}
&\int_{\Omega_2} \phi(f(y))v(y)d\mu_2(y) - \int_{\Omega_1} \phi(A_k f(x))u(x)d\mu_1(x) \qquad (7)\\
&= \sum_{s=1}^{m} \frac{\phi^{(s)}(\alpha)}{s!} \left(\int_{\Omega_2} (f(y)-\alpha)^s v(y)d\mu_2(y) - \int_{\Omega_1} (A_k f(x)-\alpha)^s u(x)d\mu_1(x) \right)\\
&+ \sum_{r=0}^{n-m-2} \sum_{s=0}^{r} \frac{(-1)^{r-s}(\beta-\alpha)^{r-s}\phi^{(m+1+r)}(\beta)}{(m+1+s)!(r-s)!} \left(\int_{\Omega_2} (f(y)-\alpha)^{m+1+s} v(y)d\mu_2(y) \right.\\
&\quad \left. - \int_{\Omega_1} (A_k f(x)-\alpha)^{m+1+s} u(x)d\mu_1(x) \right)\\
&+ \int_{\alpha}^{\beta} \left(\int_{\Omega_2} G_{mn}(f(y),t)v(y)d\mu_2(y) - \int_{\Omega_1} G_{mn}(A_k f(x),t)u(x)d\mu_1(x) \right) \phi^{(n)}(t)dt.
\end{aligned}
$$

We continue with the following result that is an upper bound for generalized Hardy's inequality.

Theorem 4. *Suppose that all the assumptions of Theorem 3 hold. Let (p, q) be a pair of conjugate exponents, that is $1 \leqslant p, q \leqslant \infty$, $\frac{1}{p} + \frac{1}{q} = 1$. Then*

$$
\begin{aligned}
&\left| \int_{\Omega_2} \phi(f(y))v(y)d\mu_2(y) - \int_{\Omega_1} \phi(A_k f(x))u(x)d\mu_1(x) \right.\\
&- \sum_{s=1}^{m} \frac{\phi^{(s)}(\alpha)}{s!} \left(\int_{\Omega_2} (f(y)-\alpha)^s v(y)d\mu_2(y) - \int_{\Omega_1} (A_k f(x)-\alpha)^s u(x)d\mu_1(x) \right)\\
&- \sum_{r=0}^{n-m-2} \sum_{s=0}^{r} \frac{(-1)^{r-s}(\beta-\alpha)^{r-s}\phi^{(m+1+r)}(\beta)}{(m+1+s)!(r-s)!} \left(\int_{\Omega_2} (f(y)-\alpha)^{m+1+s} v(y)d\mu_2(y) \right.\\
&\quad \left. \left. - \int_{\Omega_1} (A_k f(x)-\alpha)^{m+1+s} u(x)d\mu_1(x) \right) \right|\\
&\leqslant \left\| \phi^{(n)} \right\|_p \left(\int_{\alpha}^{\beta} \left| \int_{\Omega_2} v(y)G_{mn}\left(f(y),t\right)d\mu_2(y) - \int_{\Omega_1} u(x)G_{mn}\left(A_k f(x),t\right)d\mu_1(x) \right|^q dt \right)^{\frac{1}{q}}.
\end{aligned} \tag{8}
$$

The constant on the right-hand side of (8) is sharp for $1 < p \leqslant \infty$ and the best possible for $p = 1$.

We continue with a particular case of Green's function $G_{mn}(u, t)$ defined by (6).

For $n = 2$, $m = 1$, we have

$$G_{12}(u,t) = \begin{cases} u - t, & \alpha \leqslant t \leqslant u \\ 0, & u \leqslant t \leqslant \beta \end{cases}, \tag{9}$$

If we choose $n = 2$ and $m = 1$ in Theorem 4, we get the following corollary.

Corollary 5. *Let $\phi \in C^2([\alpha,\beta])$ and (p,q) be a pair of conjugate exponents, that is $1 \leqslant p,q \leqslant \infty$, $\frac{1}{p} + \frac{1}{q} = 1$. Then*

$$\begin{aligned} &\left| \int_{\Omega_2} \phi(f(y))v(y)d\mu_2(y) - \int_{\Omega_1} \phi(A_k f(x))u(x)d\mu_1(x) \right| \qquad (10) \\ &\leqslant \|\phi''\|_p \left(\int_\alpha^\beta \left| \int_{\Omega_2} v(y)G_{12}\left(f(y),t\right)d\mu_2(y) - \int_{\Omega_1} u(x)G_{12}\left(A_k f(x),t\right)d\mu_1(x) \right|^q dt \right)^{\frac{1}{q}}. \end{aligned}$$

The constant on the right hand side of (10) is sharp for $1 < p \leqslant \infty$ and the best possible for $p = 1$.

Remark 6. *If we additionally suppose that ϕ is convex, then the difference $\int_{\Omega_2} \phi(f(y))v(y)d\mu_2(y) - \int_{\Omega_1} \phi(A_k f(x))u(x)d\mu_1(x)$ is non-negative and we have*

$$\begin{aligned} 0 &\leqslant \int_{\Omega_2} \phi(f(y))v(y)d\mu_2(y) - \int_{\Omega_1} \phi(A_k f(x))u(x)d\mu_1(x) \qquad (11) \\ &\leqslant \|\phi''\|_p \left(\int_\alpha^\beta \left| \int_{\Omega_2} v(y)G_{12}\left(f(y),t\right)d\mu_2(y) - \int_{\Omega_1} u(x)G_{12}\left(A_k f(x),t\right)d\mu_1(x) \right|^q dt \right)^{\frac{1}{q}}. \end{aligned}$$

In sequel we consider some particular cases of this result.

Example 7. Let $\Omega_1 = \Omega_2 = (0,b)$, $0 < b \leqslant \infty$, replace $d\mu_1(x)$ and $d\mu_2(y)$ by the Lebesque measures dx and dy, respectively, and let $k(x,y) = 0$ for $x < y \leqslant b$. Then A_k coincides with the Hardy operator H_k defined by

$$H_k : H_k f(x) := \frac{1}{K(x)} \int_0^x f(t)k(x,t)\,dt, \tag{12}$$

where

$$K(x) := \int_0^x k(x,t)\,dt < \infty.$$

If also $u(x)$ is replaced by $u(x)/x$ and $v(x)$ by $v(x)/x$, then

$$\begin{aligned} 0 &\leqslant \int_0^b v(y)\phi\left(f(y)\right)\frac{dy}{y} - \int_0^b u(x)\phi\left(H_k f(x)\right)\frac{dx}{x} \\ &\leqslant \|\phi''\|_p \left(\int_\alpha^\beta \left| \int_0^b v(y)G_{12}\left(f(y),t\right)\frac{dy}{y} - \int_0^b u(x)G_{12}\left(H_k f(x),t\right)\frac{dx}{x} \right|^q dt \right)^{\frac{1}{q}}. \end{aligned}$$

We continue with the result that involves Hardy–Hilbert's inequality.
If $p > 1$ and f is a non-negative function such that $f \in L^p(\mathbb{R}_+)$, then

$$\int_0^\infty \left(\int_0^\infty \frac{f(x)}{x+y}\, dx \right)^p dy \leqslant \left(\frac{\pi}{\sin\left(\frac{\pi}{p}\right)} \right)^p \int_0^\infty f^p(y)\, dy. \tag{13}$$

Inequality (13) is sometimes called Hilbert's inequality even if Hilbert himself only considered the case $p = 2$.

Example 8. Let $\Omega_1 = \Omega_2 = (0, \infty)$, replace $d\mu_1(x)$ and $d\mu_2(y)$ by the Lebesque measures dx and dy, respectively. Let $k(x, y) = \frac{(\frac{y}{x})^{-1/p}}{x+y}$, $p > 1$ and $u(x) = \frac{1}{x}$. Then $K(x) = K = \frac{\pi}{\sin(\pi/p)}$ and $v(y) = \frac{1}{y}$. Let $\phi(u) = u^p$, $\prod_{i=1}^{k}(p - i + 1) \geqslant 0$, replace $f(y)$ with $f(y)y^{\frac{1}{p}}$ then the following result follows

$$\begin{aligned} 0 &\leqslant \int_0^\infty f^p(y)dy - K^{-p} \int_0^\infty \left(\int_0^\infty \frac{f(y)}{x+y} dy \right)^p dx \\ &\leqslant \|\phi''\|_p \left(\int_\alpha^\beta \left| \int_0^\infty G_{12}\left(f(y)y^{\frac{1}{p}}, t\right) \frac{dy}{y} - \int_0^\infty G_{12}\left(A_k f(x), t\right) \frac{dx}{x} \right|^q dt \right)^{\frac{1}{q}} \end{aligned}$$

where

$$A_k f(x) = \frac{\sin(\pi/p)}{\pi} \int_0^\infty \frac{f(y)}{x+y} x^{\frac{1}{p}} dy.$$

We also mention Pólya–Knopp's inequality,

$$\int_0^\infty \exp\left(\frac{1}{x} \int_0^x \ln f(t)\, dt \right) dx < e \int_0^\infty f(x)\, dx, \tag{14}$$

for positive functions $f \in L^1(\mathbb{R}_+)$. Pólya–Knopp's inequality may be considered as a limiting case of Hardy's inequality since (14) can be obtained from (1) by rewriting it with the function f replaced with $f^{\frac{1}{p}}$ and then by letting $p \to \infty$.

Example 9. By applying (11) with $\phi(x) = e^x$, and f replaced by $\ln f^p$, $p > 0$ we obtain that

$$\begin{aligned} 0 &\leqslant \int_{\Omega_2} f^p(y)v(y)d\mu_2(y) - \int_{\Omega_1} \left[\exp\left(\frac{1}{K(x)} \int_{\Omega_2} k(x, y) \ln f(y) d\mu_2(y) \right) \right]^p u(x) d\mu_1(x) \\ &\leqslant \|\phi''\|_p \left(\int_\alpha^\beta \left| \int_{\Omega_2} v(y) G_{12}\left(\ln f^p(y), t\right) d\mu_2(y) - \int_{\Omega_1} u(x) G_{12}\left(A_k f(x), t\right) d\mu_1(x) \right|^q dt \right)^{\frac{1}{q}} \end{aligned} \tag{15}$$

where $k(x, y)$, $K(x)$, $u(x)$ and $v(y)$ are defined as in Theorem 1 and

$$A_k f(x) = \frac{p}{K(x)} \int_{\Omega_2} k(x, y) \ln f(y) d\mu_2(y).$$

More results and all proofs can me found in [2].

Acknowledgments

This paper is dedicated to Professor Manuel López-Pellicer on the occasion of his 81st birth anniversary.

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Sharpened and generalized versions of the q-Steffensen inequality

Ksenija Smoljak Kalamir

We present several new contributions to the study of Steffensen-type inequalities in the setting of q-calculus. We give weaker conditions under which the q-Steffensen inequality remains valid, thereby extending its applicability to a broader class of functions. Further, we provide sharpened, refined and generalized versions which improve the classical Steffensen's inequality.

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KEYWORDS: Steffensen's inequality, $q-$integral, weaker conditions

Introduction

Quantum calculus, also known as $q-$calculus, is a generalization of classical calculus in which the notion of limits is replaced by differences depending on a parameter $q \in (0,1)$. $q-$calculus has deep connections with several areas of mathematics and theoretical physics. Its applications can be found in many fields in mathematics, such as analysis, $q-$series, $q-$hypergeometric functions, orthogonal polynomials, combinatorics, number theory, etc. In mathematical physics, it appears in quantum mechanics, statistical mechanics, and the theory of integrable systems. More details about $q-$calculus can be found in [2].

The classical Steffensen inequality is a useful tool in real analysis and integral inequalities (see [5, 9]). In the last decades, numerous extensions and generalizations of Steffensen-type inequalities have been studied, including their discrete analogues, weighted versions, and inequalities in the framework of q-calculus. The q-Steffensen inequality, in particular, has recently attracted attention due to its connections with q-analysis, approximation theory, and operator inequalities.

The classical Steffensen inequality states:

Theorem 1 ([9]). *Suppose that f is decreasing and g is integrable on $[a,b]$ with $0 \leqslant g \leqslant 1$ and $\lambda = \int_a^b g(t)dt$. Then we have*

$$\int_{b-\lambda}^{b} f(t)dt \leqslant \int_a^b f(t)g(t)dt \leqslant \int_a^{a+\lambda} f(t)dt.$$

In [1] Gauchman proved $q-$Steffensen's inequality and in [6] Rajković et al. improved Gauchman's result considering the $q-$integrals on $(0,b)$ when they are represented by infinite sums. Hence, they obtained the following result:

Theorem 2 ([6]). *Let $0 < q < 1$, $b > 0$, f and g are both $q-$integrable functions on $[0,b]$, f is nonnegative and decreasing and $0 \leqslant g(x) \leqslant 1$ for each $x \in [0,b]$ and $\lambda = \int_0^b g(x)d_qx$. Let $l, k \in \mathbb{N}_0$ be such that*

$$l = \lfloor \log_q(1-\lambda/b) \rfloor, \quad k = \lfloor \log_q(\lambda/b) \rfloor.$$

Then

$$\int_{bq^l}^{b} f(x)d_qx \leqslant \int_0^b f(x)g(x)d_qx \leqslant \int_0^{bq^k} f(x)d_qx. \tag{1}$$

Main results

Instead of the condition $0 \leqslant g \leqslant 1$ in the classical Steffensen's inequality, Milovanović and Pečarić obtained weaker conditions for the function g (see [3]). In [7] we obtained weaker conditions for the function g in the $q-$Steffensen inequality (1).

In the following, we present only the results concerning the right-hand side Steffensen's inequality. The results for the left-hand side are analogous.

Theorem 3 (*cf.* [7])**.** *Let $0 < q < 1$, $b > 0$. Let f and g be $q-$integrable functions on $[0, b]$ such that f is nonnegative and decreasing and $\lambda = \int_0^b g(x)d_qx$. Let $k \in \mathbb{N}_0$ be such that $k = \lfloor \log_q(\lambda/b) \rfloor$. If*

$$\int_0^{qx} g(t)d_qt \leqslant qx \quad \text{and} \quad \int_{qx}^{b} g(t)d_qt \geqslant 0, \quad \text{for every } x \in [0, b],$$

then

$$\int_0^b f(x)g(x)d_qx \leqslant \int_0^{bq^k} f(x)d_qx.$$

In [4] Pečarić proved a generalization of Steffensen's inequality which was studied in many papers concerning Steffensen's inequality (see [5]). Its $q-$analogue is given in the following theorem.

Theorem 4 (*cf.* [7])**.** *Let $0 < q < 1$, $b > 0$. Let f, g and h be $q-$integrable functions on $[0, b]$ such that h is positive, f is nonnegative, f/h is decreasing and $0 \leqslant g \leqslant 1$ on $[0, b]$. Let $k \in \mathbb{N}_0$ be such that*

$$\int_0^{bq^k} h(x)d_qx \geqslant \int_0^b h(x)g(x)d_qx. \tag{2}$$

Then

$$\int_0^b f(x)g(x)d_qx \leqslant \int_0^{bq^k} f(x)d_qx. \tag{3}$$

In the following theorem we give a refined version of a result given in [7].

Theorem 5 ([8])**.** *Let $0 < q < 1$, $b > 0$. Let f, g, h and κ be $q-$integrable functions on $[0, b]$ such that κ is positive, f is nonnegative, f/κ is decreasing and $0 \leqslant g \leqslant h$ on $[0, b]$. Let $k \in \mathbb{N}_0$ be such that (6) holds. Then the following inequality is valid*

$$\begin{aligned} \int_0^b f(x)g(x)d_qx &\leqslant \int_0^{bq^k} \left(f(x)h(x) - \left[\frac{f(x)}{\kappa(x)} - \frac{f(bq^k)}{\kappa(bq^k)} \right] \kappa(x)[h(x) - g(x)] \right) d_qx \\ &\leqslant \int_0^{bq^k} f(x)h(x)d_qx. \end{aligned} \tag{4}$$

In the following theorem we give sharpened and generalized version of $q-$Steffensen's inequality proved in [8] which was motivated by results given in [10].

Theorem 6 (*cf.* [8])**.** *Let $0 < q < 1$, $b > 0$. Let f, g, h and Ψ be $q-$integrable functions on $[0, b]$ such that f is decreasing and nonnegative and $0 \leqslant \Psi \leqslant g \leqslant h - \Psi$ on $[0, b]$. Let $k \in \mathbb{N}_0$ be such that $\int_0^{bq^k} h(x)d_qx \geqslant \int_0^b g(x)d_qx$ holds. Then the following inequality is valid*

$$\int_0^b f(x)g(x)d_qx \leqslant \int_0^{bq^k} f(x)h(x)d_qx - \int_0^b |f(x) - f(bq^k)|\Psi(x)d_qx. \tag{5}$$

Let us conclude by the weaker conditions for the refinement given in Theorem 5.

Theorem 7 ([8])**.** *Let $0 < q < 1$, $b > 0$. Let f, g, h and κ be $q-$integrable functions on $[0, b]$ such that κ is positive, f is nonnegative and f/κ is decreasing. Let $k \in \mathbb{N}_0$ be such that*

$$\int_0^{bq^k} \kappa(x)h(x)d_qx \geqslant \int_0^b \kappa(x)g(x)d_qx. \tag{6}$$

If

$$\int_0^{qx} \kappa(t)g(t)d_qt \leqslant \int_0^{qx} \kappa(t)h(t)d_qt, \qquad \text{for every } qx \in [0, bq^k] \tag{7}$$

and

$$\int_{qx}^b \kappa(t)g(t)d_qt \geqslant 0, \qquad \text{for every } qx \in [bq^k, b], \tag{8}$$

hold, then

$$\begin{aligned} \int_0^b f(x)g(x)d_qx &\leqslant \int_0^{bq^k} \left(f(x)h(x) - \left[\frac{f(x)}{\kappa(x)} - \frac{f(bq^k)}{\kappa(bq^k)} \right] \kappa(x)[h(x) - g(x)] \right) d_qx \\ &\leqslant \int_0^{bq^k} f(x)h(x)d_qx. \end{aligned} \tag{9}$$

Remark 8. *If $q \to 1$, then result presented here reduce to the results for classical Stefeffensen's inequality.*

Acknowledgments

This paper is dedicated to Professor Manuel López-Pellicer on the occasion of his 81st birth anniversary.

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Unveiling the mathematical marvels: Exploring differential and integral equations for multivariate Hermite-Frobenius-Genocchi polynomials

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In this article, a novel family of multi-variate Hermite-Frobenius-Genocchi polynomials is constructed and its several characterizations are observed. The properties of these polynomials, such as recurrence relations and shift operators, are investigated. Differential equations, partial differential equations, and integrodifferential equations satisfied by these polynomials are derived using the factorization method. Additionaly, the Volterra integral equation is derived for these multi-variate Hermite-Frobenius-Genocchi polynomials, which enhances the understanding and application of the factorization method in physics and engineering.

KEYWORDS: Multi-variate Hermite-Frobenius-Genocchi polynomials, recurrence relation, shift operators, differential equations, Volterra integral equation

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Global stability of Wright-type equations with negative Schwarzian

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Simplicity of the 37/24-global stability criterion announced by E.M. Wright in 1955 and rigorously proved by B. Bánhelyi et al in 2014 for the delayed logistic equation raised the question of its possible extension for other population models. In our study, we answer this question by extending the 37/24- stability condition for the Wright-type equations with decreasing smooth nonlinearity f which has a negative Schwarzian and satisfies the standard negative feedback and boundedness assumptions. The proof contains the construction and careful analysis of qualitative properties of certain bounding relations. To validate our conclusions, these relations are evaluated at finite sets of points; for this purpose, we systematically use interval analysis.

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Projections of generalized helices in 3-dimensional Lorentz-Minkowski space

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In this paper we present the plane projections of generalized helices in the three-dimensional Lorentz–Minkowski space. We consider two principal classes: helices lying on a lightlike cone, and spherical helices lying on either a pseudosphere or a hyperbolic plane. It is shown that projections of cone–helices appear as Euclidean or Lorentzian logarithmic spirals, while projections of spherical helices appear as Euclidean or Lorentzian cycloidal curves.

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KEYWORDS: Lorentz-Minkowski space, generalized helix, projection, logarithmic spiral, cycloidal curve

Introduction

A generalized helix, also known as a curve of constant slope, is defined as a space curve whose tangent vector makes a constant angle with a fixed direction, called the axis of the helix. In classical Euclidean geometry these curves are characterized by the constant ratio of their torsion to curvature. Beyond the Euclidean setting, generalized helices have been extensively studied in spaces with indefinite metrics, particularly in the Lorentz–Minkowski space, [3].

A natural problem is to investigate generalized helices with additional geometric constraints, such as containment in quadratic surfaces. In Euclidean geometry, spherical helices project onto epicycloids [2]. The analogous problems in Lorentz–Minkowski space are richer, due to the causal structure of the space and the different types of induced geometries in planes.

In this work we bring together two classes of constrained generalized helices: helices lying on a lightlike cone and helices lying on a pseudosphere or a hyperbolic plane.

We consider their plane projections, which turn out to be logarithmic spirals in the case of the cone, and cycloidal curves in the case of the pseudosphere or hyperbolic plane. The presented results are published in [1, 4].

Preliminaries

Let $\mathbb{R}^3_1$ denote the three-dimensional Lorentz–Minkowski space, that is, the vector space $\mathbb{R}^3$ equipped with the indefinite bilinear form

$$\langle x, y\rangle = x_1y_1 + x_2y_2 - x_3y_3,$$

for $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$. A vector can be spacelike, timelike, or lightlike depending on the sign of its squared pseudo-norm $\langle x, x\rangle$, $\langle x, x\rangle > 0$, $\langle x, x\rangle < 0$ or $\langle x, x\rangle = 0$ respectively. The zero vector is a spacelike vector. The causal character of a regular curve is determined by the causal character of its velocity vector. For a unit-speed curve $c(s)$ in $\mathbb{R}^3_1$, the Frenet frame (T, N, B) and Frenet equations are defined analogously to the Euclidean case, adapted to the causal character of the curve and its normal vectors, [3].

A curve c is a generalized helix if there exists a constant vector $u \neq 0$ such that

$$\langle T(s), u\rangle = \text{const.}$$

Equivalently, for spacelike or timelike curves with non-lightlike normals, the ratio τ/κ is constant, [3].

The orthogonal projection of a curve c onto a plane with unit normal u is given by

$$\tilde{c} = c - \delta\langle c, u\rangle u, \qquad \delta = \langle u, u\rangle = \pm 1.$$

The following relation between the curvatures κ of c and $\tilde{\kappa}$ of its projection is stated in [4].

Theorem 1. *Let c be a (unit speed) spacelike generalized helix with respect to a unit spacelike or timelike vector u. Let $\tilde{c}$ be the projection of c onto a plane orthogonal to u. Then $\tilde{c}$ has a constant speed. Furthermore, if the principal normals of c and $\tilde{c}$ are of the same causal non-null character, the curvature of c and of $\tilde{c}$ are related by*

$$\tilde{\kappa}^2(1 - \delta\alpha^2)^2 = \kappa^2, \qquad \alpha = \langle T, u\rangle.$$

The causal character of a projection curve depends on the causal character of the axis of a generalized helix and character of a plane spanned by the axis and a tangent vector of the helix, [4].

Theorem 2. *Let c be a (unit speed) spacelike generalized helix with respect to a unit spacelike or timelike vector u. Then $\tilde{c}$ is:*

1. *spacelike, if u timelike or if u is spacelike such that u and T span a spacelike plane;*
2. *timelike, if u is spacelike such that u and T span a timelike plane.*

Plane Projections of Generalized Helices on a Lightlike Cone

A lightlike cone with vertex p is defined as

$$\text{LC}(p) = \{q \in \mathbb{R}^3_1 : \langle q - p, q - p\rangle = 0\}.$$

It is a degenerate quadratic surface, a rotational cone with axis x_3 from the Euclidean perspective.

As shown in [4], if $c(s)$ is a unit-speed curve on $\text{LC}(p)$, then its curvature κ and torsion τ satisfy

$$\rho\tau = \pm\rho', \qquad \rho = 1/\kappa.$$

If in addition c is a generalized helix with constant $A = \tau/\kappa$, then

$$\kappa(s) = \pm\frac{1}{As}, \qquad \tau(s) = \pm\frac{1}{s}.$$

Using the relation between the curvatures κ of c and $\tilde{\kappa}$ of its projection from Theorem 1., we can obtain the curvature $\tilde{\kappa}$. Since $\tilde{c}$ is a planar curve, i.e. $\tilde{\tau} = 0$, we can reconstruct the projection curve $\tilde{c}$ from its natural equation. The projections of such helices onto coordinate planes orthogonal to the axis turn out to be logarithmic spirals. Projection onto the **spacelike xy-plane** yields a Euclidean logarithmic spiral, while the projection onto a **timelike xz-plane** yields a Lorentzian logarithmic spiral, [4].

Plane Projections of Spherical Generalized Helices

Spherical curves in Lorentz–Minkowski space are curves lying either on a pseudosphere

$$S_1^2(p,r) = \{q : (q-p)\cdot(q-p) = r^2\},$$

or on a hyperbolic plane

$$H^2(p,r) = \{q : (q-p)\cdot(q-p) = -r^2\}.$$

Both are non-degenerate quadrics and can be regarded as analogues of spheres in Lorentzian geometry.

As shown in [1], a unit-speed generalized helix lying on $S_1^2(p,r)$ or $H^2(p,r)$ has curvature and torsion given by relations of the type

$$\kappa^2(s) = \frac{1}{\pm r^2 + A^2 s^2}, \qquad \tau^2(s) = \frac{A^2}{\pm r^2 + A^2 s^2},$$

depending on the causal character of the curve.

The orthogonal projections of spherical generalized helices lead to cycloidal curves:

- For timelike axes, projections are **Euclidean epi-/hypo-/paracycloids**.
- For spacelike axes, projections are **Lorentzian epi-/hypo-/hyper-/paracycloids**.

In particular, if c lies on a pseudosphere with timelike axis, its projection onto the xy-plane is a Euclidean hypocycloid, with natural equation

$$\frac{\tilde{s}^2}{a^2} + \frac{\tilde{\rho}^2}{b^2} = 1.$$

If c lies on a hyperbolic plane with spacelike axis, projections may appear as Lorentzian epicycloids or hypercycloids, with natural equations of the form

$$\frac{\tilde{s}^2}{c^2} - \frac{\tilde{\rho}^2}{d^2} = \pm 1.$$

Conclusion

We have provided a unified view of projections of generalized helices in 3-dimensional Lorentz–Minkowski space. Generalized helices lying on a lightlike cone yield logarithmic spirals as their projections, while spherical generalized helices on a pseudosphere or hyperbolic plane yield cycloidal curves. Explicit parametrizations and natural equations of these projections were presented. These results underline the deep connection between intrinsic curve properties and induced plane geometries in semi-Riemannian spaces.

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Quasi-partial controlled metric spaces: tight/loose variants, topology

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In this paper, we introduce a generalization of metric spaces called quasi-partial controlled metric spaces (QPCM), which combine and extend features of partial metric spaces, quasi-metric spaces, and controlled metric spaces. We define two variants—*tight* and *loose* QPCM—and explore their fundamental properties. We establish relationships between QPCM and existing metric structures, including extended b-metric spaces (in the senses of Kamran and Aydi). We develop the topological theory of two-sided q-convergence, completeness, and Cauchy sequences, and compare the local bases generated by q-balls and the symmetrization d_q-balls. Worked examples include a purely loose (not tight) QPCM and a tight QPCM beyond constant-coefficient classes. The results lay a foundation for fixed point theorems in QPCM, providing a unified framework encompassing several classical and modern generalizations.

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KEYWORDS: Quasi-partial metric space, controlled metric space, extended b-metric, topology, convergence, fixed point theory

Introduction

Metric spaces have been extensively generalized to accommodate various mathematical and applied needs. Notable generalizations include partial metric spaces [7], which allow non-zero self-distances and are useful in domain theory and computer science; quasi-metric spaces [9], which lack symmetry and model asymmetric processes; and controlled metric-type spaces [8, 1], where the triangle inequality is governed by control functions. More recent developments include extended b-metric spaces [5, 2] and M-metric spaces [6].

The motivation for this work stems from the need to unify and extend these generalizations. Specifically, we introduce *quasi-partial controlled metric* (QPCM) spaces that incorporate:

- non-zero self-distances (as in partial metrics),
- asymmetry (as in quasi-metrics),
- control functions modulating the triangle inequality (as in controlled metrics).

This framework captures a broader range of phenomena while retaining enough structure to prove meaningful fixed point results. Fixed point theory in such spaces has applications in differential equations, computer algorithm complexity, and network analysis [3, 4, 2].

In this paper we define two variants of QPCM (tight and loose), supply examples, and explore their relationships with existing metrics. We then study their topological properties and prepare for fixed point results. Our work extends results from [1, 10, 4] and provides a foundation for further applications.

Preliminaries

We recall key notions that will be used throughout.

Definition 1 (Partial metric [7]). *A partial metric on a set X is a function $p : X \times X \to [0, \infty)$ satisfying for all $x, y, z \in X$:*

$x = y$ *iff* $p(x, x) = p(x, y) = p(y, y)$;
$p(x, x) \leqslant p(x, y)$;
$p(x, y) = p(y, x)$;
$p(x, y) \leqslant p(x, z) + p(z, y) - p(z, z)$.

Definition 2 (Quasi-metric [9]). *A quasi-metric on X is a function $q : X \times X \to [0, \infty)$ satisfying for all $x, y, z \in X$:*

1. $x = y$ *iff* $q(x, y) = 0$;
2. $q(x, y) \leqslant q(x, z) + q(z, y)$.

Definition 3 (Extended b-metric (Kamran) [5, 2]). *A function $d : X \times X \to [0, \infty)$ with a control function $\theta : X \times X \to [1, \infty)$ is an extended b-metric if $\forall x, y, z \in X$:*

1. $d(x, y) = 0$ *iff* $x = y$;
2. $d(x, y) = d(y, x)$;
3. $d(x, y) \leqslant \theta(x, y) [d(x, z) + d(z, y)]$.

QPCM: Axioms and triangles

We now introduce the main concept of this paper.

Definition 4 (Standing axioms). *Let $X \neq \varnothing$. A function $q : X \times X \to [0, \infty)$ and a control function $\theta : X^3 \to [1, \infty)$ are assumed to satisfy, for all $x, y, z \in X$:*

$$\text{(Q1)} \quad q(x, x) \leqslant q(x, y) \quad \text{and} \quad q(y, y) \leqslant q(x, y), \tag{1}$$
$$\text{(M)} \quad \text{if } q(x, x) = q(x, y) \text{and} q(y, x) = q(y, y), \text{ then } x = y. \tag{2}$$

We consider two triangle inequality variants:

$$\text{(Tight QPCM)} \qquad q(x, y) \leqslant \theta(x, z, y)\big(q(x, z) + q(z, y) - q(z, z)\big), \tag{3}$$
$$\text{(Loose QPCM)} \qquad q(x, y) \leqslant \theta(x, z, y)\big(q(x, z) + q(z, y)\big) - q(z, z). \tag{4}$$

The triple (X, q, θ) is called a ***quasi-partial controlled metric space*** *(qpcm space) if it satisfies (Q1), (M), and either (3) or (4). Specifically, we refer to* ***tight*** *or* ***loose*** *qpcm spaces based on the triangle inequality used.*

Remark 5. *The symmetry condition $q(x, y) = q(y, x)$ is not required. If it holds, we call the space* ***symmetric****.*

Definition 6 (Symmetrization). *Define the symmetric functional:*

$$d_q(x, y) := q(x, y) + q(y, x) - q(x, x) - q(y, y)$$

By (Q1), $d_q(x, y) \geqslant 0$.

Remark 7 (Zero diagonal). *If $q(u, u) = 0$ for all $u \in X$, tight and loose triangles coincide: $q(x, y) \leqslant \theta(x, z, y)\big(q(x, z) + q(z, y)\big)$.*

Main results

Lemma 8. *[For all $u, v \in X$,*

$$0 \leqslant q(u,v) - q(u,u) \leqslant d_q(u,v), \tag{5}$$
$$0 \leqslant q(v,u) - q(v,v) \leqslant d_q(u,v), \tag{6}$$
$$|q(u,u) - q(v,v)| \leqslant d_q(u,v). \tag{7}$$

Proof. From (Q1), $q(u,v) \geqslant q(u,u)$ and $q(v,u) \geqslant q(v,v)$, so the left inequalities in (5)–(6) hold. By definition

$$d_q(u,v) = \big(q(u,v) - q(u,u)\big) + \big(q(v,u) - q(v,v)\big),$$

with both summands nonnegative, hence the right inequalities in (5)–(6). For (7), assume w.l.o.g. $q(u,u) \geqslant q(v,v)$. Then by (Q1), $q(v,u) \geqslant q(u,u)$ and

$$d_q(u,v) = q(u,v) + q(v,u) - q(u,u) - q(v,v) \;\geqslant\; q(u,u) - q(v,v) = |q(u,u) - q(v,v)|.$$

The other case is symmetric.

The following proposition shows why the framework of Quasi-Partial Controlled Metric (QPCM) spaces is more general than partial metrics, quasi metrics, quasi partial metrics, extended b-metric etc.

Proposition 9. *If (X, D) is a quasi-metric, extended b-metric, or controlled metric-type space with $D(x,x) = 0$ for all x, then (X, q, θ) is both tight and loose QPCM by setting $q := D$ and choosing θ accordingly.*

Proof. When $D(x,x) = 0$, tight and loose triangles coincide. For a quasi-metric, $D(x,y) \leqslant D(x,z) + D(z,y)$, so take $\theta \equiv 1$. For an extended b-metric, $D(x,y) \leqslant \Theta(x,z)[D(x,z) + D(z,y)]$, so set $\theta(x,z,y) := \Theta(x,z)$. In controlled metric-type settings with two factors, take θ as any dominating coefficient to absorb the sum; the zero diagonal ensures equivalence of tight/loose forms.

Examples

Example 10. *Let the set be $X = \mathbb{R}$. We define the function $q : X \times X \to [0, \infty)$ as follows:*

$$q(x,y) = |x||y-x| + \max(|x|, |y|)$$

*This space becomes a **tight QPCM** by constructively defining the control function $\theta : X^3 \to [1, \infty)$:*

$$\theta(x,z,y) := \max\left\{1, \frac{q(x,y)}{q(x,z) + q(z,y) - q(z,z)}\right\}$$

This space satisfies all axioms for a tight QPCM. Axiom (Q1), $q(x,x) \leqslant q(x,y)$, holds since $q(x,x) = |x|$ which is always less than or equal to the $\max(|x|, |y|)$ term within $q(x,y)$. Axiom (M), $x = y \iff (q(x,x) = q(x,y)$ and $q(y,y) = q(y,x))$, is also satisfied because the condition $q(x,x) = q(x,y)$ forces $|x||y-x| = 0$, which together with its symmetric counterpart implies $x = y$. Finally, the tight triangle inequality holds by the very definition of the control function θ, which is constructively chosen to ensure the inequality is always satisfied.

Proof of Non-existence of a Constant Coefficient s

We will now show that this space is not a *quasi-partial b-metric*, meaning there cannot exist a **constant** coefficient $s \geqslant 1$ that bounds it.

Assume, for the sake of contradiction, that there exists a constant $s \geqslant 1$ such that the following inequality holds $\forall x, y, z \in \mathbb{R}$:

$$q(x, y) \leqslant s \cdot (q(x, z) + q(z, y) - q(z, z))$$

We choose a specific intermediate point, $z = 0$, to test this assumption. We calculate the terms on the right-hand side:

$$\begin{aligned} q(x, 0) &= |x||0 - x| + \max(|x|, |0|) = |x|^2 + |x| \\ q(0, y) &= |0||y - 0| + \max(|0|, |y|) = 0 + |y| = |y| \\ q(0, 0) &= |0||0 - 0| + \max(|0|, |0|) = 0 \end{aligned}$$

Substituting these into the main inequality, we get:

$$|x||y - x| + \max(|x|, |y|) \leqslant s \cdot (|x|^2 + |x| + |y|)$$

Now, let us analyze the behavior of this inequality for large values. We fix a large value for $x > 0$ and choose y to be much larger than x ($y \gg x > 0$). In this case:

- Left-Hand Side (LHS): $LHS = x(y - x) + y = xy - x^2 + y \approx xy + y = y(x + 1)$.
- Right-Hand Side (RHS): $RHS = s(x^2 + x + y) \approx s \cdot y$.

Thus, the inequality becomes approximately:

$$y(x + 1) \lesssim s \cdot y$$

Dividing by y (which is positive), we get:

$$x + 1 \lesssim s$$

This result shows that the required coefficient s must be greater than $x + 1$. However, we can choose x to be arbitrarily large. For example, if we choose $x = 100$, we would need $s \geqslant 101$. If we choose $x = 1000$, we would need $s \geqslant 1001$.

This is a **contradiction** to the assumption that s is a **constant** and fixed coefficient. Therefore, such a coefficient does not exist. ■

The constructed space (X, q, θ) is a clear example of a **tight QPCM**does not belong to the simpler class of quasi-partial b-metric spaces.

Example 11. *Let the set be* $X = \{a, b, c\}$. *Define the function* $q : X \times X \to [0, \infty)$ *as follows:*

- $q(a, a) = 1$, $q(b, b) = 1$, $q(c, c) = 10$
- $q(a, b) = q(b, a) = 25$
- $q(a, c) = q(c, a) = 10$
- $q(b, c) = q(c, b) = 10$

Define a non-constant control function $\theta : X^3 \to [1, \infty)$ *piecewise:*

$$\theta(x, z, y) = \begin{cases} 2 & \textit{if } (x, z, y) = (a, c, b) \\ 5 & \textit{otherwise} \end{cases}$$

This space is a genuinely loose QPCM that is not tight. Not Tight: The tight inequality $q(x,y) \leqslant \theta(x,z,y)(q(x,z)+q(z,y)-q(z,z))$ fails for the triple $(x,y,z) = (a,b,c)$.

$$q(a,b) \leqslant \theta(a,c,b)\cdot(q(a,c)+q(c,b)-q(c,c))$$
$$25 \leqslant 2\cdot(10+10-10) \implies 25 \leqslant 20 \quad \text{(False)}$$

Since the inequality fails, the space is not tight.

The loose inequality $q(x,y) \leqslant \theta(x,z,y)(q(x,z)+q(z,y)) - q(z,z)$ holds. For the critical case $(x,y,z) = (a,b,c)$:

$$q(a,b) \leqslant \theta(a,c,b)\cdot(q(a,c)+q(c,b)) - q(c,c)$$

$$25 \leqslant 2\cdot(10+10) - 10 \implies 25 \leqslant 30 \quad \text{(True)}$$

For all other triples, the large coefficient $\theta = 5$ ensures the inequality holds.

Topology in QPCM: balls and inclusions

Convergence and Cauchy

Below we give the definition of two sided q-convergent and two sided q-Cauchy.

Definition 12. *1. A sequence (x_n) is two-sided q-convergent to $x \in X$ if $q(x,x_n) \to q(x,x)$ and $q(x_n,x) \to q(x,x)$.*

2. (x_n) is two-sided q-Cauchy *if the limits* $\lim_{n,m} q(x_n,x_m)$ *and* $\lim_{n,m} q(x_m,x_n)$ *exist and are finite (and equal).*

3. (X,q) is two-sided q-complete if every two-sided q-Cauchy sequence converges two-sidedly.

Local bases: forward/backward/two-sided q-balls and d_q-balls

Fix $x \in X$, $\varepsilon, r > 0$. Define

$$\begin{aligned}
B_q^{\rightarrow}(x;\varepsilon) &:= \{y \in X :\ q(x,y) < q(x,x)+\varepsilon\},\\
B_q^{\leftarrow}(x;\varepsilon) &:= \{y \in X :\ q(y,x) < q(x,x)+\varepsilon\},\\
B_q(x;\varepsilon) &:= B_q^{\rightarrow}(x;\varepsilon) \cap B_q^{\leftarrow}(x;\varepsilon),\\
B_{d_q}(x;r) &:= \{y \in X :\ d_q(x,y) < r\}.
\end{aligned}$$

Sharp inclusions between q- and d_q-balls

Proposition 13. *For all $x \in X$ and $r > 0$,*

$$B_{d_q}(x;r) \ \subseteq\ B_q(x;2r).$$

Proof. Let $y \in B_{d_q}(x;r)$, so $d_q(x,y) < r$. By Lemma 8,

$$0 \leqslant q(x,y) - q(x,x) \leqslant d_q(x,y) < r$$

so $y \in B_q^{\rightarrow}(x;r)$. Also,

$$0 \leqslant q(y,x) - q(y,y) \leqslant d_q(x,y) < r \quad \text{and} \quad |q(y,y) - q(x,x)| \leqslant d_q(x,y) < r,$$

hence $q(y,x) < q(y,y) + r < q(x,x) + 2r$, so $y \in B_q^{\leftarrow}(x;2r)$. Therefore $y \in B_q(x;2r)$.

Proposition 14. *Let* $\widetilde{B}_q(x;\varepsilon) := \{y \in X : \ q(x,y) < q(x,x) + \varepsilon, \ q(y,x) < q(x,x) + \varepsilon, \ |q(y,y) - q(x,x)| < \varepsilon\}$. *Then*

$$\widetilde{B}_q(x;\varepsilon) \ \subseteq \ B_{d_q}(x;3\varepsilon).$$

If $q(\cdot,\cdot)$ *has zero diagonal, this simplifies to* $B_q(x;\varepsilon) \subseteq B_{d_q}(x;2\varepsilon)$.

Proof. For $y \in \widetilde{B}_q(x;\varepsilon)$ we have

$$q(x,y) - q(x,x) < \varepsilon, \qquad q(y,x) - q(x,x) < \varepsilon, \qquad |q(y,y) - q(x,x)| < \varepsilon.$$

Then

$$q(y,x) - q(y,y) = (q(y,x) - q(x,x)) + (q(x,x) - q(y,y)) < \varepsilon + \varepsilon = 2\varepsilon.$$

Thus

$$d_q(x,y) = (q(x,y) - q(x,x)) + (q(y,x) - q(y,y)) < \varepsilon + 2\varepsilon = 3\varepsilon$$

If $q(x,x) = 0$ for all x, then $d_q(x,y) = q(x,y) + q(y,x) < \varepsilon + \varepsilon = 2\varepsilon$, proving the simplified inclusion.

Conclusion

We introduced tight and loose quasi-partial controlled metric (QPCM) spaces, unifying several metric generalizations by incorporating non-zero self-distances, asymmetry, and a variable triangle inequality. We provided concrete examples (including a loose-but-not-tight space), analyzed the induced topologies via q- and d_q-balls, and proved sharp inclusion relations. These foundations support further development of fixed point theorems (e.g., Banach-type results under orbital control of θ), to be pursued in subsequent work.

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Fixed point theorem via uniform orbital control in tight QPCM spaces

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We prove a Banach-type fixed point theorem in *tight quasi-partial controlled metric* (tight QPCM) spaces. The distance-like map q is allowed to be non-symmetric and to admit nonzero self-distances; the triangle inequality is controlled by a function $\theta \geqslant 1$. Our proof does *not* pass through the symmetrization d_q; instead, we argue directly in q using: (i) geometric decay of the Picard "edge" terms $e_n = q(x_n, x_{n+1})$ from contractivity; (ii) a forward/backward telescoping scheme for $q(x_n, x_m)$ driven by the tight triangle; and (iii) a local endpoint estimate to identify the limit as a fixed point. The only quantitative hypothesis is a *uniform orbital bound* $\theta \leqslant C$ along the Picard orbit with $Ck < 1$ for the contraction constant $k \in [0, 1)$. Under two-sided q-completeness we obtain existence, uniqueness and Picard convergence to a fixed point u with $q(u, u) = 0$. We conclude with a concrete example of a genuinely quasi-partial, tight QPCM (non-symmetric, nonzero self-distance) and a contraction that satisfies all hypotheses with $\theta \equiv 1$.

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Introduction

Classical Banach's contraction principle [2] guarantees existence and uniqueness of a fixed point for a strict contraction on a complete metric space. In recent decades, numerous generalizations have relaxed various metric axioms to better model asymmetry, nonzero self-distances, or variable inflation of the triangle inequality. Foundational among these are the concepts of partial metric spaces, which allow for nonzero self-distances [5], and quasi-metric spaces, which handle asymmetry [6]. Building on these ideas, the triangle inequality has been generalized through various control mechanisms, leading to the development of controlled metric spaces and extended or generalized b-metric spaces [1, 3].

In this work we operate in a setting we call a *tight quasi-partial controlled metric space* (tight QPCM): a pair (q, θ) on X, where $q : X \times X \to [0, \infty)$ may be non-symmetric and may have $q(x, x) > 0$, the small self-distance axiom (Q1) holds, and the triangle inequality comes in a *tight* controlled form

$$q(x, y) \;\leqslant\; \theta(x, z, y)\big(q(x, z) + q(z, y) - q(z, z)\big), \qquad \theta \geqslant 1.$$

We also assume a seperation axiom (Sep) : $q(x, x) = q(x, y)$ and $q(y, y) = q(y, x)$ imply $x = y$. Our convergence/completeness notion is the standard *two-sided* q-*convergence* from the partial/quasi-partial literature, see e.g., [4]: a sequence $\{x_n\}$ converges to u if both $q(x_n, u)$ and $q(u, x_n)$ converge to $q(u, u)$, and completeness means every two-sided q-Cauchy sequence has such a limit.

A common approach is to pass to the symmetrization $d_q(x,y) = q(x,y) + q(y,x) - q(x,x) - q(y,y)$ and try to work metrically. However, d_q generally lacks a nice triangle inequality in this generality, and (X, d_q) need not be complete even when (X, q) is complete in the two-sided sense. To avoid these issues, we present a *direct* proof in the q-framework.

The main quantitative hypothesis beyond contractivity is a *uniform orbital control*: the control coefficient θ is bounded by a constant $C \geqslant 1$ on *triples drawn from the Picard orbit*, with $Ck < 1$ (where k is the contraction constant). This ensures that the telescoping inequalities generated by the tight triangle form a convergent geometric series. Under q-completeness, the orbit converges to some u with $q(u,u) = 0$; a final, localized bound involving the pairs (u, Tu) along the orbit forces $Tu = u$ via (Sep); uniqueness is immediate from contractivity.

Preliminaries

We fix the axioms and basic definitions used throughout.

Definition 1. *Let $X \neq \varnothing$, $q : X \times X \to [0, \infty)$ and $\theta : X^3 \to [1, \infty)$. We say (X, q, θ) is a* tight quasi-partial controlled metric space *if, for all $x, y, z \in X$,*

(Q1) *(*Small self-distance*) $q(x,x) \leqslant q(x,y)$ and $q(y,y) \leqslant q(x,y)$.*

(Sep) *(*Separation*) If $q(x,x) = q(x,y)$ and $q(y,y) = q(y,x)$, then $x = y$.*

(Tight) *(*Tight triangle*) $q(x,y) \leqslant \theta(x,z,y)\big(q(x,z) + q(z,y) - q(z,z)\big)$.*

A sequence (x_n) is two-sided q-Cauchy *if the two-variable limits* $\lim_{m,n} q(x_m, x_n)$ *and* $\lim_{m,n} q(x_n, x_m)$ *exist and are equal. The space is* q-complete *if every two-sided q-Cauchy sequence converges (two-sided) to some $u \in X$ and*

$$\lim_n q(x_n, u) = \lim_n q(u, x_n) = \lim_{m,n} q(x_m, x_n) = q(u,u).$$

Below is the definition of q-contraction and orbital control

Definition 2. *A mapping $T : X \to X$ is a q-*contraction *if $\exists\, k \in [0,1)$ such that*

$$q(Tx, Ty) \;\leqslant\; k\, q(x,y) \qquad (\forall x, y \in X). \tag{1}$$

Given $x_0 \in X$, define the Picard orbit $x_n := T^n x_0$. We assume a uniform orbital bound*:*

$$\sup\{\theta(x_i, x_j, x_\ell) : i, j, \ell \in \mathbb{N}_0\} \;\leqslant\; C \geqslant 1, \qquad \text{with } Ck < 1. \tag{2}$$

For the fixed-point identification we also require the endpoint local bound

$$\sup_{n \in \mathbb{N}_0} \{\theta(u, x_{n+1}, Tu),\ \theta(Tu, x_{n+1}, u)\} \;\leqslant\; C, \tag{3}$$

which is automatic if θ is globally bounded by C, else it is assumed explicitly.

Main results

Throughout, fix $x_0 \in X$ and write $x_n := T^n x_0$ for $n \in \mathbb{N}_0$. Introduce the basic shorthands

$$e_n := q(x_n, x_{n+1}) \qquad \text{and} \qquad d_n := q(x_n, x_n).$$

In the following lemma we prove that we have a control over diagonals and an edge decay.

Lemma 3. *For all $n \in \mathbb{N}_0$,*

$$e_{n+1} \leqslant k\, e_n, \qquad e_n \leqslant k^n e_0, \qquad 0 \leqslant d_n \leqslant e_n \leqslant k^n e_0.$$

Consequently, $e_n \to 0$ and $d_n \to 0$ as $n \to \infty$.

Proof. By (1) with $(x, y) = (x_n, x_{n+1})$,

$$e_{n+1} = q(Tx_n, Tx_{n+1}) \;\leqslant\; k\, q(x_n, x_{n+1}) = k\, e_n.$$

Inductively, $e_n \leqslant k^n e_0$. Using (Q1),

$$0 \leqslant d_n = q(x_n, x_n) \;\leqslant\; q(x_n, x_{n+1}) = e_n \;\leqslant\; k^n e_0.$$

Since $k \in [0, 1)$, $k^n e_0 \to 0$, hence $e_n \to 0$ and $d_n \to 0$.

We will now prove a forward telescoping estimate.

Lemma 4. *Assume* (2)*. Then for all integers $m > n \geqslant 0$,*

$$q(x_n, x_m) \;\leqslant\; \frac{C}{1 - Ck}\, k^n\, e_0.$$

Proof. Fix $m > n$ and set $A_j := q(x_j, x_m)$ for $j = n, \ldots, m$. Apply (Tight) with $(x, y, z) = (x_j, x_m, x_{j+1})$:

$$A_j \;\leqslant\; \theta(x_j, x_{j+1}, x_m)\Big(q(x_j, x_{j+1}) + q(x_{j+1}, x_m) - q(x_{j+1}, x_{j+1})\Big).$$

By (2), $\theta(x_j, x_{j+1}, x_m) \leqslant C$. Dropping $-q(x_{j+1}, x_{j+1}) \leqslant 0$ yields

$$A_j \;\leqslant\; C\big(e_j + A_{j+1}\big), \qquad j = n, \ldots, m - 1.$$

Unwinding the recurrence,

$$A_n \leqslant \sum_{i=0}^{m-n-1} C^{i+1} e_{n+i} + C^{m-n} A_m.$$

By (Q1), $A_m = q(x_m, x_m) \leqslant e_m$. Using Lemma 3, $e_{n+i} \leqslant k^{n+i} e_0$ and $e_m \leqslant k^m e_0$, hence

$$A_n \leqslant Ck^n e_0 \sum_{i=0}^{m-n-1} (Ck)^i + (Ck)^{m-n} k^n e_0 \;\leqslant\; \frac{C}{1 - Ck}\, k^n e_0.$$

Likewise, the following lemma establishes a backward telescoping estimate.

Lemma 5. *Under* (2)*, for all integers $m > n \geqslant 0$,*

$$q(x_m, x_n) \;\leqslant\; \frac{C}{1 - Ck}\, k^n\, e_0.$$

Proof. Fix $m > n$ and set $B_j := q(x_j, x_n)$ for $j = n, \ldots, m$. Apply (Tight) with $(x, y, z) = (x_j, x_n, x_{j+1})$:

$$B_j \leqslant \theta(x_j, x_{j+1}, x_n)\Big(q(x_j, x_{j+1}) + q(x_{j+1}, x_n) - q(x_{j+1}, x_{j+1})\Big).$$

Using (2) and dropping $-q(x_{j+1}, x_{j+1}) \leqslant 0$,

$$B_j \;\leqslant\; C\big(e_j + B_{j+1}\big), \qquad j = n, \ldots, m - 1.$$

Unwinding upward and using the base $B_n = q(x_n, x_n) \leqslant e_n$ (by (Q1)) gives

$$B_m \leqslant \sum_{i=0}^{m-n-1} C^{i+1} e_{m-1-i} + C^{m-n} e_n \;\leqslant\; Ck^n e_0 \sum_{i=0}^{m-n-1} (Ck)^i + (Ck)^{m-n} k^n e_0 \;\leqslant\; \frac{C}{1 - Ck}\, k^n e_0.$$

Below we prove that the Picard orbit is two sided q-Cauchy.

Lemma 6. *Assume* (2). *Then the Picard orbit* (x_n) *is two-sided q-Cauchy and*

$$\lim_{m,n\to\infty} q(x_m, x_n) = 0.$$

Proof. Given $\varepsilon > 0$, choose N such that $\dfrac{C}{1-Ck}k^N e_0 < \varepsilon$. Then for all $m > n \geqslant N$, Lemmas 4 and 5 imply

$$0 \leqslant q(x_n, x_m) < \varepsilon, \qquad 0 \leqslant q(x_m, x_n) < \varepsilon.$$

Thus both directed limits vanish, hence the two-variable limit is 0.

In the following lemma we prove the existence of a limit.

Lemma 7. *Assume q-completeness. Then there exists $u \in X$ such that*

$$\lim_{n\to\infty} q(x_n, u) = \lim_{n\to\infty} q(u, x_n) = \lim_{m,n\to\infty} q(x_m, x_n) = q(u, u) = 0.$$

Proof. By Lemma 6, (x_n) is two-sided q-Cauchy and $\lim_{m,n} q(x_m, x_n) = 0$. By q-completeness, there exists $u \in X$ with

$$\lim_n q(x_n, u) = \lim_n q(u, x_n) = \lim_{m,n} q(x_m, x_n) = q(u, u).$$

Therefore $q(u, u) = 0$ and both one-sided limits vanish.

The following lemma identifies the fixed point.

Lemma 8. *Assume the endpoint local bound* (3). *Then $Tu = u$.*

Proof. Apply (Tight) with $(x, y, z) = (u, Tu, x_{n+1})$:

$$q(u, Tu) \;\leqslant\; \theta(u, x_{n+1}, Tu)\Big(q(u, x_{n+1}) + q(x_{n+1}, Tu) - q(x_{n+1}, x_{n+1})\Big).$$

By (3), $\theta(u, x_{n+1}, Tu) \leqslant C$. We estimate each term:

- $q(u, x_{n+1}) \to q(u, u) = 0$ by Lemma 7;
- $q(x_{n+1}, Tu) = q(Tx_n, Tu) \leqslant k\, q(x_n, u) \to k\, q(u, u) = 0$ by (1) and Lemma 7;
- $q(x_{n+1}, x_{n+1}) = d_{n+1} \to 0$ by Lemma 3.

Hence $q(u, Tu) \leqslant C \cdot 0 = 0$, so $q(u, Tu) = 0$. A symmetric application with $(x, y, z) = (Tu, u, x_{n+1})$ and $\theta(Tu, x_{n+1}, u) \leqslant C$ gives $q(Tu, u) = 0$. By (Q1), $q(Tu, Tu) \leqslant q(Tu, u) = 0$, thus $q(Tu, Tu) = 0$. Using (Sep) with $x = u$ and $y = Tu$ (we have $q(u, u) = q(u, Tu) = 0$ and $q(Tu, Tu) = q(Tu, u) = 0$) yields $Tu = u$.

Now we prove that the fixed point is unique.

Lemma 9. *If u and v are fixed points of T, then $u = v$.*

Proof. If $Tu = u$ and $Tv = v$, then $q(u, v) = q(Tu, Tv) \leqslant k\, q(u, v)$ by (1). Since $k < 1$, $q(u, v) = 0$. Similarly $q(v, u) = 0$. By (Q1), $q(u, u) = q(v, v) = 0$, and (Sep) implies $u = v$.

Theorem 10. *Let (X, q, θ) satisfy* (Q1), (Sep), (Tight) *and be q-complete. Let $T : X \to X$ be a q-contraction* (1) *with constant $k \in [0, 1)$. Assume the orbital bound* (2) *with $Ck < 1$, and the endpoint local bound* (3). *Then the Picard orbit (x_n) converges in the two-sided q-sense to the unique fixed point u of T, and $q(u, u) = 0$.*

Proof. By Lemmas 4–6, the orbit is two-sided q-Cauchy with double limit 0. By q-completeness and Lemma 7, $x_n \to u$ with $q(u, u) = 0$. Lemma 8 gives $Tu = u$. Uniqueness is Lemma 9.

A Genuinely Quasi-Partial, Tight QPCM Example with a Non-Constant Control Function

We now provide an example that satisfies all the hypotheses of Theorem 10, featuring a genuinely quasi-partial structure and a non-constant control function θ.

Example 11. *Let $X = [0, \infty)$ and define the functions q and θ as follows:*

$$q(x, y) := x + |x - y|, \qquad \theta(x, z, y) := 1 + x.$$

We first verify that (X, q, θ) is a tight QPCM.

- (Q1) Small self-distance*: For any $x, y \in X$, $q(x, x) = x \leqslant x + |x - y| = q(x, y)$. Also, $q(y, y) = y \leqslant x + |x - y|$ since $y - x \leqslant |y - x| = |x - y|$.*
- (Sep) Separation*: If $q(x, x) = q(x, y)$, then $x = x + |x - y|$, which implies $|x - y| = 0$, so $x = y$. The second condition in (Sep) is then trivially satisfied.*
- (Tight) Tight triangle *with $\theta(x, z, y) = 1 + x$: We need to show $q(x, y) \leqslant (1 + x)(q(x, z) + q(z, y) - q(z, z))$. The right-hand side is:*

  $$(1 + x)\big((x + |x - z|) + (z + |z - y|) - z\big) = (1 + x)(x + |x - z| + |z - y|).$$

  By the standard triangle inequality, $|x - y| \leqslant |x - z| + |z - y|$. Therefore,

  $$q(x, y) = x + |x - y| \leqslant x + |x - z| + |z - y|.$$

  Since $x \geqslant 0$, we have $1 + x \geqslant 1$, which ensures that

  $$x + |x - z| + |z - y| \leqslant (1 + x)(x + |x - z| + |z - y|).$$

  Combining these inequalities confirms that the (Tight) axiom holds.

The space is a tight QPCM which is non-symmetric (since $q(x, y) \neq q(y, x)$ for $x \neq y$) and has nonzero self-distance $q(x, x) = x$.

Example 12. *On the space (X, q, θ) from the previous Example, define the mapping $T : X \to X$ by*

$$T(x) := \frac{1}{3}x.$$

For any $x, y \in X$, we have

$$q(Tx, Ty) = q\left(\frac{x}{3}, \frac{y}{3}\right) = \frac{x}{3} + \left|\frac{x}{3} - \frac{y}{3}\right| = \frac{1}{3}(x + |x - y|) = \frac{1}{3}q(x, y).$$

Thus, T is a q-contraction with constant $k = 1/3$. We now verify the uniform orbital bound condition (2). Let the initial point be $x_0 = 1$. The Picard orbit is $x_n = T^n x_0 = (1/3)^n$. The uniform orbital bound C is

$$C = \sup\{\theta(x_i, x_j, x_\ell) : i, j, \ell \in \mathbb{N}_0\} = \sup_{i \in \mathbb{N}_0}\{1 + x_i\}.$$

Since the sequence $\{x_n\}$ is decreasing, its supremum is achieved at $i = 0$. Therefore,

$$C = 1 + x_0 = 1 + 1 = 2.$$

We check the crucial condition $Ck < 1$:

$$Ck = (2)\left(\frac{1}{3}\right) = \frac{2}{3} < 1.$$

The condition is satisfied. The endpoint local bound (3) also holds, since the limit of the orbit is the fixed point $u = 0$, and $\theta(u, x_{n+1}, Tu) = \theta(0, x_{n+1}, 0) = 1 + 0 = 1 \leqslant C$. All hypotheses of Theorem 10 are met with a non-constant control function θ, and the theorem guarantees convergence to the unique fixed point $u = 0$.

Example 13. *On $X = [0, \infty)$ with q as in Example 2.9 ,but with $\theta(x, z, y) := 1$ define $T(x) := \alpha x$ with a fixed $\alpha \in (0, 1)$. Then, for all $x, y \in X$,*

$$q(Tx, Ty) = \alpha x + |\alpha x - \alpha y| = \alpha\,(x + |x - y|) = \alpha\, q(x, y).$$

Thus T is a q-contraction with constant $k = \alpha < 1$. Since $\theta \equiv 1$, the orbital bound holds with $C = 1$ and $Ck = \alpha < 1$, and the endpoint bound is automatic. The Picard orbit is $x_n = \alpha^n x_0$, so

$$e_n = q(x_n, x_{n+1}) = \alpha^n x_0 + |\alpha^n x_0 - \alpha^{n+1} x_0| = \alpha^n x_0 + \alpha^n (1-\alpha) x_0 = (2-\alpha)\alpha^n x_0 \to 0,$$

and $d_n = q(x_n, x_n) = \alpha^n x_0 \to 0$. By Theorem 10, the unique fixed point is $u = 0$ and $q(u, u) = 0$.

Conclusion

We gave a direct, q-native proof of a Banach-type fixed point theorem in tight QPCM spaces under a natural *uniform orbital control* on the triangle coefficient. The key technical step is a forward/backward telescoping scheme based on the tight triangle and the geometric decay of edge terms supplied by contractivity. No detour through the symmetrization d_q (nor any completeness of (X, d_q)) is required. The method adapts to other contraction patterns whenever the same two ingredients are available: (i) a one-step decay on the "edges" $q(x_n, x_{n+1})$ and (ii) an orbital control that keeps the accumulated triangle coefficients in a geometric regime.

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This paper is dedicated to Professor Manuel López-Pellicer on the occasion of his 81st birth anniversary.

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A study on asymptotic approximation of positive linear operators associated with Appell type polynomials

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In this presentation, we derive a linear positive sequence of operators using the generating function of a certain Appell-type polynomial. In the context of the Korovkin-type approximation, we compute its moments and central moments to ensure uniform convergence. We also estimate the rate of convergence for smooth functions and study the asymptotic behaviour of the operator sequence.

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KEYWORDS: Bivariate Szasz operators, Bernoulli polynomials, Voronovskaya theorem

Introduction

Approximation theory has played a central role in analysis since the late 19th century. One of the cornerstones of this field is the Weierstrass Approximation Theorem, which asserts that every continuous function defined on a closed and bounded interval can be uniformly approximated by polynomials. This fundamental result laid the groundwork for the systematic study of positive linear operators and their approximation properties. Building on Weierstrass' theorem, S. N. Bernstein introduced in 1912 the famous Bernstein polynomials, which provided a constructive proof of the theorem. These operators not only ensured uniform approximation but also introduced a probabilistic interpretation, which inspired many subsequent developments in approximation theory. Another milestone is Voronovskaya's theorem, which investigates the asymptotic behaviour of approximation processes. In particular, it provides precise information about the rate at which operators approximate a function, thereby linking approximation theory to asymptotic analysis. Within this rich framework, much attention has been devoted to constructing new classes of positive linear operators associated with orthogonal polynomials, special functions, and various generalizations. Among such generalizations, Appell-type polynomials play a prominent role. A polynomial sequence $\{p_k(x)\}_{k=0}^{\infty}$ is referred to as an Appell family, [1], if it fulfills the differential relation

$$\frac{d}{dx}p_k(x) = k\,p_{k-1}(x), \qquad k \geqslant 1.$$

Such families are completely characterized by the exponential generating function which provides a direct connection between the generating function and the coefficients of the sequence. Following [2], the construction of Appell sequences can be extended by considering the reciprocal of the generating function coefficient $A(t)$. In particular, replacing $A(t)$ with $1/A(t)$ gives rise to the so-called Bernoulli polynomials of order (-1). Starting from this definition, the operator underlying our work was first introduced in [4]. Inspired by recent progress on univariate and bivariate operators, we extend these ideas to a new framework. Earlier, [5] studied bivariate Szász-Kantorovich operators, [6] developed generalized bivariate Bernstein-type operators, and [7] analyzed univariate and bivariate Bernstein–Kantorovich operators with their GBS extensions. Building on these studies, our aim is to construct a bivariate analogue of a univariate operator we previously examined. For this purpose, we define a suitable domain, introduce a bivariate modulus of continuity to estimate convergence, and analyze both the rate of approximation and asymptotic behaviour. Our results provide a new class of bivariate operators associated with Appell-type polynomials, enriching multivariate approximation theory.

Results and Discussion

The definition of the operator under consideration was originally provided in [3]. In this section, we recall this operator and fundamental lemmas concerning its properties.

Definition 1. *[3] Let $\mathcal{S}_n : C([0,1]) \to C([0,1])$. For $n \in \mathbb{N}$ and $h \in C([0,1])$, define*

$$\mathcal{S}_n(g;x) := \frac{e^{-nx}}{e-1} \sum_{k=0}^{n} \frac{b_k(nx)}{k!} h(\frac{k}{n}) \tag{1}$$

where $\{b_k(x)\}_{k=0}^{\infty}$ are the adjoint Bernoulli polynomials, positive on $[0,\infty)$.

The following lemmas summarize properties of the operator $\mathcal{S}_n$, which will be used in the paper.

Lemma 2. *[3] The following equalities hold:*

$$\begin{aligned}
\mathcal{S}_n(1;x) &= 1,\\
\mathcal{S}_n(t;x) &= x + \frac{1}{n(e-1)},\\
\mathcal{S}_n(t^2;x) &= x^2 + \frac{e+1}{n(e-1)}x + \frac{1}{n^2},\\
\mathcal{S}_n(t^4;x) &= x^4 + \frac{6e-2}{n(e-1)}x^3 + \frac{13e-1}{n^2(e-1)}x^2 + \frac{11e+1}{n^3(e-1)}x + \frac{4e-1}{n^4(e-1)}.
\end{aligned}$$

Lemma 3. *[3] For the operators defined by (1), we have*

$$\begin{aligned}
\mathcal{S}_n(t-x;x) &= \frac{1}{n(e-1)},\\
\mathcal{S}_n\big((t-x)^2;x\big) &= \frac{x}{n} + \frac{1}{n^2},\\
\mathcal{S}_n\big((t-x)^4;x\big) &= \frac{3x^2}{n^2} + \frac{7e-3}{n^3(e-1)}\,x + \frac{4e-1}{n^4(e-1)}.
\end{aligned}$$

In the light of the definition and lemmas, we studied univariate positive linear operator to the bivariate setting. In doing so, we first construct an appropriate domain for the bivariate case that preserves the essential approximation properties while allowing for a natural generalization of the operator. Subsequently, we introduce a modulus of continuity that is suitable for bivariate functions, which enables us to rigorously analyze the rate of convergence of the proposed operator. By employing tools from approximation theory and functional analysis, we provide quantitative estimates for this rate of convergence in terms of the bivariate modulus of continuity. Finally, we investigate the asymptotic behaviour of the constructed bivariate operator. This analysis not only reveals the limiting properties of the approximation process but also establishes connections with classical results of Voronovskaya-type theorems in the bivariate framework. Through this study, we aim to contribute to the development of multivariate approximation theory by broadening the scope of operators associated with Appell-type polynomials.

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Uncovering the axial and periodic perturbations of chaotic solitons in the complex quintic Swift-Hohenberg equation

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In this research study, we explore and examine chaotic soliton solutions of the (1 + 1)-dimensional Complex Quintic Swift-Hohenberg Equation (CQSHE) via a potent analytical technique namely New Extended Direct Algebraic Method (NEDAM). This model is used to characterise complex pattern-forming dissipative systems. We have discovered that, under the condition that the coefficients are limited by specific relations, there are several chaotic soliton solutions to the targeted CQSHE which possess axial and periodic perturbations. Moreover, the offered transformative approach NEDAM converts CQSHE to a system of Nonlinear Ordinary Differential Equations (NODEs) by implementing a complex transformation. The extended Riccati NODE is then incorporated in order to assume a series form solution, which converts the resultant set NODEs into an algebraic system of equations. Solving this set of equations yields the soliton solutions in the form of exponential, rational, periodic, rational-hyperbolic and hyperbolic functional families. The perturbed dynamics of these chaotic soliton solutions are also graphically represented by a series of 3D and contour depictions. Our discoveries are significant because they lend light on the chaotic characteristic of the framework we are studying that allow us to comprehend its underpinning dynamics on a deeper level.

Keywords: Nonlinear partial differential equation, extended direct algebraic method, complex quintic Swift-Hohenberg equation, chaotic soliton, extended Riccati equation, axial and periodic perturbation

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On power of upper triangular Toeplitz matrix

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In this note, we apply combinatorial methods to compute powers of Upper Triangular Toeplitz matrix. This approach provides a systematic procedure for determining powers of companion or rational matrices, as well as more general matrix classes. Furthermore, generating functions and recurrence relations are the central tools in this computation, because they encode the iterative structure of the matrix and translate it into explicit formulas.

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KEYWORDS: Linear recurrence relations, matrix powers, Companion matrix, Jordan matrix, generating function, Binet formula, generalizations of the Fibonacci Sequence

Introduction

The study of powers of companion matrices lies at the intersection of linear algebra, combinatorics, and number theory. Computing R^m for a companion matrix R is not merely an algebraic exercise; rather, it is a key step in understanding the behavior of sequences defined by linear recurrence relations.

Two complementary tools play a decisive role here: generating functions and recurrence relations. On one hand, recurrence relations arise naturally from the defining polynomial of R and govern the entries of R^m. On the other hand, generating functions act as compact analytic encodings of these sequences, transforming their iterative structure into rational functions that reveal closed forms, asymptotics, and determinantal identities. Together, these two perspectives provide both computational efficiency and conceptual clarity, linking the abstract structure of matrix powers with explicit formulas that can be applied across a wide spectrum of mathematics.

The Fibonacci sequence has long attracted the attention of mathematicians and scientists, producing a vast collection of combinatorial identities since its introduction in the 13th century. Today, Fibonacci numbers and their generalizations appear in many areas of science and the arts, including physics, engineering, architecture, and design, due to their rich structural properties. This influence has inspired the introduction of many related sequences in the literature, such as Fibonacci polynomials, Lucas numbers and polynomials, Pell numbers and polynomials, Pell–Lucas numbers and polynomials, Jacobsthal numbers and polynomials, Jacobsthal–Lucas numbers and polynomials, and Chebyshev numbers and polynomials. Numerous authors have derived generating functions and identities for these generalizations, underscoring their central role in recurrence theory and matrix methods.

From a linear algebraic perspective, the connection between recurrence sequences and matrix theory is mediated through the structure of companion matrices. The existence of a diagonal or Jordan form is closely linked to the factorization of the minimal polynomial. These canonical forms, together with the rational canonical form, provide powerful frameworks for computing R^m and have important applications in analytic number theory and combinatorics.

Definition 1. *[9] Let p, q, a and b be elements of an an integral domain R. Then for a positive integer n, we define the recurrence relations*

$$w_{n+1} = pw_n + qw_{n-1}$$

and we say that

$$w = \{w_{n+1} = pw_n + qw_{n-1} : w_0 = a, w_1 = b, n \in \mathbb{Z}^+\}$$

is the sequence of p, q with a, b on an integral domain R denoted $w_n(a, b, p, q)$ or briefly $\{w_n\}$.

When R is Rational numbers in Definition 1, we get Horadam numbers in [9] with the initial conditions $w_0 = a$, $w_1 = b$. On the other hand, in the literature Horadam polynomials $h_n(x, a, b, p, q)$ (briefly h_n) was defined in the following

$$h_n = ph_{n-1} - qh_{n-2} \ (n \geqslant 3)$$

with the initial conditions $h_1 = a$, and $h_2 = bx$. Then clearly Horadam polynomials are obtained by the definition 1 when R is the polynomial ring of Rational numbers.

Let M be $n \times n$ matrix with entries from a field F. From [1], we recall the matrix

$$W(n) = \begin{bmatrix} w_{n+1} & w_n \\ w_n & w_{n-1} \end{bmatrix}.$$

Hence, we get that $W(0) = \begin{bmatrix} w_1 & w_0 \\ w_0 & q^{-1}(w_1 - pw_0) \end{bmatrix}$.

Lemma 2. *[1]For any integer n, we have*

$$W(n) = K.W(n-1)$$

where $K = \begin{bmatrix} p & q \\ 1 & 0 \end{bmatrix}$.

Proof. By the definition, we get that $w_{n+1} = pw_n + qw_{n-1}$ and $w_n = pw_{n-1} + qw_{n-2}$ for any integer n. Then it follows that $K.W(n-1) = W(n)$.

Now, we fix the notation $K = \begin{bmatrix} p & q \\ 1 & 0 \end{bmatrix}$.

If $w_0 = 0$ and $w_1 = 1$ then

$$W(n) = K(n) = \begin{bmatrix} k_{n+1} & k_n \\ k_n & k_{n-1} \end{bmatrix} \text{ and } W(0) = K(0) = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{q} \end{bmatrix}.$$

If $w_0 = 0$ and $q = 1$ then

$$W(n) = L(n) = \begin{bmatrix} l_{n+1} & l_n \\ l_n & l_{n-1} \end{bmatrix} \text{ and } L(0) = w_1 I.$$

Theorem 3. *[1]For any integer n, we have $K^n = k_n K + qk_{n-1}$ and*

$$K^n = \begin{bmatrix} k_{n+1} & qk_n \\ k_n & qk_{n-1} \end{bmatrix}.$$

Proof. Let $m(t) = t^2 - pt - q \in R[t]$. Then $m(t)$ is the minimal polynomial of K since q is non zero. Hence it follows that $K^2 = pK + q$ and so $K^2 = k_2K + k_1I$.

We use induction on n. Now we assume that it is holds for n. Then we have

$$\begin{aligned} K^{n+1} &= k_nKK + qk_{n-1}K \\ &= k_n(pK + q) + qk_{n-1}K \\ &= k_{n+1}K + qk_n. \end{aligned}$$

Now we focus on Upper triangular Toeplitz matrix. Let

$$T = \begin{bmatrix} a & b & c & d \\ 0 & a & b & c \\ 0 & 0 & a & b \\ 0 & 0 & 0 & a \end{bmatrix}$$

be Upper triangular Toeplitz matrix for elements a, b, c, d of a field. Then Upper triangular Toeplitz blocks of the form

$$T = aI + bN + cN^2 + dN^3, \qquad N = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad N^4 = 0,$$

have attracted significant attention in the literature. They appear as canonical models in the study of companion matrices, Hessenberg forms, and rational canonical forms. In particular, they provide explicit examples of single–eigenvalue operators whose powers can be described by polynomial–exponential sequences, mirroring the solution spaces of linear recurrence relations. Moreover, such matrices occur naturally in applications such as signal processing, combinatorial sequence analysis, and discretizations of differential operators, where banded Toeplitz and Hessenberg structures yield closed forms for transition operators and generating functions.

Theorem 4. *Let*

$$T = aI + bN + cN^2 + dN^3, \qquad N^4 = 0.$$

Then for all integers $n \geqslant 0$,

$$\begin{aligned} T^n = a^nI &+ \left(na^{n-1}b\right) N \\ &+ \left(na^{n-1}c + \binom{n}{2}a^{n-2}b^2\right) N^2 \\ &+ \left(na^{n-1}d + n(n-1)a^{n-2}bc + \binom{n}{3}a^{n-3}b^3\right) N^3. \end{aligned}$$

Proof. Write $S = bN + cN^2 + dN^3$, so $T = aI + S$. Since aI commutes with S, the binomial theorem gives $T^n = \sum_{j=0}^{n} \binom{n}{j} a^{n-j} S^j$. Because S is a polynomial in N with $N^4 = 0$, one has $S^4 = 0$, so the sum truncates at $j = 3$. A direct computation yields

$$S^2 = b^2N^2 + 2bc\,N^3, \qquad S^3 = b^3N^3.$$

Substituting into the expansion and collecting coefficients of N, N^2, N^3 gives the claimed formula.

Writing out the bands explicitly, we obtain

$$T^n = \begin{bmatrix} a^n & na^{n-1}b & na^{n-1}c + \binom{n}{2}a^{n-2}b^2 & na^{n-1}d + n(n-1)a^{n-2}bc + \binom{n}{3}a^{n-3}b^3 \\ 0 & a^n & na^{n-1}b & na^{n-1}c + \binom{n}{2}a^{n-2}b^2 \\ 0 & 0 & a^n & na^{n-1}b \\ 0 & 0 & 0 & a^n \end{bmatrix}.$$

Thus the diagonal entries are a^n, the first superdiagonal entries are $na^{n-1}b$, the second superdiagonal entries combine c and b^2 terms, and the third superdiagonal entries combine d, bc, and b^3 terms.

If we set $b = 1$, $c = d = 0$, then

$$T = aI + N = \begin{bmatrix} a & 1 & 0 & 0 \\ 0 & a & 1 & 0 \\ 0 & 0 & a & 1 \\ 0 & 0 & 0 & a \end{bmatrix} = J_4(a).$$

Hence T *is already in Jordan form* (a single Jordan block of size 4 with eigenvalue a). In particular,

$$\chi_T(x) = (x-a)^4, \qquad m_T(x) = (x-a)^4, \qquad \dim\ker(T - aI) = 1,$$

and T is not diagonalizable unless $N = 0$ (which never occurs here). The power formula specializes to

$$T^n = \sum_{k=0}^{3} \binom{n}{k} a^{n-k} N^k = \begin{bmatrix} a^n & na^{n-1} & \binom{n}{2}a^{n-2} & \binom{n}{3}a^{n-3} \\ 0 & a^n & na^{n-1} & \binom{n}{2}a^{n-2} \\ 0 & 0 & a^n & na^{n-1} \\ 0 & 0 & 0 & a^n \end{bmatrix}.$$

(For $a = 0$, this reduces to the nilpotent Jordan block N, with $T^4 = 0$.)

Let

$$q(x) = (x-a)^4 = x^4 - 4ax^3 + 6a^2x^2 - 4a^3x + a^4.$$

Its companion matrix. Then we have that

$$R := \begin{bmatrix} 0 & 0 & 0 & -a^4 \\ 1 & 0 & 0 & 4a^3 \\ 0 & 1 & 0 & -6a^2 \\ 0 & 0 & 1 & 4a \end{bmatrix}.$$

Thus $\chi_R(x) = (x-a)^4$, and every scalar sequence extracted linearly from R^m satisfies the quartic recurrence

$$u_{n+4} = 4a\,u_{n+3} - 6a^2\,u_{n+2} + 4a^3\,u_{n+1} - a^4\,u_n. \tag{1}$$

Let $J := aI_4 + N$ with

$$N = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad N^4 = 0.$$

As in the cubic case, $R \sim J$ (a single Jordan chain of length 4), and the binomial expansion yields the *quartic* version of equation (2):

$$J^m = \sum_{k=0}^{3} \binom{m}{k} a^{m-k} N^k = \begin{bmatrix} a^m & ma^{m-1} & \binom{m}{2}a^{m-2} & \binom{m}{3}a^{m-3} \\ 0 & a^m & ma^{m-1} & \binom{m}{2}a^{m-2} \\ 0 & 0 & a^m & ma^{m-1} \\ 0 & 0 & 0 & a^m \end{bmatrix}. \tag{2}$$

Consequently $R^m = P\,J^m\,P^{-1}$ for some invertible P.

Define $W(m) := R^m$ and let W_I be the initial Hankel block formed from (w_0, w_1, w_2, w_3) as in the general construction. Then the following are the $n = 4$ specializations of the standard identities.

Theorem 5. *For all integers m, m' one has:*

1. $W(m) = a_0\,W(m-1) + a_1\,W(m-2) + a_2\,W(m-3) + a_3\,W(m-4)$, *where*
$$(a_0, a_1, a_2, a_3) = \left(4a,\ -6a^2,\ 4a^3,\ -a^4\right);$$
2. $W(m) = R^m\,W_I$;
3. $W(m+m') = R^m\,W(m')$.

Proof. It is clear from the definitions

Let$M := R - aI_4$ and $v := e_1 = (1, 0, 0, 0)^{\mathsf{T}}$. Then a Jordan chain of length 4 is

$$v_0 := v, \quad v_1 := Mv, \quad v_2 := M^2 v, \quad v_3 := M^3 v,$$

with

$$v_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad v_1 = \begin{bmatrix} -a \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad v_2 = \begin{bmatrix} a^2 \\ -2a \\ 1 \\ 0 \end{bmatrix}, \quad v_3 = \begin{bmatrix} -a^3 \\ 3a^2 \\ -3a \\ 1 \end{bmatrix}.$$

We define P by stacking these vectors order:

$$P = \begin{bmatrix} v_3 & v_2 & v_1 & v_0 \end{bmatrix} = \begin{bmatrix} -a^3 & a^2 & -a & 1 \\ 3a^2 & -2a & 1 & 0 \\ -3a & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \qquad \det P = 1.$$

Then

$$P^{-1}RP = J \quad \text{and} \quad R = P\,J\,P^{-1}.$$

For later use, we also record the explicit inverse:

$$P^{-1} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 3a \\ 0 & 1 & 2a & 3a^2 \\ 1 & a & a^2 & a^3 \end{bmatrix}.$$

Proof. Since v_0, v_1, v_2, v_3 form a Jordan chain for $M = R - aI_4$ (i.e. $Mv_0 = v_1$, $Mv_1 = v_2$, $Mv_2 = v_3$, $Mv_3 = 0$), the similarity in the displayed basis follows from the standard Jordan reduction: in the basis $\{v_3, v_2, v_1, v_0\}$, the matrix of R is $J_4(a)$.

Theorem 6. *Let a be an elelemt of a fieald and $q(x) = (x-a)^4 = x^4 - 4ax^3 + 6a^2x^2 - 4a^3x + a^4$ and let R be the companion matrix of q. Then:*

1. *for every integers $m \geqslant 0$ and every matrix entry function $w_{ij}(m) := (R^m)_{ij}$, the sequence $(w_{ij}(m))_{m\geqslant 0}$ satisfies the quartic recurrence*

$$w_{ij}(m+4) = 4a\, w_{ij}(m+3) - 6a^2\, w_{ij}(m+2) + 4a^3\, w_{ij}(m+1) - a^4\, w_{ij}(m). \quad (3)$$

2. *Let S be$R - aI_4$. Then the binomial expansion is that*

$$R^n = \sum_{k=0}^{3} \binom{n}{k} a^{n-k} S^k \qquad (n \in \mathbb{Z}_{\geqslant 0}). \quad (4)$$

   Equivalently, with the basis sequences, we may write

$$R^n = a^n I_4 + n\, a^{n-1} S + 2!\binom{n}{2} a^{n-2} S^2 + 3!\binom{n}{3} a^{n-3} S^3.$$

3. *If $W(m) := R^m$ and W_I denotes the initial Hankel-type block built from (w_0, w_1, w_2, w_3) as in the general n-th order construction, then*

$$W(m) = R^m W_I, \qquad W(m+m') = R^m W(m') \quad (5)$$

   and each 4×4 contiguous Hankel minor scales by a factor a^{4m} under the shift $n \mapsto n+m$ (quartic analogue of the cubic case).

Proof. (1) By construction, $\chi_R(x) = q(x) = (x-a)^4$, hence the Cayley–Hamilton theorem yields $q(R) = 0$, . Taking (i,j)-entries of R^n, we get (3) for each fixed (i,j), exactly as in the cubic case of the note.

(2) Since $R = aI_4 + S$ and S is a polynomial in the nilpotent shift (Jordan) with $S^4 = 0$, the binomial theorem in the endomorphism ring truncates at $k = 3$, giving (4). The alternative form follows by grouping with $\kappa_n^{(k)}$.

(3) The identities (5) are the $n = 4$ specializations of the standard n-th order relations for the Hankel blocks and the companion dynamics; see the general statements proved for arbitrary order.

Let $w_m := (R^m)_{1,1}$ where R is the Frobenius companion matrix of $q(x) = (x-a)^4$. Then we have that

Corollary 7. (1) *The sequence w_m satisfies in (1)*

$$w_{m+4} = 4a\, w_{m+3} - 6a^2\, w_{m+2} + 4a^3\, w_{m+1} - a^4\, w_m, \qquad m \geqslant 0,$$

with initial conditions

$$w_0 = 1, \qquad w_1 = 0, \qquad w_2 = 0, \qquad w_3 = 0.$$

(2)

$$w_m = \left(1 - m + \binom{m}{2} - \binom{m}{3}\right) a^m, \qquad m \geqslant 0.$$

Theorem 8. *Let $a \in F$, and let $(w_n)_{n\geqslant 0}$ be sequance in (1) with initial values $w_0, w_1, w_2 \in F$. If $W(x) := \sum_{n\geqslant 0} w_n x^n$ denotes the generating function, then*

$$W(x) = \frac{w_0 + (w_1 - 3a\, w_0)x + (w_2 - 3a\, w_1 + 3a^2 w_0)x^2}{1 - 3ax + 3a^2x^2 - a^3x^3}.$$

Proof. Let $W(x) = \sum_{n \geqslant 0} w_n x^n$. W Multiplying both sides of the recurrence by x^{m+4} and summing over $m \geqslant 0$ gives

$$\big(1 - 4ax + 6a^2x^2 - 4a^3x^3 + a^4x^4\big)W(x) = w_0 + (w_1 - 4aw_0)x + (w_2 - 4aw_1 + 6a^2w_0)x^2 \\ +(w_3 - 4aw_2 + 6a^2w_1 - 4a^3w_0)x^3$$

Then we get that

$$W(x) = \frac{w_0 + \big(w_1 - 3a\,w_0\big)x + \big(w_2 - 3a\,w_1 + 3a^2w_0\big)x^2}{1 - 3ax + 3a^2x^2 - a^3x^3}.$$

If we get the initials $(w_0, w_1, w_2, w_3) = (1, 0, 0, 0)$, we have that

$$W(x) = \frac{1 - 3ax + 3a^2x^2}{1 - 3ax + 3a^2x^2 - a^3x^3}.$$

Conversely, expanding the rational expression as a formal series and equating coefficients reproduces the same recurrence and initial values, so the correspondence is exact.
The generating function

$$W(x) = \sum_{n \geqslant 0} w_n x^n$$

plays a fundamental role in analyzing the powers of the companion matrix R. The sequence (w_n) is generated by the action of R on the initial state vector, so the coefficients of $W(x)$ encode the entire sequence of iterates of R. Expressing $W(x)$ as a rational function immediately shows that (w_n) satisfies a linear recurrence whose coefficients coincide with those of the characteristic polynomial of R. From the algebraic and analytic structure of $W(x)$, one can derive closed formulas, asymptotic behavior, and determinantal identities. In this way, the generating function serves as a bridge between the abstract notion of the matrix power R^m and explicit formulas for the sequence (w_n), providing both an effective computational tool and a conceptual explanation of how recurrence relations are governed by the minimal polynomial of R.

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Evolving relocation rules for container relocation problem with genetic programming

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The Container Relocation Problem is a combinatorial optimization problem of great importance in maritime container yards. This problem is NP-hard, and as such, it can only be solved exactly for small instances or cases with specific characteristics. Therefore, heuristic approaches are most commonly used to address it, ranging from complex metaheuristics to simple heuristics such as relocation rules. Although relocation rules are simple heuristics whose solution quality generally cannot match that of complex metaheuristics, they are frequently used in practice due to their speed and applicability in dynamic environments. In this talk, we provide an overview of the automated development of relocation rules for this problem using genetic programming.

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Keywords: Container relocation problem, genetic programming, relocation rules, training models

Introduction

The importance of international goods transport has been increasing annually, and consequently, storage and loading in ports have become key factors in maritime transport [7]. The Container Relocation Problem (CRP) [8] is a crucial problem in optimising storage. During storage, due to space limitations, containers are stacked side by side and on top of each other, thus forming S stacks and H tiers. Since containers are typically loaded in a specific order, which is often not known in advance, situations frequently arise where the next container to be retrieved is blocked by other containers. These blocking containers must be relocated first to access the target one.

This problem is NP-hard, and numerous methods have been developed to address it. Due to its NP classification, the problem cannot be solved exactly in most cases; therefore, heuristic approaches are employed, ranging from complex metaheuristics [10, 2] to simple heuristics, such as Relocation Rules (RRs) [1, 9, 11]. Although RRs cannot match the solution quality of metaheuristics, they are widely used in practice due to their simplicity, speed, and applicability in dynamic environments.

RRs consist of two components: a relocation scheme (RS) and a priority function (PF). The RS is a general framework that ensures all constraints are satisfied. Specifically, if the next target container is located at the top of its stack, it is retrieved; otherwise, the RS decides where to relocate the blocking containers. This decision is guided by the PF, which assigns a priority to each stack with available space. The stack with the highest or lowest priority is chosen depending on the rule applied. In the literature, expert-designed RRs can be found; however, the development process is typically reduced to lengthy trial-and-error procedures.

Since Genetic Programming (GP) has been successfully applied as a hyper-heuristic for automatically developing scheduling rules in various problems, the idea is to use this approach to the CRP as well. In this talk, we present an overview of applying GP to the development of RRs and discuss the obtained results.

Application of GP to the Development of Relocation Rules

To develop relocation rules (RRs), it is necessary to select a solution representation and provide the algorithm with building blocks for constructing rules. Genetic Programming typically represents solutions using a flexible tree structure, where the set of elements that can appear in nodes forms the primitive set.

The primitive set consists of a function set and a terminal set. The function set contains operators that connect building elements, while the terminal set consists of problem-specific features. Function set elements are placed in non-leaf nodes in the tree, whereas terminals are placed in leaf nodes. In [3], basic arithmetic operations were used as the function set, along with six terminals.

The relocation rules developed using GP were compared with existing expert-designed RRs on well-known benchmarks. The results demonstrated that GP-generated rules outperform existing ones, providing strong motivation for further research and subsequent studies that extended this approach to other CRP variants and different GP settings [4, 5, 6].

Conclusion

This talk has provided only a partial overview of an area that has recently attracted a lot of researchers - the application of GP as a hyper-heuristic to the CRP. The achieved results indicate that GP offers a promising approach for the automated development of relocation rules. We expect this field to continue developing, and we also anticipate that various machine learning methods will be increasingly applied within these approaches.

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Second-order cone programming for robust and sparse feature selection with ℓ_p-quasi-norms

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Support Vector Machines (SVM) are widely used in classification tasks but are sensitive to outliers and uncertainty in the data. To address this, we consider a robust variant of the classical SVM formulation by incorporating chance constraints on the data distributions of each class. These constraints are converted into deterministic second-order cone constraints via the multivariate Chebyshev inequality.

In this work, we propose a non-convex extension of this robust formulation by replacing the standard Euclidean norm with a concave pseudo-ℓ_p-norm ($0 < p < 1$) in the objective function. This promotes sparsity in the weight vector, effectively enabling feature selection while maintaining robustness (Chen et al., 2010; Tan et al., 2013; Tian et al., 2010). The resulting problem is a non-convex second-order cone program (SOCP), which is challenging to solve with standard convex optimization tools.

To tackle this, we develop an iterative algorithm tailored to this structure and prove its convergence under mild conditions, which has been inspired by Chen et al., 2014. Our method is particularly well-suited for high-dimensional classification problems where interpretability and robustness are crucial.

We validate the approach on real-world medical datasets related to cancer detection. The numerical experiments demonstrate that our model achieves high classification accuracy with a reduced number of features, offering a competitive and interpretable alternative to classical and robust SVM approaches.

This work contributes to bridging the gap between robust optimization and sparse learning, with promising applications in medical diagnostics and other domains involving uncertainty and high dimensionality.

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Keywords: Robust optimization, feature selection, support vector machine, second-order cone, nonconvex programming, cancer detection

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Extended codiskcyclic C_0-semigroups and their properties

Mansooreh Moosapoor

In this paper, we introduce a new concept in the structure of C_0-semigroups that is named extended codiskcyclicity, and we state some results about it. A C_0-semigroup $(T_t)_{t\geqslant 0}$ on a complex separable Banach space X is called an extended codiskcyclic C_0-semigroup if there exists $x \in X$, and a real number $r > 1$ so that $Eorb((T_t)_{t\geqslant 0}, x, r) = \{\beta T_t x : \beta \in \mathbb{C}, |\beta| \geqslant r, t \geqslant 0\}$ is dense in X. We establish that if x is an extended codiskcyclic vector for $(T_t)_{t\geqslant 0}$, then $T_p x$ for any $p > 0$ is an extended codiskcyclic vector for $(T_t)_{t\geqslant 0}$. Also, we prove that for a C_0-semigroup $(T_t)_{t\geqslant 0}$, extended codiskcyclicity is equivalent to codiskcyclicity. Moreover, we mention that an element x is an extended codiskcyclic vector for $(T_t)_{t\geqslant 0}$ if and only if x is a codiskcyclic vector for it.

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KEYWORDS: Extended codiskcyclicity, codiskcyclicity, diskcyclicity, semigroups

Introduction

There are some well known concepts for operators in the theory of dynamical systems. Assume that X is a separable complex Banach space, and $B(X)$ is the set of bounded linear operators on X. An operator $T \in B(X)$ is cyclic if $span\{T^n x : n \in \mathbb{N} \cup \{0\}\}$ is dense in X, and it is said supercyclic if $\{\alpha T^n x : \alpha \in \mathbb{C}, n \in \mathbb{N} \cup \{0\}\}$ is dense in X [3]. For decades Much research has been conducted on these operators and related concepts, some of which can be seen in [7].

Some newer concepts like diskcyclicity and codiskcyclicity are defined lately. An operator $T \in B(X)$ is diskcyclic if $\{\alpha T^n x : |\alpha| \leqslant 1, n \in \mathbb{N} \cup \{0\}\}$ is dense in X [2]. This concept first introduced in [5], and investigated more in [4] and [2]. An operator $T \in B(X)$ is codiskcyclic if there exists $x \in X$ so that $\{\alpha T^n x : |\alpha| \geqslant 1, n \in \mathbb{N} \cup \{0\}\}$ is dense in X [5].

This concepts are defined for C_0-semigroups, either. Recall that $(T_t)_{t\geqslant 0}$ is a C_0-semigroup on X if the operators T_t on X have the following properties:

(1) $T_0 = I$,

(2) for any $t, s \geqslant 0$ and $x \in X$, $T_{t+s} = T_t T_s$,

(3) for any $t \geqslant 0$ and $x \in X$, $\lim_{s\to t} T_s x = T_t x$.

If for a vector $x \in X$, $\mathbb{D}orb((T_t)_{t\geqslant 0}, x) = \{\alpha T_t x : t \geqslant 0, |\alpha| \leqslant 1\}$ is dense in X, then $(T_t)_{t\geqslant 0}$ on X is called a diskcyclic C_0-semigroup [1]. One can see various criteria for diskcyclicity of C_0-semigroups in [8].

A C_0-semigroup $(T_t)_{t\geqslant 0}$ on X is named codiskcyclic if there exists $x \in X$ so that $\{\alpha T_t x : |\alpha| \geqslant 1, t \geqslant 0\}$ is dense in X [1]. It is proved in [1] that these C_0-semigroups can be found only on spaces with $dim(X) = 1$ or $dim(X) = \infty$. one can also see [6] for newer concepts.

In this paper, we define a new concept for C_0-semigroups that is named extended codiskcyclicity. We investigate properties of these C_0-semigroups and study their relations with codiskcyclic C_0-semigroups that are stated in Section 2 of this paper.

Main results

We begin this section with our main definition.

Definition 1. *Suppose $(T_t)_{t\geqslant 0}$ is a C_0-semigroup on X. Then $(T_t)_{t\geqslant 0}$ is called an extended codiskcyclic C_0-semigroup if there exists $x \in X$, and a real number $r > 1$ so that*

$$Eorb((T_t)_{t\geqslant 0}, x, r) = \{\beta T_t x : \beta \in \mathbb{C}, |\beta| \geqslant r, t \geqslant 0\}$$

is dense in X. In this case, x is named an extended codiskcyclic vector for $(T_t)_{t\geqslant 0}$.

Lemma 2. *Suppose $(T_t)_{t\geqslant 0}$ is a C_0-semigroup on X. If for some $x \in X$ and $r_0 > 1$, $Eorb((T_t)_{t\geqslant 0}, x, r_0)$ is dense in X, then for any $1 < r < r_0$, $Eorb((T_t)_{t\geqslant 0}, x, r)$ is dense in X.*

Proof. Consider $1 < r < r_0$. Then

$$\{\beta T_t x : \beta \in \mathbb{C}, |\beta| \geqslant r_0, t \geqslant 0\} \subseteq \{\beta T_t x : \beta \in \mathbb{C}, |\beta| \geqslant r, t \geqslant 0\}.$$

Therefore,

$$Eorb((T_t)_{t\geqslant 0}, x, r_0) \subseteq Eorb((T_t)_{t\geqslant 0}, x, r). \tag{1}$$

By hypothesis, $Eorb((T_t)_{t\geqslant 0}, x, r_0)$ is dense in X. Hence, (1) asserts that $Eorb((T_t)_{t\geqslant 0}, x, r)$ is also dense in X.

Proposition 3. *Suppose $(T_t)_{t\geqslant 0}$ is a C_0-semigroup on X. Assume there exists $x \in X$, and $r > 1$ so that*

$$\overline{\{\beta T_t x : \beta \in \mathbb{C}, |\beta| \geqslant r, t \geqslant 0\}} = X.$$

Then for any non-empty open set U of X the set

$$\{t \in \mathbb{R}^{\geqslant 0} : \alpha T_t x \in U \text{ for some } \alpha \in \mathbb{C}, |\alpha| \geqslant r\}$$

is an infinite set.

Proof. Assume $U \subseteq X$ is a non-empty open set. By hypothesis, there exists $\alpha \in \mathbb{C}$ with $|\alpha| \geqslant r$, and $s \geqslant 0$ so that $\alpha T_s x \in U$. Hence, $x \in T_s^{-1}(\alpha^{-1}U)$. Therefore, $T_s^{-1}(\alpha^{-1}U)$ is a non-empty open set. Another by hypothesis, there exists $\beta \in \mathbb{C}$ with $|\beta| \geqslant r$, and $p \geqslant 0$ so that $\beta T_p x \in T_s^{-1}(\alpha^{-1}U)$. Hence $\alpha\beta T_p T_s x \in U$. Since, $T_p T_s = T_{p+s}$, this indicates

$$\alpha\beta T_{p+s} x \in U. \tag{2}$$

Consider $\gamma = \alpha\beta$ and $q = p + s$. Then $|\gamma| = |\alpha\beta| \geqslant r^2 \geqslant r$. Hence, $\gamma T_q x \in U$, and $q > s$. Consequently, we can find infinite real numbers that belong to the set that is specified.

By using Proposition 3 we show in the next theorem that while we find an extended codiskcyclic vector for a C_0-semigroup, we have many of them.

Theorem 4. *Suppose $(T_t)_{t\geqslant 0}$ is a C_0-semigroup on X. If $x \in X$ is an extended codiskcyclic vector for $(T_t)_{t\geqslant 0}$, then $T_p x$ for any $p > 0$ is an extended codiskcyclic vector for $(T_t)_{t\geqslant 0}$, either.*

Proof. By hypothesis, $x \in X$ is an extended codiskcyclic vector for $(T_t)_{t\geqslant 0}$. Hence, $r > 1$ exists so that $\{\beta T_t x : \beta \in \mathbb{C}, |\beta| \geqslant r, t \geqslant 0\}$ is dense in X. Consider $p > 0$. Assume $U \subseteq X$ is a non-empty open set. Therefore, there exists $\alpha \in \mathbb{C}$ with $|\alpha| \geqslant r$, and $s \geqslant 0$ so that

$$\alpha T_s x \in U. \tag{3}$$

By Proposition 3, we can assume that $s > p$. So, we can write $s = (s-p)+p$. Hence, it is concluded from (3) that

$$\alpha T_{(s-p)+p} x \in U. \tag{4}$$

Consequently, $\alpha T_{(s-p)} T_p x \in U$. If consider $q = s - p$, then $\alpha T_q(T_p x) \in U$. Therefore, $T_p x$ is an extended codiskcyclic vector for $(T_t)_{t\geqslant 0}$.

We establish in the next theorem that the concept of extended codiskcyclicity and the concept of codiskcyclicity is equivalent.

Theorem 5. *Suppose $(T_t)_{t\geqslant 0}$ is a C_0-semigroup on X. Then $(T_t)_{t\geqslant 0}$ is an extended codiskcyclic C_0-semigroup if and only if $(T_t)_{t\geqslant 0}$ is a codiskcyclic C_0-semigroup.*

Proof. Assume $(T_t)_{t\geqslant 0}$ is an extended codiskcyclic C_0-semigroup. Hence, there exists $x \in X$, and $r > 1$ so that

$$\overline{\{\beta T_t x : \beta \in \mathbb{C}, |\beta| \geqslant r, t \geqslant 0\}} = X. \tag{5}$$

Assume $y \in X$, and $\varepsilon > 0$. By (5), there exists $\alpha \in \mathbb{C}$ with $|\alpha| \geqslant r$, and $t \geqslant 0$ so that $\|\alpha T_t x - y\| < \varepsilon$. But $|\alpha| \geqslant r$ and $r > 1$. So, there exists $\alpha \in \mathbb{C}$ with $|\alpha| \geqslant 1$ so that

$$\|\alpha T_t x - y\| < \varepsilon.$$

That means $(T_t)_{t\geqslant 0}$ is codiskcyclic since y is an arbitrary element of X.

Now, presume that $(T_t)_{t\geqslant 0}$ is codiskcyclic. Hence, there exists $x \in X$ so that

$$\overline{\{\beta T_t x : \beta \in \mathbb{C}, |\beta| \geqslant 1, t \geqslant 0\}} = X. \tag{6}$$

Assume $y \in X$, and $\varepsilon > 0$. Consider $r_0 > 1$. By (6), there is $\alpha \in \mathbb{C}$ with $|\alpha| \geqslant 1$, and $t \geqslant 0$ so that $\|\alpha T_t x - \frac{y}{r_0}\| < \frac{\varepsilon}{r_0}$. Hence

$$\|(r_0\alpha) T_t x - y\| < \varepsilon. \tag{7}$$

But $|r_0\alpha| = r_0|\alpha| \geqslant r_0$. So, if we consider $\beta := r_0\alpha$, then $|\beta| \geqslant r_0$ and

$$\|\beta T_t x - y\| < \varepsilon.$$

That means $(T_t)_{t\geqslant 0}$ is extended codiskcyclic since y is an arbitrary element of X.

Considering the process of proving Theorem 5, we can state the following result.

Corollary 6. *Suppose $(T_t)_{t\geqslant 0}$ is a C_0-semigroup on X. Then x is an extended codiskcyclic vector for $(T_t)_{t\geqslant 0}$ if and only if x is a codiskcyclic vector for it.*

Conclusion

We defined the concept of extended codiskcyclicity and investigated its properties in this paper. We studied the relations between extended codiskcyclic and codiskcyclic C_0-semigroups. Theorem 5 stated that a codiskcyclic C_0-semigroup is an extended codiskcyclic C_0-semigroup. Therefore, we can generalize a wide range of research conducted on codiskcyclic C_0-semigroups to extended codiskcyclic C_0-semigroups. We suggest that the direct sum of these C_0-semigroups be examined. Also, their relationship with other types of C_0-semigroups be investigated.

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Explicit rate of convergence in strong laws of large numbers for random variables with double indices

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In classical probability theory, the strong law of large numbers (SLLN) establishes conditions under which sample averages converge almost surely to the expected value. Motivated by the proof mining approach and recent developments in convergence rate analysis by Neri (2025), this paper investigates quantitative versions of the SLLN for sequences of random variables indexed by pairs of natural numbers, i.e., double-indexed families. By constructing a suitable family of rate functions based on parametrized curves in $\mathbb{R}^2_+$, we derive explicit convergence rates for double-indexed random variables. We show that the rate of convergence along a specific subsequence implies the rate of convergence of the whole sequence.

Further, we apply these general results to obtain convergence rates in the SLLN for both pairwise independent and quasi-uncorrelated random variables, the latter being a broader class which includes pairwise negatively dependent sequences.

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KEYWORDS: Laws of large numbers, double indices, rate of convergence

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A study on degenerate Peters-type Simsek polynomials of the second kind using p-adic integrals

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In this paper, we focus on the higher order degenerate Peters-type Simsek polynomials of the second kind. By applying p-adic integrals, including the Volkenborn integral and the fermionic p-adic integral, to the formula given in [3], we obtain some identities related to the Daehee and Changhee numbers of the first kind. Finally, we also present relations involving the Bernoulli numbers and the Euler numbers.

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KEYWORDS: Stirling numbers, Bernoulli and Euler numbers, Daehee numbers and Changhee numbers, degenerate Peters-type Simsek numbers and polynomials, generating functions, p-adic integrals

Introduction

The following notations and definitions will be used in the throughout of this paper:

Let $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{Z}_p$ denote the set of positive integers, integers, real numbers, complex numbers and p-adic integers, respectively. Also, let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

The Stirling numbers of the first kind are defined by

$$\frac{(\log(1+t))^k}{k!} = \sum_{v=0}^{\infty} S_1(v,k) \frac{t^v}{v!} \tag{1}$$

and

$$(x)_v = \sum_{k=0}^{v} S_1(v,k) x^k,$$

where $(x)_v = x(x-1)(x-2)\dots(x-v+1)$ and $k \in \mathbb{N}_0$ (*cf.* [1]- [16]).

The higher order degenerate Peters-type Simsek numbers of the second kind $Y_{v,2}^{(k)}(\lambda \mid \eta)$, and the higher order degenerate Peters-type Simsek polynomials of the second kind $Y_{v,2}^{(k)}(x;\lambda \mid \eta)$, are defined by

$$\left(\frac{2}{\frac{\lambda^2}{\eta}\log(1+\eta t) + 2(\lambda-1)}\right)^k = \sum_{v=0}^{\infty} Y_{v,2}^{(k)}(\lambda \mid \eta) \frac{t^v}{v!} \tag{2}$$

and

$$\left(\frac{2}{\frac{\lambda^2}{\eta}\log(1+\eta t) + 2(\lambda-1)}\right)^k \left(1+\frac{\lambda}{\eta}\log(1+\eta t)\right)^x = \sum_{v=0}^{\infty} Y_{v,2}^{(k)}(x;\lambda \mid \eta) \frac{t^v}{v!} \tag{3}$$

where $k \in \mathbb{N}_0$, $\lambda \in \mathbb{C}$ (or $\mathbb{R}$) and $\eta \in \mathbb{R} \setminus \{0\}$ (*cf.* [3]).

When $x = 0$ in (3), we have

$$Y_{v,2}^{(k)}(0; \lambda \mid \eta) = Y_{v,2}^{(k)}(\lambda \mid \eta).$$

For $\eta \to 0$, the Eqs. (2) and (3) reduce to the higher order Peters-type Simsek numbers of the second kind $Y_{v,2}^{(k)}(\lambda)$ and the higher order Peters-type Simsek polynomials of the second kind $Y_{v,2}^{(k)}(x; \lambda)$:

$$\lim_{\eta \to 0} Y_{v,2}^{(k)}(\lambda \mid \eta) = Y_{v,2}^{(k)}(\lambda) \quad \text{and} \quad \lim_{\eta \to 0} Y_{v,2}^{(k)}(x; \lambda \mid \eta) = Y_{v,2}^{(k)}(x; \lambda),$$

for more details, see [3, 11], and also [14].

We briefly recall some definitions and properties of p-adic integrals, which are significant in p-adic calculus, mathematics, and mathematical physics.

Let $\mathbb{K}$ be field with a complete valuation and $C^1(\mathbb{Z}_p \to \mathbb{K})$ be set of continuous derivative functions:

$$\left\{ f : \mathbb{Z}_p \to \mathbb{K};\ f(x) \text{ is differentiable and } \frac{d}{dx} f(x) \text{ is continuous} \right\}$$

(*cf.* [12]).

The Volkenborn integral (or the bosonic p-adic integral) is defined by

$$\int_{\mathbb{Z}_p} f(x)\, d\mu_1(x) = \lim_{N \to \infty} \frac{1}{p^N} \sum_{x=0}^{p^N - 1} f(x),$$

where $\mu_1(x)$ denotes the Haar distribution $\mu_1(x) = \mu_1\left(x + p^N \mathbb{Z}_p\right) = \frac{1}{p^N}$ (*cf.* [5, 6, 12, 13, 15]).

Using the Volkenborn integral, the Daehee numbers of the first kind are given by

$$D_n = \frac{(-1)^n n!}{n+1} = \int_{\mathbb{Z}_p} (x)_n\, d\mu_1(x), \tag{4}$$

where $n \in \mathbb{N}_0$ (*cf.* [12]; see also [13, 15]).

The fermionic p-adic integral is defined by

$$\int_{\mathbb{Z}_p} f(x)\, d\mu_{-1}(x) = \lim_{N \to \infty} \sum_{x=0}^{p^N - 1} (-1)^x f(x),$$

where $\mu_{-1}(x) = \mu_{-1}\left(x + p^N \mathbb{Z}_p\right) = (-1)^x$ (*cf.* [7]- [9], [13, 15]).

Using the fermionic p-adic integral, the Changhee numbers of the first kind are given by

$$Ch_n = \frac{(-1)^n n!}{2^n} = \int_{\mathbb{Z}_p} (x)_n\, d\mu_{-1}(x), \tag{5}$$

where $n \in \mathbb{N}_0$ (*cf.* [12]; see also [13, 15]).

Main results

In this section, we present some identities and relations for the polynomials $Y_{v,2}^{(k)}(x;\lambda \mid \eta)$ by using the p-adic integrals techniques. These results also involve the numbers $Y_{v,2}(\lambda \mid \eta)$, the Bernoulli numbers, the Euler numbers, the Changhee numbers of the first kind, the Daehee numbers of the first kind and the Stirling numbers of the first kind.

In [3, Theorem 2.5.], the author gave the following identity including the polynomials $Y_{v,2}^{(k)}(x;\lambda \mid \eta)$, the numbers $Y_{v,2}^{(k)}(\lambda \mid \eta)$ and the numbers $S_1(v,k)$ as follows:

$$Y_{v,2}^{(k)}(x;\lambda \mid \eta) = \sum_{r=0}^{v}\binom{v}{r}\sum_{m=0}^{v-r}(x)_m\, Y_{r,2}^{(k)}(\lambda \mid \eta)\, S_1(v-r,m)\,\lambda^m\eta^{v-r-m}. \tag{6}$$

By applying the Volkenborn integral to the Eq. (6), we get

$$\begin{aligned}\int_{\mathbb{Z}_p} Y_{v,2}^{(k)}(x;\lambda \mid \eta)\, d\mu_1(x) &= \sum_{r=0}^{v}\binom{v}{r}\sum_{m=0}^{v-r} Y_{r,2}^{(k)}(\lambda \mid \eta)\\ &\quad\times S_1(v-r,m)\,\lambda^m\eta^{v-r-m}\int_{\mathbb{Z}_p}(x)_m\, d\mu_1(x).\end{aligned}$$

Combining the above equation with the Eq. (4), we arrive at the following theorem:

Theorem 1. *For $v,k \in \mathbb{N}_0$, we have*

$$\int_{\mathbb{Z}_p} Y_{v,2}^{(k)}(x;\lambda \mid \eta)\, d\mu_1(x) = \sum_{r=0}^{v}\binom{v}{r}\sum_{m=0}^{v-r} Y_{r,2}^{(k)}(\lambda \mid \eta)\, S_1(v-r,m)\,\frac{\lambda^m D_m}{\eta^{r+m-v}}$$

or equivalently

$$\int_{\mathbb{Z}_p} Y_{v,2}^{(k)}(x;\lambda \mid \eta)\, d\mu_1(x) = \sum_{r=0}^{v}\binom{v}{r}\sum_{m=0}^{v-r} Y_{r,2}^{(k)}(\lambda \mid \eta)\, S_1(v-r,m)\,\frac{(-\lambda)^m\eta^{v-r-m}m!}{m+1}.$$

By applying the fermionic p-adic integral to the Eq. (6), we obtain

$$\begin{aligned}\int_{\mathbb{Z}_p} Y_{v,2}^{(k)}(x;\lambda \mid \eta)\, d\mu_{-1}(x) &= \sum_{r=0}^{v}\binom{v}{r}\sum_{m=0}^{v-r} Y_{r,2}^{(k)}(\lambda \mid \eta)\\ &\quad\times S_1(v-r,m)\,\lambda^m\eta^{v-r-m}\int_{\mathbb{Z}_p}(x)_m\, d\mu_{-1}(x).\end{aligned}$$

Combining the above equation with the Eq. (5), we obtain the following theorem:

Theorem 2. *For $v,k \in \mathbb{N}_0$, we have*

$$\int_{\mathbb{Z}_p} Y_{v,2}^{(k)}(x;\lambda \mid \eta)\, d\mu_{-1}(x) = \sum_{r=0}^{v}\binom{v}{r}\sum_{m=0}^{v-r} Y_{r,2}^{(k)}(\lambda \mid \eta)\, S_1(v-r,m)\,\frac{\lambda^m Ch_m}{\eta^{r+m-v}}$$

or equivalently

$$\int_{\mathbb{Z}_p} Y_{v,2}^{(k)}(x;\lambda \mid \eta)\, d\mu_{-1}(x) = \sum_{r=0}^{v}\binom{v}{r}\sum_{m=0}^{v-r} Y_{r,2}^{(k)}(\lambda \mid \eta)\, S_1(v-r,m)\,\frac{(-\lambda)^m m!}{2^m\eta^{m+r-v}}.$$

The Daehee numbers of the first kind and the Changhee numbers of the first kind are also related to the Stirling numbers of the first kind, as well as to the Bernoulli and Euler numbers. These relationships are given by

$$D_n = \sum_{l=0}^{n} S_1\left(n, l\right) B_l \tag{7}$$

and

$$Ch_n = \sum_{l=0}^{n} S_1\left(n, l\right) E_l, \tag{8}$$

where B_l denotes the Bernoulli numbers and E_l denotes the Euler numbers (*cf.* [5, 13, 15]).

Combining the Theorems 1 and 2 with the Eqs. (7) and (8), we arrive at the following result:

Corollary 3. *For $v, k \in \mathbb{N}_0$, we have*

$$\begin{aligned} \int_{\mathbb{Z}_p} Y_{v,2}^{(k)}\left(x; \lambda \mid \eta\right) d\mu_1\left(x\right) &= \sum_{r=0}^{v} \sum_{m=0}^{v-r} \sum_{l=0}^{m} \binom{v}{r} S_1\left(m, l\right) Y_{r,2}^{(k)}\left(\lambda \mid \eta\right) B_l \\ &\quad \times S_1\left(v-r, m\right) \lambda^m \eta^{v-r-m} \end{aligned}$$

and

$$\begin{aligned} \int_{\mathbb{Z}_p} Y_{v,2}^{(k)}\left(x; \lambda \mid \eta\right) d\mu_{-1}\left(x\right) &= \sum_{r=0}^{v} \sum_{m=0}^{v-r} \sum_{l=0}^{m} \binom{v}{r} S_1\left(m, l\right) Y_{r,2}^{(k)}\left(\lambda \mid \eta\right) E_l \\ &\quad \times S_1\left(v-r, m\right) \lambda^m \eta^{v-r-m}. \end{aligned}$$

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A-prime ideals: A new class of prime ideals

Nejma Ugljanin *[1] *and Mustafa Alkan* [2]

In this note we focus on some generalization of prime ideals and prime submodules over a commutative ring. Then we review some results about them. In this work we introduce and study a finer notion, the A-prime ideals, and compare it systematically with these established variants.

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KEYWORDS: Prime ideals, prime submodule, generalization of prime submodules, A-prime ideals, prime ideals

Introduction

Prime ideals constitute a cornerstone of commutative algebra and algebraic geometry. In ring theory, they extend the classical concept of prime numbers from the integers to arbitrary commutative rings. For an ideal $P \subset R$, primality ensures that the quotient R/P is an integral domain, making prime ideals indispensable for studying the structure of rings via their factor rings. In this sense, prime ideals capture the interaction between multiplication and addition in R.

Beyond their intrinsic importance, prime ideals also govern global ring-theoretic properties. For instance, the nilradical $\sqrt{0}$ of a ring R coincides with the intersection of all prime ideals of R. This identity provides a canonical decomposition of nilpotent behavior and highlights how prime ideals detect non-reduced elements in a ring. Moreover, several fundamental classes of rings, such as Noetherian, Artinian, or Dedekind domains, are often described in terms of conditions on their prime ideals, including finiteness of chains or uniqueness of ideal factorization into prime powers.

From a geometric viewpoint, the set of prime ideals $\operatorname{Spec}(R)$, endowed with the Zariski topology, forms the *spectrum* of R. Here, closed subsets take the form $V(I) = \{ P \in \operatorname{Spec}(R) \mid I \subseteq P \}$ for ideals $I \subseteq R$. This construction translates algebraic data into a topological framework and provides the foundation for modern algebraic geometry. In this picture, prime ideals play the role of "points," while the structure of $\operatorname{Spec}(R)$ encodes the arithmetic and geometric complexity of the ring.

Numerous generalizations of prime ideals appear in the literature [1, 3, 4, 6, 7, 8, 9, 10, 11, 12], including *strongly prime*, *weakly prime*, *S-prime*, and 2-*absorbing* ideals, each modifying the absorption property in a distinct way (for example, restricting to nonzero products or requiring pairwise absorption when $abc \in I$) [5, 6, 9].

In this paper we investigate a refinement called an *A-prime ideal.* An ideal I of R is A-prime if, for all $x, y \in R$ with $xy \in I$, there exists a corrector element $c \in R \backslash (I + Rx)$ such that $cx \in I$ (or symmetrically with respect to y). This modification allows one to detect and control the local effect of zero divisors, thereby extending prime-like behavior to rings that are not reduced. Every prime ideal is A-prime, showing that the concept generalizes classical primeness.

Theorem 1. *An ideal I of a commutative ring R is A-prime if and only if the factor ring R/I is an A-domain.*

This correspondence establishes a bridge between the study of A-prime ideals and the global structure of rings. Concrete examples, such as $12\mathbb{Z}$ and $6\mathbb{Z}$ in $\mathbb{Z}$, demonstrate that A-prime ideals may appear even when classical primeness fails, while cases like $4\mathbb{Z}$ illustrate the limits of this refinement.

The notion of primeness extends naturally from ideals to modules. Given an R-module M and a proper submodule $N \subset M$, the submodule N is called *prime* if $rm \in N$ (with $r \in R, m \in M$) implies $m \in N$ or $r \in (N :_R M)$. Prime submodules have been extensively studied since they inherit many structural features of prime ideals while capturing refined module-theoretic information. In parallel with the ideal case, generalizations such as weakly prime, φ-prime, and S-prime submodules have been introduced, each relaxing the absorption condition in distinct ways.

Motivated by A-prime ideals, we introduce and study *A-prime submodules*. A proper submodule N of M is called A-prime if, whenever $rm \in N$ with $m \notin N$, there exists a corrector element $t \in R\backslash\big((N :_R M) + Rr\big)$ such that $0 \neq tr \in (N :_R M)$. Clearly, every prime submodule is A-prime, but the converse need not hold. Thus, the A-prime condition refines primeness to include a wider range of submodules. This parallel between ideals and submodules provides fertile ground for new concepts: in the following sections, we recall classical generalizations of prime ideals, then extend the framework to prime submodules and finally present their A-prime counterparts.

Generalizations of Prime Ideals

In this note, we use the notions R to be a commutative ring with identity. Then we review some generalization of prime ideals and some useful results without the proof in the literature. One useful generalization obtained by enlarging where the elements a and b lie is the notion of a strong prime ideal.

Definition 2. *[4, 11] Let R be a commutative ring with identity and P a proper ideal. If R is a domain with quotient field K, then P is* strongly prime *if for all $a, b \in K$ with $ab \in P$, one has $a \in P$ or $b \in P$.*

Theorem 3. *[11] Let R be a commutative ring and P a strongly prime ideal of R. If $S \subseteq R$ is a multiplicatively closed subset with $P \cap S = \varnothing$, then the extension $S^{-1}P$ is a strongly prime ideal of the localization $S^{-1}R$.*

Theorem 4. *[4, 11] Let R be a commutative ring and P a proper ideal of R. Then P is strongly prime if and only if R/P is an integral domain whose ideals are linearly ordered by inclusion. In particular, the quotient R/P is a valuation domain.*

For an ideal $I \subseteq R$ of a commutative ring, the *radical* of I is defined as

$$\sqrt{I} = \{\, r \in R \mid r^n \in I \text{ for some } n \geqslant 1 \,\}.$$

It measures the nilpotent behavior inside I, collecting all elements of R whose powers eventually fall into I.

In the Zariski topology on $\operatorname{Spec}(R)$, closed sets are defined by

$$V(I) = \{\, P \in \operatorname{Spec}(R) \mid I \subseteq P \,\},$$

for ideals $I \subseteq R$.

Theorem 5. *[4, 11] For any ideal I of R,*

$$V(I) \;=\; V(\sqrt{I}).$$

Thus, the closed sets in the Zariski topology depend only on the radical of an ideal. In particular, the closed subsets of $\mathrm{Spec}(R)$ are in bijection with radical ideals of R. Moreover, the radical of an ideal preserves the associated algebraic set, underscoring the algebra–geometry correspondence.

In particular, the nilradical of R, namely $\sqrt{0}$, is the intersection of all prime ideals of R. This result highlights the central role of prime ideals in determining the structure of radicals.

Theorem 6. *[4, 11] If P is a strongly prime ideal of R, then P is prime. In particular,*

$$\sqrt{P} = P.$$

Thus every strongly prime ideal is a radical ideal.

Some other generalizations are related to the fact that the product of the element with another element is in the ideal.

Definition 7. *[10] A proper ideal P of R is* weakly prime *if for $a, b \in R$ with $ab \in P \backslash \{0\}$, one has $a \in P$ or $b \in P$.*

Theorem 8. *[10] If P is weakly prime, then its radical $\sqrt{P}$ is a prime ideal .*

Theorem 9. *[10] If P is a weakly prime ideal of a commutative ring R, then its radical $\sqrt{P}$ is a prime ideal. In particular, every weakly prime ideal is contained in some minimal prime ideal.*

Theorem 10. *[8] Let P be a weakly prime ideal of R. If $ab \in P$ with $a \notin P$, then there exists $n \geqslant 1$ such that $b^n \in P$. Thus weakly prime ideals exhibit a closure property under powers of elements not absorbed at once.*

The concept of S-prime ideals is useful for studying the properties of ideals in rings that do not behave as nicely as integral domains .

Definition 11. *[1], [8] Let S be a multiplicatively closed subset of R. A proper ideal P disjoint from S is called S-*prime *if there exists $s \in S$ such that for $a, b \in R$ with $ab \in P$, either $sa \in P$ or $sb \in P$.*

Theorem 12. *[8] An ideal P is S-prime in R iff $S^{-1}P$ is a prime ideal in the localization $S^{-1}R$* [7, 1].

Definition 13. *[6], [7] A proper ideal I of a ring R is* 2-absorbing *if for $a, b, c \in R$ with $abc \in I$, one has $ab \in I$ or $ac \in I$ or $bc \in I$.*

Theorem 14. *[6], Let R be a Noetherian domain. Then every 2-absorbing ideal I of R is of the form*

$$I = P^n \quad \text{or} \quad I = P^n Q^m,$$

where P, Q are distinct prime ideals of R and $n, m \geqslant 1$.

Theorem 15. *[6] If I is a 2-absorbing ideal of R, then its radical $\sqrt{I}$ is either a prime ideal or the intersection of two distinct prime ideals.*

Unlike classical generalizations such as S-prime ideals, weakly prime ideals, or 2-absorbing ideals, which primarily relax multiplicative absorption properties, the notion of an A-prime ideal is based on the existence of a corrector element $c \in R \backslash (I + Rx)$ such that $cx \in I$ whenever $xy \in I$.

Definition 16 (A-prime)**.** *[14] Let R be a commutative ring with identity. An ideal I of R is* A-prime *if for all $x, y \in R$ with $xy \in I$, either there exists $c \in R \backslash (I + Rx)$ such that $cx \in I$, or there exists $c \in R \backslash (I + Ry)$ such that $cy \in I$.*

Theorem 17. *[14] Every prime ideal is A-prime. Moreover, I is A-prime iff R/I is an A-domain.*

Example 18. *Let $R = \mathbb{Z}$ be a ring of integers.*

(i) Consider the ideal $I = 21\mathbb{Z}$. If $ab \in 21\mathbb{Z}$ with $a, b \notin 21\mathbb{Z}$, then one of a, b is divisible by 3 and the other is divisible by 7. Without loss of generality assume $a \in 3\mathbb{Z}$ and $b \in 7\mathbb{Z}$. Then we can choose $c = 7$, since $c \notin 15\mathbb{Z} + 3\mathbb{Z} = 3\mathbb{Z}$, but $c \cdot a = 7a \in 21\mathbb{Z}$. Hence $21\mathbb{Z}$ is an A-prime ideal of $\mathbb{Z}$.

(ii) Consider the ideal $J = 25\mathbb{Z}$. Note that $5 \cdot 5 = 25 \in J$, with $4 \notin J$. The definition requires some $c \in \mathbb{Z}\backslash(J + 5\mathbb{Z}) = \mathbb{Z}\backslash 5\mathbb{Z}$ such that $c \cdot 5 \in 25\mathbb{Z}$. But $c \cdot 5$ is a multiple of 16 precisely when c is a multiple of 5, contradicting $c \notin 5\mathbb{Z}$. Thus no such c exists, and $25\mathbb{Z}$ is not an A-prime ideal.

Generalizations of Prime Submodules

In this note, we use the notions R to be a commutative ring with identity. Then we review some generalization of prime ideals and some useful results without the proof in the literature. One useful generalization obtained by enlarging where the elements a and b lie is the notion of a strong prime ideal.

Definition 19. *[4, 11] Let R be a commutative ring with identity and P a proper ideal. If R is a domain with quotient field K, then P is* strongly prime *if for all $a, b \in K$ with $ab \in P$, one has $a \in P$ or $b \in P$.*

Theorem 20. *[11] Let R be a commutative ring and P a strongly prime ideal of R. If $S \subseteq R$ is a multiplicatively closed subset with $P \cap S = \varnothing$, then the extension $S^{-1}P$ is a strongly prime ideal of the localization $S^{-1}R$.*

Theorem 21. *[4, 11] Let R be a commutative ring and P a proper ideal of R. Then P is strongly prime if and only if R/P is an integral domain whose ideals are linearly ordered by inclusion. In particular, the quotient R/P is a valuation domain.*

For an ideal $I \subseteq R$ of a commutative ring, the *radical* of I is defined as

$$\sqrt{I} = \{\, r \in R \mid r^n \in I \text{ for some } n \geqslant 1 \,\}.$$

It measures the nilpotent behavior inside I, collecting all elements of R whose powers eventually fall into I.

In the Zariski topology on $\operatorname{Spec}(R)$, closed sets are defined by

$$V(I) = \{\, P \in \operatorname{Spec}(R) \mid I \subseteq P \,\},$$

for ideals $I \subseteq R$.

Theorem 22. *[4, 11] For any ideal I of R,*

$$V(I) \;=\; V(\sqrt{I}).$$

Thus, the closed sets in the Zariski topology depend only on the radical of an ideal. In particular, the closed subsets of $\operatorname{Spec}(R)$ are in bijection with radical ideals of R. Moreover, the radical of an ideal preserves the associated algebraic set, underscoring the algebra–geometry correspondence.

In particular, the nilradical of R, namely $\sqrt{0}$, is the intersection of all prime ideals of R. This result highlights the central role of prime ideals in determining the structure of radicals.

Theorem 23. *[4, 11] If P is a strongly prime ideal of R, then P is prime. In particular,*

$$\sqrt{P} = P.$$

Thus every strongly prime ideal is a radical ideal.

Some other generalizations are related to the fact that the product of the element with another element is in the ideal.

Definition 24. *[10] A proper ideal P of R is* weakly prime *if for $a, b \in R$ with $ab \in P\backslash\{0\}$, one has $a \in P$ or $b \in P$.*

Theorem 25. *[10] If P is weakly prime, then its radical $\sqrt{P}$ is a prime ideal .*

Theorem 26. *[10] If P is a weakly prime ideal of a commutative ring R, then its radical $\sqrt{P}$ is a prime ideal. In particular, every weakly prime ideal is contained in some minimal prime ideal.*

Theorem 27. *[8] Let P be a weakly prime ideal of R. If $ab \in P$ with $a \notin P$, then there exists $n \geqslant 1$ such that $b^n \in P$. Thus weakly prime ideals exhibit a closure property under powers of elements not absorbed at once.*

The concept of S-prime ideals is useful for studying the properties of ideals in rings that do not behave as nicely as integral domains .

Definition 28. *[1], [8] Let S be a multiplicatively closed subset of R. A proper ideal P disjoint from S is called S-*prime *if there exists $s \in S$ such that for $a, b \in R$ with $ab \in P$, either $sa \in P$ or $sb \in P$.*

Theorem 29. *[8] An ideal P is S-prime in R iff $S^{-1}P$ is a prime ideal in the localization $S^{-1}R$* [7, 1].

Definition 30. *[6], [7] A proper ideal I of a ring R is* 2-absorbing *if for $a, b, c \in R$ with $abc \in I$, one has $ab \in I$ or $ac \in I$ or $bc \in I$.*

Theorem 31. *[6], Let R be a Noetherian domain. Then every 2-absorbing ideal I of R is of the form*

$$I = P^n \quad \text{or} \quad I = P^n Q^m,$$

where P, Q are distinct prime ideals of R and $n, m \geqslant 1$.

Theorem 32. *[6] If I is a 2-absorbing ideal of R, then its radical $\sqrt{I}$ is either a prime ideal or the intersection of two distinct prime ideals.*

Unlike classical generalizations such as S-prime ideals, weakly prime ideals, or 2-absorbing ideals, which primarily relax multiplicative absorption properties, the notion of an A-prime ideal is based on the existence of a corrector element $c \in R\backslash(I + Rx)$ such that $cx \in I$ whenever $xy \in I$.

Definition 33 (A-prime)**.** *[14] Let R be a commutative ring with identity. An ideal I of R is* A-prime *if for all $x, y \in R$ with $xy \in I$, either there exists $c \in R\backslash(I + Rx)$ such that $cx \in I$, or there exists $c \in R\backslash(I + Ry)$ such that $cy \in I$.*

Theorem 34. *[14] Every prime ideal is A-prime. Moreover, I is A-prime iff R/I is an A-domain.*

Example 35. *Let $R = \mathbb{Z}$ be a ring of integers.*

(i) *Consider the ideal* $I = 21\mathbb{Z}$. *If* $ab \in 21\mathbb{Z}$ *with* $a, b \notin 21\mathbb{Z}$, *then one of* a, b *is divisible by* 3 *and the other is divisible by* 7. *Without loss of generality assume* $a \in 3\mathbb{Z}$ *and* $b \in 7\mathbb{Z}$. *Then we can choose* $c = 7$, *since* $c \notin 15\mathbb{Z} + 3\mathbb{Z} = 3\mathbb{Z}$, *but* $c \cdot a = 7a \in 21\mathbb{Z}$. *Hence* $21\mathbb{Z}$ *is an A-prime ideal of* $\mathbb{Z}$.

(ii) *Consider the ideal* $J = 25\mathbb{Z}$. *Note that* $5 \cdot 5 = 25 \in J$, *with* $4 \notin J$. *The definition requires some* $c \in \mathbb{Z}\backslash(J + 5\mathbb{Z}) = \mathbb{Z}\backslash 5\mathbb{Z}$ *such that* $c \cdot 5 \in 25\mathbb{Z}$. *But* $c \cdot 5$ *is a multiple of* 16 *precisely when* c *is a multiple of* 5, *contradicting* $c \notin 5\mathbb{Z}$. *Thus no such* c *exists, and* $25\mathbb{Z}$ *is not an A-prime ideal.*

Generalizations of Prime Submodules

In this note, we let R denote a commutative ring with identity and M an R-module. We review some generalizations of prime submodules and present several useful results from the literature, without including their proofs. We begin with the notion of weakly prime submodules, which extends the concept of prime submodules.

Definition 36. *[17] Let* N *be a proper submodule of an* R*-module* M. *Then* N *is called a* weakly prime submodule *if for each submodule* K *of* M *and elements* $a, b \in R$, *the inclusion* $abK \subseteq N$ *implies that either* $aK \subseteq N$ *or* $bK \subseteq N$.

Corollary 37. *[17] Let* M *be an* R*-module and* N *a proper submodule of* M. *Then* N *is a prime submodule if and only if* N *is primary and weakly prime.*

Theorem 38. *[17] Let* M *be an* R*-module and* N *a proper submodule of* M. *The following statements are equivalent:*

(i) N *is a weakly prime submodule.*

(ii) *For any* $x, y \in M$, *if* $(N : x) \neq (N : y)$, *then*

$$N = (N + Rx) \cap (N + Ry).$$

Corollary 39. *[17] Let* M *be an* R*-module,* N *a weakly prime submodule of* M, *and* $x, y \in M$.

(i) *If* $rx \in N$ *where* $r \in R$, *then*

$$N = (N + Rx) \cap (N + Rry).$$

(ii) *If* N *is an irreducible submodule, then* N *is a prime submodule.*

Proposition 40. *[17] Let* A_i, $1 \leqslant i \leqslant n$, *be a finite collection of ideals of a ring* R, *and let* M *be the free* R*-module* $\bigoplus_{i=1}^{n} R$. *Then* $\bigoplus_{i=1}^{n} A_i$ *is a weakly prime submodule of* M *if and only if*

$$\{\, A_i \mid A_i \neq R \,\}$$

is a non-empty chain of prime ideals of R.

Beyond weakly prime submodules, one can consider more general frameworks such as φ-prime submodules.

Definition 41. *[18] Let* R *be a commutative ring with non-zero identity and* M *be a unitary* R*-module. Let* $S(M)$ *denote the set of all submodules of* M, *and let*

$$\varphi : S(M) \longrightarrow S(M) \cup \{\varnothing\}$$

be a function. We say that a proper submodule P *of* M *is a* prime submodule relative to φ *(or a* φ-prime submodule*) if for* $a \in R$ *and* $x \in M$, *the condition* $ax \in P\backslash\varphi(P)$ *implies that either* $a \in (P :_R M)$ *or* $x \in P$.

Theorem 42. *[18] Let R be a commutative ring and M be an R-module. Let $\varphi : S(M) \to S(M) \cup \{\varnothing\}$ be a function and P be a φ-prime submodule of M such that*

$$(P :_R M)P \nsubseteq \varphi(P).$$

Then P is a prime submodule of M.

Corollary 43. *[18] Let P be a weak prime submodule of M such that*

$$(P :_R M)P \neq 0.$$

Then P is a prime submodule of M.

Corollary 44. *[18] Let P be a φ-prime submodule of M such that*

$$\varphi(P) \subseteq (P :_R M)^2 P.$$

Then for each $a \in R$ and $x \in M$, the condition

$$ax \in P \setminus \bigcap_{i=1}^{\infty} (P :_R M)^i P$$

implies that either $a \in (P :_R M)$ or $x \in P$. In other words, P is a φ_ω-prime submodule.

Corollary 45. *[18] Let M be an R-module and P be a φ-prime submodule of M. Then*

$$(P :_R M) \subseteq \sqrt{(\varphi(P) :_R M)} \quad \textit{or} \quad \sqrt{(\varphi(P) :_R M)} \subseteq (P :_R M).$$

If

$$(P :_R M) \subsetneq \sqrt{(\varphi(P) :_R M)},$$

then P is not a prime submodule of M; while if

$$\sqrt{(\varphi(P) :_R M)} \subsetneq (P :_R M),$$

then P is a prime submodule of M.

Moreover, if $\varphi(P)$ is a radical submodule of M, either $(P :_R M) = (\phi(P) :_R M)$ or P is a prime submodule of M.

Another important line of generalization arises by introducing multiplicatively closed subsets, leading to the notion of S-prime submodules.

Definition 46. *[19] Let $S \subseteq R$ be a multiplicatively closed subset (m.c.s.), and let P be a submodule of M with*

$$(P :_R M) \cap S = \varnothing.$$

*Then P is said to be an S-*prime submodule *if there exists $s \in S$ such that whenever $am \in P$ (for $a \in R$, $m \in M$), one has either*

$$sa \in (P :_R M) \quad \textit{or} \quad sm \in P.$$

Proposition 47. *[19] Assume that $S \subseteq R$ is a multiplicatively closed subset and M is an R-module. Then:*

(i) If $P \in Spec({}_RM)$ provided that $(P :_R M)$ and S are disjoint, then $P \in Spec_S({}_RM)$. In fact, if $S \subseteq u(R)$ and $P \in Spec_S({}_RM)$, then $P \in Spec({}_RM)$.

(ii) *If $S_1 \subseteq S_2$ are m.c.s. of R and $P \in Spec_{S_1}({}_RM)$, then $P \in Spec_{S_2}({}_RM)$in case $(P :_R M) \cap S_2 = \varnothing$.*

(iii) *$P \in Spec_S({}_RM)$ if and only if $P \in Spec_{S*}({}_RM)$.*

(iv) *If $P \in Spec_S({}_RM)$, then $S^{-1}P$ is a prime submodule of $S^{-1}M$.*

Lemma 48. *[19] Suppose P is a submodule of M and S is a multiplicatively closed subset (m.c.s.) of R satisfying*

$$(P :_R M) \cap S = \varnothing.$$

Then the following are equivalent:

(i) *$P \in Spec_S({}_RM)$.*

(ii) *There exists $s \in S$, and $JN \subseteq P$, implies $sJ \subseteq (P :_R M)$ or $sN \subseteq P$ for each ideal J of R and submodule N of M.*

Corollary 49. *[19] Suppose S is a multiplicatively closed subset (m.c.s.) of R and let P be an ideal of R such that*

$$P \cap S = \varnothing.$$

Then the following are equivalent:

(i) *$P \in Spec_S(R)$.*

(ii) *There exists $s \in S$ such that, for each pair of ideals I, J of R with $IJ \subseteq P$, either $sI \subseteq P$ or $sJ \subseteq P$.*

Proposition 50. *[19] Suppose $f : M \to M'$ is an R-homomorphism. Then:*

(i) *If $P' \in Spec_S({}_RM')$ and $(f^{-1}(P') :_R M) \cap S = \varnothing$, then*

$$f^{-1}(P') \in Spec_S({}_RM).$$

(ii) *If f is an epimorphism and $P \in Spec_S({}_RM)$ with $Ker(f) \subseteq P$, then*

$$f(P) \in Spec_S({}_RM').$$

Several new generalizations of prime submodules have also been introduced in the literature. In particular, Sevim, Arabacı, Tekir, and Koç [20] formulated definitions using trace and rejection operators, which take into account the interaction of a module with its homomorphism space. Through these tools, generalized forms of classical prime structures are obtained via trace functions.

For a commutative ring R with identity and an R-module M, the authors introduced two preradical functions associated with a submodule $N \subseteq M$:

$$\mathrm{tr}_N^M(X) = \sum\{f(N) \mid f \in \mathrm{Hom}_R(M, X)\}, \quad \mathrm{rej}_N^M(X) = \bigcap\{f^{-1}(N) \mid f \in \mathrm{Hom}_R(X, M)\}.$$

Using these operators, they defined the *module product*

$$N_M L = \mathrm{tr}_N^M(L),$$

which captures the interaction between submodules N and L of M. On this basis, notions of prime and semiprime submodules were redefined in terms of full invariance.

Definition 51. *[20] Let M be an R-module and Q a proper submodule of M. Then Q is called a* fully prime submodule *if Q is fully invariant and, for any fully invariant submodules $N, L \subseteq M$,*

$$N_M L \subseteq Q \quad \Rightarrow \quad N \subseteq Q \text{ or } L \subseteq Q.$$

Lemma 52. *[20] Let N be a fully invariant submodule of an R-module M. If every fully invariant submodule of M is idempotent in M, then every fully invariant submodule of N is idempotent in N.*

Lemma 53. *[20] The following conditions hold for submodules $N, L \subseteq M$:*

(i) $(NL^{-1})_M L \subseteq N$;

(ii) If $K \subseteq M$ is any submodule such that $K_M L \subseteq N$, then $K \subseteq NL^{-1}$;

(iii) $(N \cap L)L^{-1} = NL^{-1}$;

(iv) $N \subseteq (N_M L)L^{-1}$;

(v) If N is a fully invariant submodule of M, then $N \subseteq NL^{-1}$.

The above notions illustrate how the classical definition of primeness in modules can be relaxed in several different directions, each capturing a particular weakening of the absorption property. Nevertheless, all these approaches share the feature that they modify the absorption rule only internally (by excluding zero products, by localizing with respect to S, or by requiring full invariance). Inspired by the recently introduced concept of A-prime ideals, we now extend this refinement to the module setting. The idea is to keep the absorption condition for submodules but allow the presence of a corrector element that compensates when $rm \in N$ with $m \notin N$. This leads naturally to the following definition.

Definition 54. *[14] Let R be a ring, M an R-module, and N a proper submodule of M. Then N is called an A-*prime submodule *if, for every $r \in R$ and $m \in M$, the condition $rm \in N$ implies that either $m \in N$ or there exists an element $t \in R \backslash \big((N : M) + Rr\big)$ such that $0 \neq tr \in (N : M)$.*

Proposition 55. *[14] Every prime submodule is an A-prime submodule.*

Theorem 56. *[14] Let R be a ring, M an R-module, and N a prime submodule of M. If N is an A-prime submodule of M, then $(N : M)$ is an A-prime ideal of R.*

Example 57. *Let $R = \mathbb{Z}$, $M = \mathbb{Z} \bigoplus \mathbb{Z}$ be a $\mathbb{Z}$-module, and let $N = (2, 4)\mathbb{Z}$. Then N is neither a prime submodule nor an A-prime submodule. Indeed, $N = \{(2k, 4k) \mid k \in \mathbb{Z}\}$, and taking $m = (1, 2) \in M$ and $r = 1$, we see that $rm = (1, 2) \notin N$ although $r \notin (N : M)$ and $m \notin N$, so N fails to be prime. For the A-prime condition, note that $(2, 4) = 2(1, 2) \in N$, with $r = 2$ and $m = (1, 2)$. Here $(N : M) = \{0\}$, and to satisfy the definition one would need a $t \in \mathbb{Z} \backslash ((N : M) + 2\mathbb{Z})$ with $t2 \in (N : M)$, which is impossible. Thus N is not A-prime.*

Acknowledgments

This paper is dedicated to Professor Manuel López-Pellicer on the occasion of his 81st birth anniversary.

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On the cumulative distribution function of McKay I_ν Bessel random variable

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The main aim of this talk is to introduce two new representation formulae for the cumulative distribution function of McKay I_ν Bessel random variable and its generalization which was presented by McNolty using the second mean–value theorems for definite integrals and properties of the Lambert W function. Our newly derived formulae could offer notable practical utility, particularly in cases where computing McNolty's distribution function proves computationally demanding. Additionally, we establish its connection to the incomplete Lipschitz–Hankel integral and the Rice I_e–function, both of which are very useful in telecommunications.

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Introduction

Probability distributions involving Bessel functions have attracted considerable attention for mathematicians. The earliest findings on this subject can be traced back to the early work of A. T. McKay in 1932 [3] and R. G. Laha in 1954 who considered two classes of continuous distributions known as Bessel function distributions. One is based on the modified Bessel function of the first kind, I_ν, of order ν, while another is defined using the modified Bessel function of the second kind, K_ν, of the same order ν. Later, in 1973, McNolty proposed a generalization of the first type distribution [4] which has density of the form

$$f_I(x;a,b;\nu) = \frac{\sqrt{\pi}(b^2-a^2)^{\nu+1/2}}{(2a)^\nu\Gamma\left(\nu+\frac{1}{2}\right)} \mathrm{e}^{-bx}x^\nu I_\nu(ax), \qquad x \geqslant 0,$$

where $\nu > -1/2$, $b > a > 0$, and the appropriate cumulative distribution function is

$$F_I(x;a,b;\nu) = \frac{\sqrt{\pi}(b^2-a^2)^{\nu+1/2}}{(2a)^\nu\Gamma\left(\nu+\frac{1}{2}\right)} \int_0^x \mathrm{e}^{-bt}t^\nu I_\nu(at)\,\mathrm{d}t, \qquad x \geqslant 0. \tag{1}$$

Given the broad range of applications of random variables following the McNolty's variant of McKay I_ν Bessel distribution, the corresponding cumulative distribution function has been the subject of investigation in the mathematical literature. We will introduce two new representation formulae for such cumulative distribution function with the help of the second mean–value theorems for definite integrals, one of Bonnet type [6] and another relying on the stronger version of the Okamura's variant [7]. A point, characteristic for the mean–value theorems, will be expressed in terms of the Lambert W function [5]. Namely, Lambert and also Euler studied solutions of a transcendental equation of the form

$$x\mathrm{e}^x = a.$$

The unique real–valued concave increasing solution to such an equation exists for $a \in [-\mathrm{e}^{-1}, \infty)$ and is known as the Lambert W function [5]. It satisfies $W(-\mathrm{e}^{-1}) = -1$, $W(0) = 0$, $W'(0) = 1$ and $W(a) \sim \log a$ as $a \to \infty$. Moreover, for $-\mathrm{e}^{-1} < a < 0$ there are two real solutions, while enabling complex values of a we get infinitely many solutions [1].

Furthermore, we will form a connection between the observed cumulative distribution function and both the incomplete Lipschitz–Hankel integral of the first kind modified Bessel functions

$$I_{e_{\mu,\nu}}(z; a, b) = \int_0^z \mathrm{e}^{-bt} t^\mu I_\nu(at)\,\mathrm{d}t,$$

where $a, b > 0$, $z, \nu, \mu \in \mathbb{C}$ and $\Re(\mu + \nu) > -1$, and the Rice I_e–function

$$I_e(k, x) = \int_0^x \mathrm{e}^{-t}\, I_0(kt)\,\mathrm{d}t, \qquad x \geqslant 0,\ 0 \leqslant k \leqslant 1.$$

The presented results are published in [2].

Main results

Using the second mean–value theorem for definite integrals for Riemann integrable input functions and properties of the Lambert W function, we obtain the following result [2]:

Theorem 1. *Let $\nu > -1/2$, $b > a > 0$ and $x > 0$.*

1. *Then there exists the point $c_\nu \in (0, 1]$ such that*

$$F_I(x; a, b; \nu) = \frac{\sqrt{\pi}(b^2 - a^2)^{\nu+1/2} x^\nu \mathrm{e}^{-bxc_\nu}}{2^\nu\, c_\nu\, a^{\nu+1}\, \Gamma\left(\nu + \frac{1}{2}\right)}\, I_{\nu+1}(ax).$$

An explicit formula for the point c_ν is given by

$$c_\nu = -\frac{1}{bx}\, W\!\left(\frac{axI_\nu(ax)}{I_{\nu+1}(ax)} \exp\left(-bx + \frac{axI_\nu(ax)}{I_{\nu+1}(ax)}\right)\right) + \frac{a}{b}\, \frac{I_\nu(ax)}{I_{\nu+1}(ax)}. \tag{2}$$

2. *There holds the limit*

$$\lim_{x\to 0} \frac{c_\nu}{x} = \frac{2\nu + 1}{2\nu + 2}\,, \qquad \nu > -1/2\,.$$

Similarly, for Lebesgue integrable input functions, we have [2]:

Theorem 2. *For all $\nu > 1/2$, $b > a > 0$ and $x > 0$ there holds*

$$F_I(x; a, b; \nu) = \frac{\sqrt{\pi}(b^2 - a^2)^{\nu+1/2} x^\nu}{(2a)^\nu\, b\, \Gamma\left(\nu + \frac{1}{2}\right)}\, I_\nu(ax)\, (\mathrm{e}^{-bxc_{\nu-1}} - \mathrm{e}^{-bx})\,.$$

Since the incomplete Lipschitz–Hankel integral and the Rice I_e–function have a certain role in telecommunications, we point out their connection to (1) in the following results [2]:

Corollary 3. *For all $x \geqslant 0$, $b > a > 0$ and $\nu > -1/2$ it is*

$$I_{e_{\nu,\nu}}(x;a,b) = \frac{x^\nu\, \mathrm{e}^{-bxc_\nu}}{ac_\nu}\, I_{\nu+1}(ax)\,.$$

Moreover, for $\nu > 1/2$ there holds

$$I_{e_{\nu,\nu}}(x;a,b) = \frac{x^\nu}{b}\, I_\nu(ax)\, \left(\mathrm{e}^{-bxc_{\nu-1}} - \mathrm{e}^{-bx}\right)\,.$$

The explicit expression of c_ν is described as the display (2) *of Theorem 1.*

Corollary 4. *For all $x \geqslant 0$ and $k \in (0,1)$ we have*

$$I_e(k,x) = \frac{1}{kc_0}\, \mathrm{e}^{-xc_0}\, I_1(kx).$$

Acknowledgments

This paper is dedicated to Professor Manuel López-Pellicer on the occasion of his 81st birth anniversary.

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On full-fledged recursion operators for symmetries of linearly degenerate Lax-integrable equations

Petr Vojčák

We present a construction of full-fledged recursion operators for the nonlocal symmetries of linearly degenerate Lax-integrable equations. The recursion operators we construct can be interpreted as infinite-dimensional matrices of differential functions, acting on the generating vector functions of the nonlocal symmetries via simple matrix multiplication. We also examine the algebraic properties of these operators and discuss their role in constructing the $\mathbb{Z}$-graded Lie algebras of full-fledged nonlocal symmetries.

2020 MSC: 35B06, 17B70

KEYWORDS: Recursion operators, nonlocal symmetries, $\mathbb{Z}$-graded Lie algebras

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On reduction of totally imaginary binary quartic forms

Ryotaro Okazaki

Let $F(X,Y)$ be a quartic form with integer coefficients. The representation number of unity by F is the number of pair (x,y) of integers such that $F(x,y) = 1$. The quartic form $F(X,Y)$ is totally imaginary if it does not have any real linear form as a factor. I am trying to identify all totally imaginary binary quartic field with representation number larger than 4.

Nagell introduced three families of totally imaginary binary quartic forms.

$$G_{t,h} = X^4 + 2tX^3Y + (t^2+h)X^2Y^2 + htXY^3 + Y^4 \tag{1}$$

Here, the variable t ranges over integers. The variable h takes three values -1, 0 and $+1$. Each value defines a family.

The quartic form $G_{t,h}$ is equivalent to

$$X^4 - (2u^2 - h)X^2Y^2 + (u^4 - hu^2 + 1)Y^4$$

if t is even ($t = 2u$) or

$$X^4 - 2X^3 - (2uw - 1 - h)X^2Y^2 + (2uw - h)XY^3 + (u^2w^2 - huw + 1)Y^4$$

or if t' is odd ($t = 2u+1$ and $w = u+1$).

We prove these forms are reduced in Julia's sense. Julia reduces the quartic form $F(X,Y)$ by reducing the covariant quadratic form

$$Q(X,Y) = \sum_{j=1}^{4} \frac{(X - \alpha_j Y)(X - \bar{\alpha}_j Y)}{|F_X(\alpha_j, 1)|}, \tag{2}$$

where α_1 α_2, ..., α_4 are the four roots of $F(X,1)$.

We multiply some invariant of $F(X,Y)$ to the covariant quadratic form $Q(X,Y)$ in order to find a covariant quadratic form easier both for handling and for interpretation.

2020 MSC: 11E76

KEYWORDS: Quartic form, Julia reduction

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A hybrid numerical method for solving the EW equation

Serpil Cıkıt *[1] *and Emre Kırlı* [2]

In this work, a hybrid technique which has high accuracy is applied in order to get the numerical solutions of the Equal-Width (EW) equation. To this end, the spatial integration based on compact finite difference technique of fourth order accuracy is utilized with cubic B-spline collocation process and Adam's Moulton technique is employed for the temporal calculations. The efficiency of the worked technique is examined by studying a test problem which has a solution of the EW equation. The obtained results are shown with figures and tables. The calculated three invariants are given in tables with the results from literature. Accuracy of this technique is shown with the L_∞ error norm.

2020 MSC: 65M06, 65M70, 65D07

Keywords: Equal-width equation, compact finite difference technique, cubic B-Spline, collocation method, Adams Moulton

Introduction

Equal-Width Equation

Many phenomena that happen in nature are described with nonlinear partial differential equations (PDE). Equal-Width (EW) equation that is introduced by Morrison et.al. known as an PDE to model the wave propagation of nonlinear dispersive media [1].The analytical solutions of this equation are possible only for some limited conditions. Therefore, numerical solutions of this equation have great importance and show a significant development in recent years.

It is written in a form with the boundary conditions (BCs) and initial condition (IC).

$$w_t + ww_x - \mu w_{xxt} = 0, \qquad x \in [\alpha, \beta] \tag{1}$$

the BCs

$$\begin{array}{rlrl} w(\alpha,t) = & 0 & w(\beta,t) = & 0 \\ w_x(\alpha,t) = & 0 & w_x(\beta,t) = & 0 \end{array}, t \in (0,T] \tag{2}$$

and the IC

$$w(x,0) = f(x), \qquad x \in [\alpha, \beta] \tag{3}$$

μ shows a positive parameter in the equation.

There are various numerical methods to gain approximate results of the EW equation. Gardner et al. have worked with Petrov-Galerkin technique which contains quadratic B-spline functions [2], Zaki has employed the least-squares technique which utilizes finite elements [3], Raslan used collocation method together quintic B-spline finite elements [4], Doğan has applied the Galerkin technique [5] to obtain the solutions of the EW equation, Saka has adopted a collocation method improved with cubic

B-spline [6] and also Galerkin technique with quartic B-spline [8]. Esen has investigated the solutions of the EW equation by operating the lumped Galerkin technique which based on quadratic B-spline finite elements [7].

In this study, the numerical solutions of the EW equation are investigated with a hybrid computational method which provides a high accuracy for temporal and spatial discretization. This hybrid technique combines the finite difference technique of fourth order accuracy with the terms of cubic B-spline for the second order spatial derivatives in the equation and Adam's Moulton method is used for discretization in time. With the help of the method, the numerical results are acquired and presented in tables and figures by means of comparison.

Proposed Method

To build a mesh for the method, the intervals $[\alpha, \beta]$ and $[0, T]$ are taken. Mesh points are shown as (x_m, t_n). Where $x_m = \alpha + rh$ and $r = 0, 1, ...N$ and $t_n = n\Delta t$ and $n = 0, 1, ...M$. Here, h is the mesh interval and Δt is time step in the statement.

The analytical and approximate solutions are given for the mesh points as follows.

$$w\left(x_m, t_n\right) = w_r^n \text{ and } W\left(x_r, t_n\right) = W_r^n$$

Temporal Discrezation

Let's take $v = w - \mu w_{xx}$. Thus, Eq. (1) is rearranged as below:

$$v_t = \left(w - \mu w_{xx}\right)_t = -w w_x \tag{4}$$

Adam's Moulton method is utilized for the temporal discretization of Eq. (4):

$$v^{n+1} = v^n + \Delta t \left(\frac{5}{12} v_t^{n+1} + \frac{2}{3} v_t^n - \frac{1}{12} v_t^{n-1}\right) + O\left(\Delta t^4\right) \tag{5}$$

and Eq. (4) is written as

$$\begin{aligned} & w^{n+1} - \mu w_{xx}^{n+1} + \frac{5\Delta t}{12}\left(w^{n+1} w_x^{n+1}\right) \\ = \; & w^n - \mu w_{xx}^n - \frac{2\Delta t}{3}\left(w^n w_x^n\right) \\ & + \frac{\Delta t}{12}\left(w^{n-1} w_x^{n-1}\right) \end{aligned} \tag{6}$$

Spatial discretization

The conventional cubic B-spline functions $B_r^3(x)$, $m = -1, ..., N+1$, constructed at the mesh points x_m, are shown in the following manner

$$B_j^3(x) = \frac{1}{h^3} \begin{cases} (x - x_{m-2})^3, & x_{m-2} \leqslant x < x_{m-1} \\ h^3 + 3h^2(x - x_{m-1}) + & \\ 3h(x - x_{m-1})^2 - & \\ 3(x - x_{m-1})^3, & , x_{m-1} \leqslant x < x_m \\ h^3 + 3h^2(x_{m+1} - x) + & \\ 3h(x_{m+1} - x)^2 - & \\ 3(x_{m+1} - x)^3, & , x_m \leqslant x < x_{m+1} \\ (x_{m+2} - x)^3, & , x_{m+1} \leqslant x < x_{m+2} \\ 0, & otherwise. \end{cases} \tag{7}$$

Let us write approximate solution $W(x,t)$ for the spatial integration of the Eq. (6):

$$W(x,t) = \sum_{j=-1}^{N+1} \delta_j B_j^3 \tag{8}$$

where $\delta_j(t)$ that will be computed shows temporal variables.

Employing Eqs. (7) and (8), the unknown function W and its derivative terms on the spatial interval $[x_m, x_{m+1}]$ are obtained as

$$\begin{aligned} &W(x_m,t) = \delta_{m-1} + 4\delta_m + \delta_{m+1} \\ &W'(x_m,t) = \tfrac{-3}{h}(\delta_{m-1} - \delta_{m+1}) \\ &W''(x_m,t) = \tfrac{6}{h^2}(\delta_{m-1} - 2\delta_m + \delta_{m+1}) \end{aligned} \tag{9}$$

First derivative terms are written with fourth order finite difference approximation below

$$\frac{1}{4}W'_{m-1} + W'_m + \frac{1}{4}W'_{m+1} = \frac{1}{h}\left[-\frac{3}{4}W_{m-1} + \frac{3}{4}W_{m+1}\right] \tag{10}$$

W'_m corresponds to the derivative of the W of order one at the mesh point x_m.

Similarly, the calculation of the second derivative terms using finite difference approximation is stated

$$\frac{1}{10}W''_{m-1} + W''_m + \frac{1}{10}W''_{m+1} = \frac{1}{h^2}\left[\frac{6}{5}W_{m-1} - \frac{12}{5}W_m + \frac{6}{5}W_{m+1}\right] \tag{11}$$

where W''_m shows the derivative of the unknown W of order two at the mesh point x_m.

Using the differential operator outlined in the Eq. (10), the second-order equation has been established

$$\frac{1}{4}W''_{m-1} + W''_m + \frac{1}{4}W''_{m+1} = \frac{1}{h}\left[-\frac{3}{4}W'_{m-1} + \frac{3}{4}W'_{m+1}\right] \tag{12}$$

Clearing off the terms W''_{m-1} and W''_{m+1} in Eqs. (11) and (12), the approximation is achieved for the second derivative [10]

$$W''_m = 2\frac{W_{m+1} - 2W_m + W_{m-1}}{h^2} - \frac{W'_{m+1} - W'_{m-1}}{2h} \tag{13}$$

In this manner, the newly derived Eq. (13) will be employed in place of the second order derivative terms for the inner points.

When the Eq.(6) is rearranged by putting the derivative terms in (9), it is obtained at the boundaries $m = 0$ and N,

$$\begin{aligned} &\delta_{m-1}^{n+1}\left(1 - \tfrac{6\mu}{h^2} - \tfrac{15\Delta t}{12h}a_1\right) + \delta_m^{n+1}\left(4 + \tfrac{12\mu}{h^2}\right) + \delta_{m+1}^{n+1}\left(1 - \tfrac{6\mu}{h^2} + \tfrac{15\Delta t}{12h}a_1\right) \\ &= \\ &W_m^n - \mu\left(W''\right)_m^n + a_2\left(W'\right)_m^n + a_3\left(W'\right)_m^{n-1} \end{aligned} \tag{14}$$

where;

$$a_1 = W_m^{n+1}, \quad a_2 = -\tfrac{2\Delta t}{3}\left(W_m^n\right), \quad a_3 = \tfrac{\Delta t}{12}\left(W_m^{n-1}\right)$$

Likewise, upon substitution of the values of $W(x,t)$ and its derivatives of order one from Eq. (9) and obtained derivative of order two from Eq. (13) into (6) at the inner points, $m = 1, 2, ..., N-1$, it is achieved,

$$\begin{aligned}
&\delta_{m-2}^{n+1}\left(-\tfrac{\mu}{2h^2}\right) + \delta_{m-1}^{n+1}\left(1 - \tfrac{4\mu}{h^2} - \tfrac{15\Delta t}{12h}a_1\right) \\
&+\delta_{m}^{n+1}\left(4 + \tfrac{9\mu}{h^2}\right) + \delta_{m+1}^{n+1}\left(1 - \tfrac{4\mu}{h^2} + \tfrac{15\Delta t}{12h}a_1\right) \\
&+\delta_{m+2}^{n+1}\left(-\tfrac{\mu}{2h^2}\right) = W_m^n - \mu\left(W''\right)_m^n + a_2\left(W'\right)_m^n + a_3\left(W'\right)_m^{n-1}
\end{aligned} \tag{15}$$

and

$$a_1 = W_m^{n+1}, \quad a_2 = -\tfrac{2\Delta t}{3}\left(W_m^n\right), \quad a_3 = \tfrac{\Delta t}{12}\left(W_m^{n-1}\right)$$

As a result, the system is obtained with $(N+1)$ equations and $(N+3)$ unknowns from Eqs. (14) and (15). With the aim of solving this system, the BCs are utilized to remove the terms

$$\delta_{-1}^{n+1}\text{and } \delta_{N+1}^{n+1}$$

in the system. Accordingly, the obtained matrix system with the dimension of $(N+1)\times(N+1)$ is figured out smoothly via Matlab program. To initiate the computation of the resulting system iteratively, the starting vectors $\delta^0 = \left(\delta_{-1}^0, ..., \delta_{N+1}^0\right)^T$ and $\delta^1 = \left(\delta_{-1}^1, ..., \delta_{N+1}^1\right)^T$ must be determined. When the conditions (3) and (2) are used, the starting vector δ^0 is computed, then the other vector δ^1 is obtained by using Crank-Nicolson technique to Eq. (4).

So, the unknown vectors $\delta^{n+1} = \left(\delta_{-1}^{n+1}, ..., \delta_{N+1}^{n+1}\right)^T$ $(n = 1, 2, ...)$ at any time which is wanted to be found can be calculated by using the former unknown vectors δ^n and δ^{n-1} iteratively. After a nonlinear system in implicit form is obtained in terms of δ, an inner algorithm is employed three times for all temporal steps.

Numerical Calculations

As a test problem , single solitary wave which is a solution of the EW equation is analyzed to perform the effectiveness of the method used. The accuracy of the technique used is inspected by figuring out the error norm L_∞ as shown below

$$L_\infty = \max_m \left|w_m - W_m\right|, \tag{16}$$

the convergence order for space and time is examined with the formulas

$$\text{order}=\frac{\log\left|\dfrac{(L_\infty)_{h_i}}{(L_\infty)_{h_{i+1}}}\right|}{\log\left|\dfrac{h_i}{h_{i+1}}\right|}, \tag{17}$$

$$\text{order}=\frac{\log\left|\dfrac{(L_\infty)_{\Delta t_i}}{(L_\infty)_{\Delta t_{i+1}}}\right|}{\log\left|\dfrac{\Delta t_i}{\Delta t_{i+1}}\right|}, \tag{18}$$

The three invariants which correspond mass, momentum and energy respectively are calculated via following formulas [9]

$$
\begin{aligned}
I_1 &= \int_{-\infty}^{\infty} w dx \approx \int_{\alpha}^{\beta} W dx, \\
I_2 &= \int_{-\infty}^{\infty} \left(w^2 + \mu \left(w_x\right)^2\right) dx \approx \int_{\alpha}^{\beta} \left(W^2 + \mu \left(W_x\right)^2\right) dx, \\
I_3 &= \int_{-\infty}^{\infty} \left(w^3 + 3w^2\right) dx \approx \int_{\alpha}^{\beta} \left(W^3 + 3W^2\right) dx.
\end{aligned}
$$

Motion of a Single Solitary Wave

A single solitary wave is expressed

$$w(x,t) = 3csech^2\left(k\left[x - \tilde{x}_0 - vt\right]\right) \tag{19}$$

v is the velocity of the solitary wave and taken as c, $3c$ is the amplitude of the solitary wave, the width of the wave pulse is shown with $k = \sqrt{\frac{1}{4\mu}}$ the BCs are set to zero and the IC is determined as

$$w(x,0) = 3csech^2\left(k\left[x - \tilde{x}_0\right]\right) \tag{20}$$

The invariant quantities are calculated by utilizing the initial condition (20) in the integrals I_1, I_2, I_3 analytically as below [2].

$$I_1 = \frac{6c}{k}, \qquad I_2 = \frac{12c^2}{k} + \frac{48k\mu c^2}{5}, \qquad I_3 = \frac{144c^3}{5k}$$

For computer calculations, the spatial interval is taken as $[0, 30]$ and the temporal interval is $0 \leqslant t \leqslant 80$. The parameters are choosen to realize the calculations as $c = 0.1$ and $c = 0.01$ respectively by taking $\mu = 1$, $\tilde{x}_0 = 10$. The results that obtained from simulation are given in Figure 1. The waves seen in the figure correspond to the moments $t = 10$, $t = 80$. It is monitored that the solitary wave propagates to the rightward direction by maintaining its same shape, velocity and amplitude.

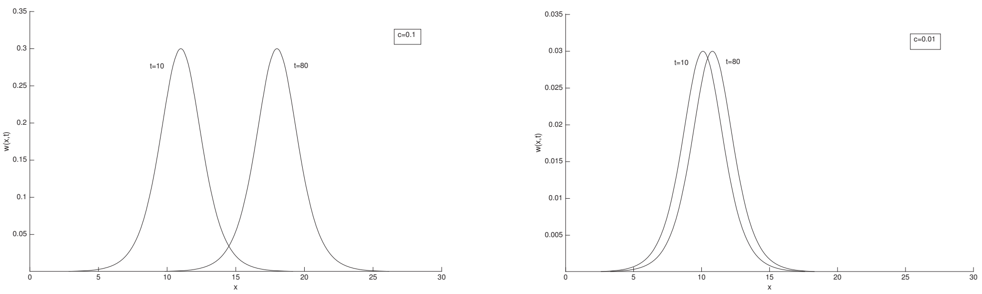

Figure 1: Simulation of motion of a single solitary wave for different amplitude values

Results obtained by means of the presented hybrid approximation are analyzed with the results obtained from the other techniques and presented in Tables 1 and 2. The comparisons approves that the developed technique gives better results than the

Table 1: Comparison of error norms L_∞ and three invariants at $t = 80$ for $\Delta t = 0.05$ and $c = 0.01$

Method	L_∞	h	I_1	I_2	I_3
Present	2.45×10^{-6}	0.05	0.119995	0.002879	0.000058
[7]	2.00×10^{-6}	0.05	0.12000	0.00288	0.000058
[5]	2.06×10^{-4}	0.05	0.12088	0.00291	0.000059
Exact			0.1200	0.00288	0.00006

Table 2: Comparison of error norms L_∞ and three invariants at $t = 80$ for $\Delta t = 0.05$ and $c = 0.1$

Method	L_∞	h	I_1	I_2	I_3
Present	7.37×10^{-6}	0.03	1.19999	0.28800	0.05760
[7]	2.10×10^{-5}	0.03	1.19995	0.28798	0.05759
[5]	1.64×10^{-2}	0.03	1.23387	0.29915	0.06097
[11]	5.15×10^{-5}	0.03	1.20004	0.28880	0.05760
[12]	9.60×10^{-6}	0.03	1.19999	0.28800	0.05760
[13]	1.26×10^{-5}	0.03	1.19999	0.28799	0.05759
Exact			1.20000	0.28800	0.05760

other techniques. Moreover, the error norms provide strong evidence that the applied technique gives highly accurate results.

The error norms L_∞ calculated for different spatial step sizes are given in Table 3 together with the spatial order of convergence for interval $-10 \leqslant x \leqslant 40$ by adjusting time step size as 0.125. It is obvious that computational order of the spatial convergence shows strong agreement with the theoretical one. Error norms decreases when the spatial step size is decreased and spatial order of convergence approaches four. Table 4 presents temporal order of covergence with different time step sizes for interval $-10 \leqslant x \leqslant 40$ together with error norms. As the time step size decreases, error norm also decreases significantly. Temporal order of convergence also approaches three.

Table 3: The error norms and spatial order of convergence for space at $t = 80$ for $t = 0.125$, $c = 0.1$

h	L_∞	Order
0.5	6.26×10^{-4}	–
0.25	3.77×10^{-5}	4.05
0.125	2.33×10^{-6}	4.02
0.0625	1.51×10^{-7}	3.94

The plot of absolute error is indicated in Figure 2 for the values $h = 0.0625$, $t = 80$, $c = 0.1$, $\Delta t = 0.25$ and interval $-10 \leqslant x \leqslant 40$ It is very clear that the absolute error is consistent with the results presented in Table 3.

Conclusion

The results obtained with the help of the introduced procedure for solving the EW equation are presented using a hybrid computational technique with high accuracy. The spatial discretization is realized with this hybrid technique which combines cubic h!

Table 4: The error norms and temporal order of convergence for time at $t = 80$ for $h = 0.05$, $c = 0.1$

Δt	L_∞	Order
2	2.54×10^{-4}	–
1	3.29×10^{-5}	2.95
0.5	4.16×10^{-6}	2.98
0.25	5.13×10^{-7}	3.02

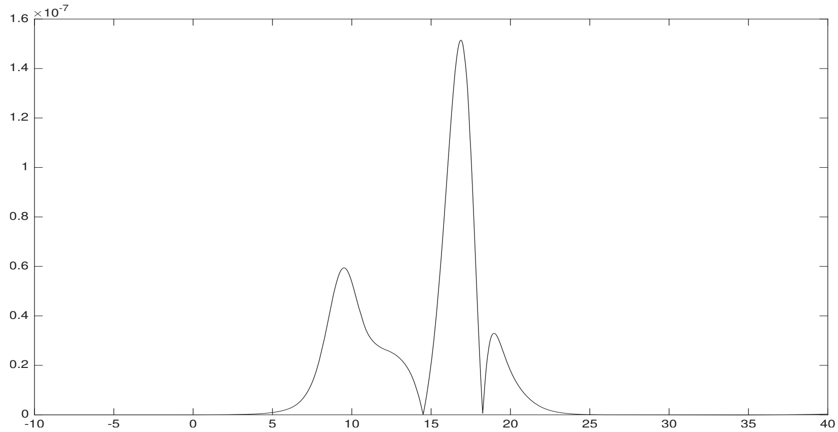

Figure 2: Absolute error at $t = 80$, $h = 0.0625$, , $c = 0.1$

B-spline functions and finite difference scheme. The developed hybrid technique requires a new adjusment for the spatial derivative terms of order two in the equation. The Adam's Moulton technique is operated in order to obtain discretization of time. The accuracy and efficency of the used technique is demonstrated by checking the error norms L_∞ and the three invariants for different values of the amplitude of wave. The evaluated results are in accordance with the corresponding theoretical values and the invariants are conserved during the computation. The spatial and temporal order of convergence are also very coherent with the theoretical values. Consequently, the hybrid method used in this study appears as a very powerful and applicable technique for the calculations of the solutions of nonlinear problems.

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Refined Fejér-type inequalities for generalized convex functions: New results on $(h, g; \alpha - m)-$convexity

Sanja Kovač

This presentation introduces new refinements of Fejér-type inequalities within the framework of generalized convexity. Specifically, we extend classical results to the class of $(h, g; \alpha - m)-$convex functions, which subsume various known convexity concepts such as $m-$convex, $h-$convex, and standard convex functions as special cases. The main contribution includes improved integral bounds for convex and concave functions weighted by symmetric integrable functions. These results not only generalize classical inequalities but also provide tighter estimates with broader applicability in mathematical analysis, particularly in the study of integral inequalities and optimization. Special cases and connections to existing theorems are also discussed, offering a unified approach to convexity-based inequalities.

2020 MSC: 26D10, 26D15, 26D20, 26D99

KEYWORDS: Fejér inequality, convex function, $(h, g; -m)-$convex function

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Combinatorial interpretation of some fubini-type polynomials in terms of Lah-numbers

Sithembele Nkonkobe

We study a class of fubini-type polynomials. We represent these fubini-type polynomials in terms of Lah numbers. The insertion of vertical bars in-between the subsets of an ordered set partition forms a barred preferential arrangement. We propose a combinatorial interpretation of these Fubini-type polynomials in terms of barred preferential arrangements.

2020 MSC: 05A15, 05A16, 05A18, 05A19

Keywords: Lah numbers, Stirling numbers, Barred preferential arrangements

Introduction

The strling numbers of the second kind $\left\{ {n \atop m} \right\}$ are well known (*cf.* [4, 15]) that they may be defined in the following way

$$\sum_{n=0}^{n} \left\{ {n \atop m} \right\} \frac{x^n}{n!} = \frac{(e^x - 1)^m}{m!}. \tag{1}$$

The numbers $\left\{ {n \atop m} \right\}$ may be interpreted combinatorial as giving the number of ways of partitioning $[n] = \{1, 2, 3, \ldots, n\}$ into m non-empty subsets (*cf.* [4, 15]).

The number of preferential arrangements P_n of the set $[n]$ is defined in the following way (*cf.* [5])

$$\sum_{n=0}^{\infty} P_n \frac{x^n}{n!} = \frac{1}{2 - e^x}. \tag{2}$$

For $n \geqslant 0$ the integer sequence given by the number of preferential arrangements goes as far back as Arthur Caycley year 1859 paper [3].

A type of Fubini polynomials F_n defined in the following way appear in the year 1991 paper [16] in connection with chains in power sets

$$\sum_{n=0}^{\infty} P_n^l \frac{x^n}{n!} = \frac{e^{lx}}{2 - e^x}. \tag{3}$$

A barred preferential arrangement is formed when one inserts a number of identical bars in-between the blocks of an ordered partition (*cf.* [1, 14]). The following are barred preferential arrangements of the sets [8], and [6], have three and two bars respectively

i) 4 1|35 67|8|,

ii) |46 123 5|.

The barred preferential arrangement in (i), the three bars induce four sections. The first section (to the left of the first bar), there are two blocks $\{4\}$, and $\{1\}$. The second section, the section in-between the first two bars has the blocks $\{3, 5\}$, and $\{6, 7\}$. The barred preferential arrangement in (*ii*) has two bars hence three sections. The first and the third sections are empty. The second section has the blocks $\{4, 6\}$, $\{1, 2, 3\}$, and $\{5\}$. The number $P_n(\lambda)$ may be defined in the following way (*cf.* [1])

$$\sum_{n=0}^{\infty} P_n(\lambda)\frac{x^n}{n!} = \frac{1}{(2-e^x)^{\lambda}}. \tag{4}$$

Definition 1 (Restricted Section [11]). *A restricted section of a barred preferential arrangement is one where elements form at most one block.*

Definition 2 (Free Section [11]). *A free section of a barred preferential arrangement is one where elements are partitioned into several ordered blocks.*

Nkonkobe and Murali in [11] studied the following generalization of (3) which is a class of fubini-polynomials,

$$\sum_{n=0}^{\infty} P^l_n(\lambda)\frac{x^n}{n!} = \frac{e^{lx}}{(2-e^x)^{\lambda}} \tag{5}$$

The polynomials $P^l_n(\lambda)$ represent the number of barred preferential arrangements of $[n]$ into l restricted sections, and λ free sections [11]. A number of researchers have shown interest on fubini-polynomials, for instance see the generalization of (5) in [12], and references therein. In the next section we will discuss several other fubini related polynomials.

Preliminaries

Related to the Fubini-polynomials are Fubini-type polynomials. There are several versions of Fubini-type polynomials, for instance

$$\sum_{n=0}^{\infty} F^{\theta}{}_n(\gamma)\frac{t^n}{n!} = \frac{2^{\gamma}e^{\theta t}}{(2-e^t)^{2\gamma}}, \tag{6}$$

The Fubini-type polynomials in (6) seems to first appear in Kilar and Simsek's paper [7]. Properties of these polynomials have been extensively studied by Kilar and Simsek in [8, 10]. Another type of Fubini-type polynomials is the following

$$\sum_{n=0}^{\infty} G(n)\frac{t^n}{n!} = \frac{e^t-1}{2-e^t}. \tag{7}$$

Equation 7 seem to first appear in (*cf.* [9]). The following generalized version of (7) was independently discovered in [1] which was combinatorially motivated in connection with special barred preferential arrangements

$$\sum_{n=0}^{\infty} G_m(n)\frac{t^n}{n!} = \left(\tfrac{e^t-1}{2-e^t}\right)^m. \tag{8}$$

Definition 3 (*cf.* [1]). *Combinatorially $G_m(n)$ gives the number of barred preferential arrangements of $[n]$ having m bars such that none of the sections of the barred preferential arrangements are empty.*

The barred preferential arrangements in Definition 3 are referred to as special barred preferential arrangements in [1]. A well-known representation of the Lah-numbers is given as follows (*cf.* [6] and references therein)

$$\sum_{n\geqslant m} L(n,m)\frac{t^n}{n!} = \frac{1}{m!}\left(\tfrac{t}{1-t}\right)^m, \tag{9}$$

where $L(n,m)=0$ for $m>n$.

Definition 4. *[Lah-distribution [2]] The Lah-numbers $L(n,m)$ combinatorially gives the number of partitions of $[n]$ into m ordered lists.*

Theorem 5 (*cf.* [13])**.** *For $n\geqslant m\geqslant 0$ we have*

$$L(n,m) = \frac{1}{m!}\sum_{j=0}^{m}\binom{m}{j}(-1)^{m-j} < j >_n . \tag{10}$$

where $< x >_n = x(x+1)(x+2)\cdots(x+n-1)$.

Theorem 6 (*cf.* [13])**.** *For $n\geqslant m\geqslant 0$ we have*

$$L(n,m) = \sum_{j=m}^{n}\left\{ {j \atop m} \right\}\left[{n \atop j} \right]. \tag{11}$$

Main results

In this section we present the main results of the study.

Theorem 7. *For $n\geqslant m\geqslant 0$ we have*

$$G_m(n) = \sum_{j\geqslant m} L(j,m)\left\{ {n \atop j} \right\} m!. \tag{12}$$

Proof. Special barred preferential arrangements may be formed in the following way. By partitioning $[n]$ into m subsets in $\left\{ {n \atop m} \right\}$. Then, m ordered lists of the j subsets may be formed in $L(j,m)$ ways. Then, $m!$ account for the fact that each of the m lists may occupy any of the m sections of a special barred preferential arrangement.

Theorem 8. *For $n\geqslant m\geqslant 0$ we have*

$$G_m(n+1) = m\sum_{i_1+i_2+i_3+i_4=n}^{n}\binom{n}{i_1,i_2,i_3,i_4}\sum_{j\geqslant m-1} L(j,m-1)\left\{ {i_1 \atop j} \right\}(m-1)!P_{i_2}P_{i_4}. \tag{13}$$

Proof. Special barred preferential arrangements of $[n+1]$ may be formed in the following way. We denote the block having $(n+1)$ by $\mathcal{C}$. We let i_1 denote the number of elements of $[n]$ that are not in the same section as the element $(n+1)$. We let i_2 elements which are in the same section as $(n+1)$ and forming blocks to the left of $\mathcal{C}$, of which by (2), there are P_{i_2} such possible blocks. We let i_3 denote elements of $[n]$ that form part of $\mathcal{C}$, and we let i_4 denote elements of $[n]$ that form blocks to the write of $\mathcal{C}$,of which by (2), there are P_{i_2} such possible blocks. The section to which $(n+1)$ belongs may be chosen in m ways. Regarding the $m-1$ other sections, $[i_1]$ into $m-1$ subsets in $\left\{ {i_1 \atop m-1} \right\}$. Then, $m-1$ ordered lists of the j subsets may be formed in $L(j,m-1)$ ways. Then, $(m-1)!$ account for the fact that each of the $m-1$ lists may occupy any of the $m-1$ sections of a special barred preferential arrangement (see Theorem 7).

Theorem 5 and (8) one gets the following.

Theorem 9. *For $n \geqslant m \geqslant 0$ we have*

$$G_m(n) = \sum_{j \geqslant m} \sum_{j=0}^{m} \binom{m}{j} (-1)^{m-j} < j >_n m \left\{ {n-1 \atop m} \right\} + \sum_{j \geqslant m} \sum_{j=0}^{m} \binom{m}{j} (-1)^{m-j} < j >_n \left\{ {n-1 \atop m-1} \right\}.$$

Theorem 9 may be proved combinatorially by basing the argument on the block having the *nth* element, and using the inclusion/exclusion principle. By (8) we have the following.

Theorem 10. *For $n, m \geqslant 0$ we have*

$$G_n(m) = \sum_{j=0}^{\infty} \binom{m+j-1}{j} \sum_{l=0}^{m+j} \binom{m+j}{l} (-1)^{m+j-l} (m+j)^n. \tag{14}$$

Conclusion

In this study we studied combinatorial identities of number of special barred preferential arrangements using the Lah numbers. Other properties of the numbers may be studied such as arithmetic properties, asymptotic expansions, and bijective proofs.

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Picard's iteration on fractional ordinary differential equations

Tzon-Tzer Lu

Picard's Iteration is the most profound technique in differential equations. It is a fixed point iteration and used to prove the existence and uniqueness of a differential equation. Moreover, its successive approximations provide a decent numerical solution for this equation.

Fractional calculus is the hottest research topic recently. Many practical problems can be modeled by fractional differential equations. So it has many applications on fluid dynamics, geology, material science, electromagnetism, astrophysics, optics, bio-sciences, economics, to name a few. Nowadays fractional calculus plays a crucial role in all disciplines.

In this talk we employ Picard's iteration to solve fractional ordinary differential equations with different types of fractional derivatives, e.g. Riemann–Liouville, Caputo, Caputo–Fabrizio, Atangana–Baleanu etc. We begin with the simplest models and approximate them by Picard's iteration. Then we test more general problems. We will report several interesting observations and its possible extension to fractional partial differential equations.

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Keywords: Picard's iteration, fractional differential equation, ordinary differential equation, fractional calculus

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Asymptotic expansion of the Gaussian and Archimedean compound of non-symmetric means

Toni Milas

We derive asymptotic expansions of the Gaussian compound mean $M \otimes_g N$ and the Archimedean compound mean $M \otimes_a N$, where M and N are arbitrary non-symmetric means which possess asymptotic expansions in terms of negative powers. We present applications to some classical means, such as neo-Pythagorean means.

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KEYWORDS: Asymptotic expansions, compound means

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Utilization of a general form of generating functions to extract a comprehensive class of numbers and polynomials

Irem Kucukoglu

In this study, in order to show how an inclusive family of special numbers and polynomials can be brought to light, it has been addressed to consider the usage of general forms of ordinary generating functions, introduced recently by Simsek [9]. In this context, several special cases of these general forms have been discussed and analyzed by picking its parameters as different values of the classical Bernoulli polynomials. Particularly, this study has been conducted to demonstrate how general forms of generating functions are used to extract a comprehensive class of numbers and polynomials and how their computation formulas are obtained.

2020 MSC: 05A15, 11B39, 11B68, 11B83, 11L99, 33F99, 65D20

Keywords: Generating functions, rational generating functions, rational functions, generalization of Fibonacci and Lucas type numbers polynomials, Bernoulli numbers and polynomials, computation formulas, finite sums, multivariate polynomials

Introduction

The main purpose of this study is to illustrate the usage of general forms of ordinary generating functions to reveal a comprehensive class of number and polynomial sequences which are generated by rational generating functions.

To carry out this study, we shall mainly prefer to utilize the positive higher-order Bernoulli numbers $B_n^{(v)}$ and the positive higher-order Bernoulli polynomials $B_n^{(v)}(x)$ whose generating functions are respectively given by

$$\left(\frac{t}{e^t-1}\right)^v = \sum_{n=0}^{\infty} B_v^{(v)} \frac{t^n}{n!} \tag{1}$$

and

$$\left(\frac{t}{e^t-1}\right)^v e^{xt} = \sum_{n=0}^{\infty} B_n^{(v)}(x) \frac{t^n}{n!}, \tag{2}$$

where $|t| < 2\pi$ (*cf.* [7]; see also [6, 8, 10]).

Note that the special case of (2) when $v = 1$ gives the generating function for the classical Bernoulli polynomials, i.e.:

$$B_n(x) = B_n^{(1)}(x),$$

and also if $v = 1$ and $x = 0$, then the equation (2) generates the classical Bernoulli numbers, i.e:

$$B_n = B_n(0) = B_n^{(1)}(0), \tag{3}$$

which are computed by the following recursive formula:

$$B_n = \sum_{j=0}^{n} \binom{n}{j} B_j, \tag{4}$$

by which one may easily obtain that $B_{2n+1} = 0$ for $n \geqslant 1$, and the first few values of the Bernoulli numbers enumerated as below:

$$\begin{aligned} &B_0 = 1, \quad B_1 = -\frac{1}{2}, \quad B_2 = \frac{1}{6}, \quad B_4 = -\frac{1}{30}, \\ &B_6 = \frac{1}{42}, \quad B_8 = -\frac{1}{30}, \quad B_{10} = \frac{5}{66}, \quad B_{10} = -\frac{691}{2730}, \end{aligned}$$

and so on (*cf.* [2, 10]).

As for the multivariate polynomials $\mathbb{Y}_n\left(P(\overrightarrow{X_m})\right)$ and $\mathbb{S}_n\left(P(\overrightarrow{X_m}); Q(\overrightarrow{X_k})\right)$, they made acquainted recently by Simsek [9], respectively as in the following general forms of ordinary generating functions, for positive integer m and nonnegative integer k:

$$\begin{aligned} \mathcal{F}\left(t, P(\overrightarrow{X_m})\right) &:= \frac{1}{1 + \sum\limits_{j=1}^{m} P_j(x_j) t^j} \\ &= \sum_{n=0}^{\infty} \mathbb{Y}_n\left(P(\overrightarrow{X_m})\right) t^n \end{aligned} \tag{5}$$

and

$$\begin{aligned} \mathcal{G}\left(t, P(\overrightarrow{X_m}); Q(\overrightarrow{X_k})\right) &:= \frac{\sum\limits_{j=0}^{k} Q_j(x_j) t^j}{1 + \sum\limits_{j=1}^{m} P_j(x_j) t^j} \\ &= \sum_{n=0}^{\infty} \mathbb{S}_n\left(P(\overrightarrow{X_m}); Q(\overrightarrow{X_k})\right) t^n, \end{aligned} \tag{6}$$

where

$$\begin{aligned} P(\overrightarrow{X_m}) &= \left(P_1(x_1), P_2(x_2), \ldots, P_m(x_m)\right), \\ Q(\overrightarrow{X_k}) &= \left(Q_0(x_0), Q_1(x_1), Q_2(x_2), \ldots, Q_k(x_k)\right) \end{aligned}$$

with any polynomials, for nonenegative integers d and c:

$$P_j(x_j) = \sum_{\omega=0}^{d} a_\omega x_j^\omega, \qquad Q_r(x_r) = \sum_{\omega=0}^{c} b_\omega x_r^\omega,$$

which are polynomials respectively in x_j and x_r such that

$$1 \leqslant j \leqslant m$$

$$0 \leqslant r \leqslant k,$$

(*cf.* [9]).

It is understood from Simsek's paper [9] that the fundamental reason he constructed these general forms is to unify and generalize the functions that generate Fibonacci type, Lucas type, Pell type and Sextet type numbers and polynomials.

However, subsequent studies conducted in [1], [3], [5] and [4] revealed that the general forms (5) and (6) therewithal encompass an infinite diverse number of generating functions when their parameters are selected in varying tuples.

Let $[\omega]$ denote the greatest integer $\leqslant \omega$. Then, the polynomials $\mathbb{Y}_n\left(P(\overrightarrow{X_m})\right)$ are computed by the following formula, for $n_1 \in \mathbb{N}_0$ and $m \in \mathbb{N}$ with $m > 1$ (see, for details, [9, p. 6, Theorem 1]):

$$\begin{aligned}\mathbb{Y}_{n_1}\left(P(\overrightarrow{X_m})\right) &= \left(\prod_{j=2}^{m}\sum_{n_j=0}^{\left[\frac{n_{j-1}}{j}\right]}\right)(-1)^{\sum\limits_{j=1}^{m} n_j} \\ &\quad\times \prod_{d=2}^{m}\binom{\sum\limits_{k=1}^{d} n_k}{n_d}\prod_{v=1}^{m}\left(P_v(x_v)\right)^{n_v},\end{aligned} \tag{7}$$

where

$$(-1)^{\sum\limits_{j=1}^{m} n_j} = (-1)^{n_1+n_2+\cdots+n_m},$$

$$\prod_{d=2}^{m}\binom{\sum\limits_{k=1}^{d} n_k}{n_d} = \binom{n_1+n_2}{n_2}\binom{n_1+n_2+n_3}{n_3}\cdots\binom{n_1+n_2+\cdots+n_m}{n_m},$$

$$\prod_{v=1}^{m}\left(P_v(x_v)\right)^{n_v} = \left(P_1(x_1)\right)^{n_1}\left(P_2(x_2)\right)^{n_2}\cdots\left(P_m(x_m)\right)^{n_m}$$

and

$$\left(\prod_{j=2}^{m}\sum_{n_j=0}^{\left[\frac{n_{j-1}}{j}\right]}\right) = \sum_{n_2=0}^{\left[\frac{n_1}{2}\right]}\sum_{n_3=0}^{\left[\frac{n_2}{2}\right]}\cdots\sum_{n_m=0}^{\left[\frac{n_{m-1}}{m}\right]}.$$

The polynomials $\mathbb{S}_n\left(P(\overrightarrow{X_m});Q(\overrightarrow{X_k})\right)$ are computed by the following formula:

$$\mathbb{S}_n\left(P(\overrightarrow{X_m});Q(\overrightarrow{X_k})\right) = \sum_{j=0}^{k} Q_j(x_j)\mathbb{Y}_{n-j}\left(P(\overrightarrow{X_m})\right) \tag{8}$$

for $n, k \in \mathbb{N}_0$ and $m \in \mathbb{N}$ (*cf.* [9, p. 9, Theorem 2]).

In [3] and [4], Kucukoglu investigated several special cases of the polynomials $\mathbb{S}_n\left(P(\overrightarrow{X_m});Q(\overrightarrow{X_k})\right)$ and $\mathbb{Y}_n\left(P(\overrightarrow{X_m})\right)$ with their generating functions by considering their components as the Euler and Genocchi numbers and polynomials.

Moreover, in the book chapter [5], written to make the usability of general forms easier to understand by the researchers, Kucukoglu and Simsek [5] wrote the polynomials $\mathbb{S}_n\left(P(\overrightarrow{X_m});Q(\overrightarrow{X_k})\right)$ and their generating functions in terms of the Bernoulli numbers and polynomials, and investigated their several special cases providing by some formulas and tables. Indeed, when the following tuples:

$$P(\overrightarrow{X_m}) = \left(B_1^{(v_1)}(x_1), B_2^{(v_2)}(x_2), \ldots, B_m^{(v_m)}(x_m)\right)$$

and

$$Q(\overrightarrow{X_k}) = \left(B_0^{(v_0)}(x_0), B_1^{(v_1)}(x_1), \ldots, B_k^{(v_k)}(x_k)\right)$$

are substituted into the equation (6), the following generating functions are obtained:

$$
\begin{aligned}
&\mathcal{G}\left(t,\left(B_1^{(v_1)}\left(x_1\right),B_2^{(v_2)}\left(x_2\right),\ldots,B_m^{(v_m)}\left(x_m\right)\right)\right.\\
&\qquad\qquad\left.;\left(B_0^{(v_0)}\left(x_0\right),B_1^{(v_1)}\left(x_1\right),\ldots,B_k^{(v_k)}\left(x_k\right)\right)\right)\\
&=\frac{\sum\limits_{j=0}^{k}B_j^{(v_j)}(x_j)t^j}{1+\sum\limits_{j=1}^{m}B_j^{(v_j)}(x_j)t^j}\\
&=\sum_{n=0}^{\infty}\mathbb{S}_n\left(\left(B_1^{(v_1)}\left(x_1\right),B_2^{(v_2)}\left(x_2\right),\ldots,B_m^{(v_m)}\left(x_m\right)\right)\right.\\
&\qquad\qquad\left.;\left(B_0^{(v_0)}\left(x_0\right),B_1^{(v_1)}\left(x_1\right),\ldots,B_k^{(v_k)}\left(x_k\right)\right)\right)t^n,
\end{aligned}
\tag{9}
$$

(see, for details, [5]).

In the next section, it is shown how the mentioned general forms can be used to extract a comprehensive class of special sequences by considering some other special values of the Bernoulli numbers in these components of the general forms.

Key findings and observations

This section presents key findings of this research and observations made on them.

Here, we begin our research with the consideration of (9) for its special case:

If we set $x_j = 1$ and $v_j = 1$ in (9), and substitute the following well-known fact (*cf.* [11]):

$$B_n(1) = (-1)^n B_n; \quad n \in \mathbb{N}_0$$

into the final equation, we achieve

$$
\begin{aligned}
&\frac{\sum\limits_{j=0}^{k}(-1)^j B_j t^j}{1+\sum\limits_{j=1}^{m}(-1)^j B_j t^j}\\
&=\sum_{n=0}^{\infty}\mathbb{S}_n\left(\left(-B_1,B_2,\ldots,(-1)^m B_m\right);\left(B_0,-B_1,\ldots,(-1)^k B_k\right)\right)t^n
\end{aligned}
\tag{10}
$$

in which case, if we take $k = 1$ and $m = 7$, we obtain

$$
\begin{aligned}
&\frac{B_0-B_1t}{1-B_1t+B_2t^2-B_3t^3+B_4t^4-B_5t^5+B_6t^6-B_7t^7}\\
&=\sum_{n=0}^{\infty}\mathbb{S}_n\left(\left(-B_1,B_2,-B_3,B_4,-B_5,B_6,-B_7\right);\left(B_0,-B_1\right)\right)t^n.
\end{aligned}
$$

By embedding the values of the Bernoulli numbers, the above equation implies

$$
\begin{aligned}
&\frac{105(2+t)}{210+105t+35t^2-7t^4+5t^6}\\
&=\sum_{n=0}^{\infty}\mathbb{S}_n\left(\left(\frac{1}{2},\frac{1}{6},0,-\frac{1}{30},0,\frac{1}{42},0\right);\left(1,\frac{1}{2}\right)\right)t^n.
\end{aligned}
\tag{11}
$$

Remark 1. *The question may be asked what the calculation formula for the numbers*

$$\mathbb{S}_n\left(\left(\frac{1}{2},\frac{1}{6},0,-\frac{1}{30},0,\frac{1}{42},0\right);\left(1,\frac{1}{2}\right)\right)$$

is. In this instance, we benefit from the equations (7) *and* (8) *to answer this question, as detailed below.*

Due to the nature of the generating functions, one may easily infer that the function on the left-hand side of the equation (11) corresponds to the ordinary generating function of the numbers

$$\mathbb{S}_n\left(\left(\frac{1}{2},\frac{1}{6},0,-\frac{1}{30},0,\frac{1}{42},0\right);\left(1,\frac{1}{2}\right)\right),$$

and by applying the equations (7) and (8), the computation formula for these numbers is given by the following theorem:

Theorem 2. *Let*

$$\begin{gathered}B_0=1,\quad B_1=-\frac{1}{2},\quad B_2=\frac{1}{6},\quad B_3=0,\\ B_4=-\frac{1}{30},\quad B_5=0,\quad B_6=\frac{1}{42},\quad B_7=0.\end{gathered}$$

Then, for $n_1 \in \mathbb{N}_0$, we have

$$\begin{aligned}&\mathbb{S}_{n_1}\left((-B_1,B_2,-B_3,B_4,-B_5,B_6,-B_7);(B_0,-B_1)\right)\\ &=\mathbb{Y}_{n_1}\left((-B_1,B_2,-B_3,B_4,-B_5,B_6,-B_7)\right)\\ &\qquad-\frac{1}{2}\mathbb{Y}_{n_1-1}\left((-B_1,B_2,-B_3,B_4,-B_5,B_6,-B_7)\right),\end{aligned}$$

where

$$\begin{aligned}&\mathbb{Y}_{n_1}\left((-B_1,B_2,-B_3,B_4,-B_5,B_6,-B_7)\right)\\ &=\sum_{n_2=0}^{\left[\frac{n_1}{2}\right]}\sum_{n_3=0}^{\left[\frac{n_2}{3}\right]}\cdots\sum_{n_7=0}^{\left[\frac{n_6}{7}\right]}\left((-1)^{\sum_{r=1}^{7}n_r}\right)\binom{n_1+n_2}{n_2}\binom{n_1+n_2+n_3}{n_3}\\ &\qquad\times\cdots\times\binom{n_1+n_2+\cdots+n_7}{n_7}\prod_{v=1}^{7}\left((-1)^v B_v\right)^{n_v}.\end{aligned}$$

If we set $x_j=\frac{1}{2}$ and $v_j=1$ in (9) and substitute the following well-known fact (*cf.* [11]):

$$B_n\left(\frac{1}{2}\right)=\left(2^{1-n}-1\right)B_n;\quad n\in\mathbb{N}_0$$

into the final equation, we achieve

$$\begin{aligned}&\frac{\sum_{j=0}^{k}\left(2^{1-j}-1\right)B_j t^j}{1+\sum_{j=1}^{m}\left(2^{1-j}-1\right)B_j t^j} \qquad (12)\\ &=\sum_{n=0}^{\infty}\mathbb{S}_n\left(\left(0,-\frac{B_2}{2},-\frac{3B_3}{4},-\frac{7B_4}{8},\ldots,\left(2^{1-m}-1\right)B_m\right);\right.\\ &\qquad\left.\left(B_0,0,-\frac{B_2}{2},-\frac{3B_3}{4},-\frac{7B_4}{8},\ldots,\left(2^{1-k}-1\right)B_k\right)\right)t^n.\end{aligned}$$

Next, we list and investigate several generating functions that arise from (12) in some of its special cases, (among others):

First, we consider a special case of (12), in which $k = 1$ and $m = 2$:

$$\frac{B_0}{1 - \frac{B_2}{2}t^2} = \sum_{n=0}^{\infty} \mathbb{S}_n\left(\left(0, -\frac{B_2}{2}\right); (B_0, 0)\right) t^n.$$

By substituting the values of the Bernoulli numbers, the above equation reduces to

$$\frac{12}{12 - t^2} = \sum_{n=0}^{\infty} \mathbb{S}_n\left(\left(0, -\frac{1}{12}\right); (1, 0)\right) t^n. \tag{13}$$

With the assumption of $|t^2| < 12$, if we apply series expansion on the left-hand side of the above equation, and then compare the coefficients of t^n on both sides of the final equation, we arrive at the following theorem:

Theorem 3. *Let n be a nonnegative integer. Then we have*

$$\mathbb{S}_n\left(\left(0, -\frac{1}{12}\right); (1, 0)\right) = \begin{cases} 0 & \text{if } n = 2r + 1 \\ (12)^{-r} & \text{if } n = 2r, \end{cases}$$

where r is a nonnegative integer.

Next, we consider another special case of (12), in which $k = 1$ and $m = 4$:

$$\begin{aligned} &\frac{B_0}{1 - \frac{B_2}{2}t^2 - \frac{3B_3}{4}t^3 - \frac{7B_4}{8}t^4} \\ &= \sum_{n=0}^{\infty} \mathbb{S}_n\left(\left(0, -\frac{B_2}{2}, -\frac{3B_3}{4}, -\frac{7B_4}{8}\right); (B_0, 0)\right) t^n. \end{aligned}$$

By substituting the values of the Bernoulli numbers into the above equation, we have

$$\frac{240}{240 - 20t^2 + 7t^4} = \sum_{n=0}^{\infty} \mathbb{S}_n\left(\left(0, -\frac{1}{12}, 0, \frac{7}{240}\right); (1, 0)\right) t^n. \tag{14}$$

In a similar manner, due to the nature of the generating functions, one may easily infer that the function on the left-hand side of the equation (14) corresponds to the ordinary generating function of the numbers

$$\mathbb{S}_n\left(\left(0, -\frac{1}{12}, 0, \frac{7}{240}\right); (1, 0)\right),$$

and by applying the equations (7) and (8), the computation formula for these numbers is given by the following theorem:

Theorem 4. *Let*

$$B_0 = 1, \quad B_1 = -\frac{1}{2}, \quad B_2 = \frac{1}{6}, \quad B_3 = 0, \quad B_4 = -\frac{1}{30}.$$

Then, for $n_1 \in \mathbb{N}_0$, we have

$$\mathbb{S}_{n_1}\left(\left(0, -\frac{B_2}{2}, -\frac{3B_3}{4}, -\frac{7B_4}{8}\right); (B_0, 0)\right) = \mathbb{Y}_{n_1}\left(\left(0, -\frac{B_2}{2}, -\frac{3B_3}{4}, -\frac{7B_4}{8}\right)\right),$$

where

$$\begin{aligned}
&\mathbb{Y}_{n_1}\left(\left(0,-\frac{B_2}{2},-\frac{3B_3}{4},-\frac{7B_4}{8}\right)\right)\\
&=\sum_{n_2=0}^{\left[\frac{n_1}{2}\right]}\sum_{n_3=0}^{\left[\frac{n_2}{3}\right]}\cdots\sum_{n_4=0}^{\left[\frac{n_3}{4}\right]}\left((-1)^{\sum\limits_{r=1}^{4}n_r}\right)\binom{n_1+n_2}{n_2}\binom{n_1+n_2+n_3}{n_3}\\
&\qquad\times\binom{n_1+n_2+n_3+n_4}{n_4}\prod_{v=1}^{4}\left(\left(2^{1-v}-1\right)B_v\right)^{n_v}.
\end{aligned}$$

In what follows, we consider another special case of (12), in which $k=1$ and $m=6$:

$$\begin{aligned}
&\frac{B_0}{1-\frac{B_2}{2}t^2-\frac{3B_3}{4}t^3-\frac{7B_4}{8}t^4-\frac{15B_5}{16}t^5-\frac{31B_6}{32}t^6}\\
&=\sum_{n=0}^{\infty}\mathbb{S}_n\left(\left(0,-\frac{B_2}{2},-\frac{3B_3}{4},-\frac{7B_4}{8},-\frac{15B_5}{16},-\frac{31B_6}{32}\right);(B_0,0)\right)t^n.
\end{aligned}$$

Substituting the values of the Bernoulli numbers into the above equation yields

$$\begin{aligned}
&\frac{6720}{6720-560t^2+196t^4-155t^6}\\
&=\sum_{n=0}^{\infty}\mathbb{S}_n\left(\left(0,-\frac{1}{12},0,\frac{7}{240},0,-\frac{31}{1344}\right);(1,0)\right)t^n.
\end{aligned}\tag{15}$$

Likewise, by the nature of the generating functions, one may easily infer that the function on the left-hand side of the equation (15) corresponds to the ordinary generating function of the numbers

$$\mathbb{S}_n\left(\left(0,-\frac{1}{12},0,\frac{7}{240},0,-\frac{31}{1344}\right);(1,0)\right),$$

and by applying the equations (7) and (8), the computation formula for these numbers is given by the following theorem:

Theorem 5. *Let*

$$B_0=1,\quad B_1=-\frac{1}{2},\quad B_2=\frac{1}{6},\quad B_3=0,$$
$$B_4=-\frac{1}{30},\quad B_5=0,\quad B_6=\frac{1}{42}.$$

Then, for $n_1\in\mathbb{N}_0$, we have

$$\begin{aligned}
&\mathbb{S}_{n_1}\left(\left(0,-\frac{B_2}{2},-\frac{3B_3}{4},-\frac{7B_4}{8},-\frac{15B_5}{16},-\frac{31B_6}{32}\right);(B_0,0)\right)\\
&=\mathbb{Y}_{n_1}\left(\left(0,-\frac{B_2}{2},-\frac{3B_3}{4},-\frac{7B_4}{8},-\frac{15B_5}{16},-\frac{31B_6}{32}\right)\right),
\end{aligned}$$

where

$$\begin{aligned}
&\mathbb{Y}_{n_1}\left(\left(0,-\frac{B_2}{2},-\frac{3B_3}{4},-\frac{7B_4}{8},-\frac{15B_5}{16},-\frac{31B_6}{32}\right)\right)\\
&=\sum_{n_2=0}^{\left[\frac{n_1}{2}\right]}\sum_{n_3=0}^{\left[\frac{n_2}{3}\right]}\cdots\sum_{n_6=0}^{\left[\frac{n_5}{6}\right]}\left((-1)^{\sum\limits_{r=1}^{6}n_r}\right)\binom{n_1+n_2}{n_2}\binom{n_1+n_2+n_3}{n_3}\\
&\qquad\cdots\times\binom{n_1+n_2+\cdots+n_6}{n_6}\prod_{v=1}^{6}\left(\left(2^{1-v}-1\right)B_v\right)^{n_v}.
\end{aligned}$$

Remark 6. *In addition to the cases discussed in this study, more cases can be diversified further and elaborated in a similar manner. In this context, analysis of other cases of* (10) *and* (12)*, for other values of k and m, is left to the reader as it can be done by using the same technique handled here. Besides, the interested readers are advised to consult the references* [9], [5], [3] *and* [4] *for a more detailed analysis and tables of the other special number and polynomial sequences reduced from the general forms given by the equations* (5) *and* (6)*.*

Conclusion

We can conclude this study by stating that we can generate an infinite number of generating functions and sequences corresponding to the unlimited number of choices for the parameters of the general forms given by the equations (5) and (6). Moreover, the terms of the corresponding sequences can be calculated by using the formulas given in the equations (7) and (8).

The fact that the general forms mentioned cover a large family of numbers and polynomials and that the sequences produced can be calculated with the existing calculation formulas is an important contribution to the literature, and will lead many researchers to perform calculations effortlessly.

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On this occasion, in recognition of Professor López-Pellicer's great contributions to mathematics, especially to the areas of functional analysis and general topology, I would like to dedicate my present study to him on the occasion of his 81st birthday, with my best wishes. I wish him a happy, fruitful, and successful life, and many healthy years to spend with his family.

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Comparison of SPIHT and CAE image compression methods in the chroma-key effect of a virtual television studio

Mirko Milošević [1] *and Branimir Jakšić* *[2]

This paper compares the performance of the traditional SPIHT image compression method with a deep learning-based method – the Convolutional Autoencoder (CAE) – in a virtual TV studio. The experiments were carried out with different values of front and back lighting. Compression quality was evaluated using PSNR and SSIM metrics. The results show that CAE provides better performance at lower bitrates and under unfavorable lighting conditions, while SPIHT remains more suitable for real-time applications due to its simplicity and speed. SPIHT retains an advantage in terms of implementation simplicity and latency.

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Keywords: Chroma key, image compression, SPIHT, CAE, PSNR, SSIM

Introduction

The chroma key procedure in a television studio replaces the physical scenery and reduces production costs. It represents a method of mixing two video signals, where a monochrome background in one video signal (live action signal from the scene) is replaced by another video signal. Replacement is carried out by a fast switching circuit that alternately includes the foreground video signal and the new background signal. The background replacement process is called keying (*cf.* [7, 5]). Insertion of the new background video signal into the monochrome background of the foreground video signal is performed during scanning, at the boundary between the subject in the foreground and the monochrome background. In this way, the impression is created that the objects in the foreground are located in the scene displayed by the second source. The monochrome background can be any color, provided that this color does not appear in the objects or participants in the foreground (*cf.* [5]). Today, blue or green colors are typically used as the background.

The virtual TV studio provides a natural relationship between participants and computer-generated scenery by correcting the generated scenery in real time and synchronizing it with data obtained from the position of the camera that captures the foreground. In this way, the logical relationship between the scenery and the participants is maintained, while also preserving the sense of depth of the image (Figure 1) (*cf.* [5]). In a virtual studio, it is also possible to generate scenery in the foreground, which can be either opaque or transparent. Computer-generated scenery visually resembles a real set, and it is possible to create a variety of decors, even surreal ones, giving the impression that the TV studio is much larger than it actually is.

To achieve a stable chroma key effect, lighting must be uniform and shadow-free. The most commonly used types are:

Figure 1: Example of a chroma-key effect in a virtual TV studio

- Front light (200–1400 lx): illuminates the subject,
- Back light (200–1400 lx): separates the subject from the background and reduces shadows.

In such environments, image compression plays a crucial role in maintaining the quality of visual output. Traditional methods, such as JPEG (Joint Photographic Experts Group) and SPIHT (Set Partitioning in Hierarchical Trees), rely on wavelet transforms and statistical optimization. They provide good results in real time, but at low bitrates and under complex lighting conditions, the quality deteriorates.

The development of Artificial Intelligence (AI) has enabled the emergence of new compression methods based on deep neural networks. CAE (Contractive Autoencoder) and GAN (Generative Adversarial Network) achieve better visual quality and higher resistance to lighting variations, but at the cost of greater computational complexity (*cf.* [2, 9]).

The goal of this paper is to compare the SPIHT method with CAE and to highlight the advantages and limitations of each method in a virtual television studio environment.

Image compression

Compression quality metrics

The most commonly used objective measures for image evaluation are MSE (Mean Square Error), SNR (Signal to Noise Ratio), and PSNR (Peak Signal to Noise Ratio) (*cf.* [6]).

MSE is defined as:

$$MSE = \frac{E}{MN} = \frac{1}{MN} \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} ||a_{ij} - a'_{ij}||^2 \tag{1}$$

where MN is the total number of pixels in each image, and the summation is defined over all pixels, while a_{ij} and a'_{ij} are the elements of the original and compressed image matrices.

The amplitude of image elements has a range determined by $[0, 2^n - 1]$, where n is the number of bits required to represent the amplitude of the original image elements. Since MSE does not consider the amplitude range, PSNR (Peak Signal to Noise Ratio) is introduced:

$$PSNR = 10 \cdot log_{10} \left(\frac{MAX_I^2}{MSE} \right) = 20 \cdot log_{10} \left(\frac{MAX_I}{\sqrt{MSE}} \right) \tag{2}$$

Here MAX_I is the maximum pixel value. When pixels are represented with B bits per sample, MAX_I is $2^B - 1$. Typical PSNR values for lossy images range from 30 to 50 dB.

SSIM (Structural Similarity Index Measure) is calculated using luminance, contrast, and structural components.

SPIHT method

SPIHT (Set Partitioning in Hierarchical Trees) is a method based on wavelet decomposition of an image. Coefficients are organized into hierarchical trees (parent–child). At each pass, significance tests are performed, transmitting the most significant bits while deferring the less significant ones.

The algorithm maintains three lists: LIS (List of Insignificant Sets), LIP (List of Insignificant Pixels), and LSP (List of Significant Pixels), and iterates through sorting, refinement, and threshold updating phases.

The advantages of SPIHT include simple implementation, high speed, and good accuracy at medium and high bitrate values (*cf.* [3]). SPIHT does not use arithmetic coding, which makes it simpler than the EZW algorithm. However, the quality of the resulting image depends on lighting and object contrast in the frame, which is the focus of this study.

SPIHT is well-known for its efficiency at medium and high bitrates and modest hardware requirements.

Pseudo-code of SPIHT algorithm (simplified):

1. Perform wavelet decomposition of the image;
2. Initialize LIS, LIP, LSP, and threshold T = 2^n;
3. Sorting: Test the significance of coefficients and sets; transmit significant bits;
4. Refinement: Transmit the next significant bits for elements in LSP;
5. Decrease threshold T $\leftarrow$ T/2; repeat steps 3–5 until the target bpp (bits per pixel) is reached;

Convolutional Autoencoder (CAE)

A CAE is a neural network that learns the representation of an image in a latent space:

- The encoder reduces the dimensionality of the image.
- The decoder reconstructs the image from the latent representation.

The advantage of CAE is its ability to preserve structure and textures at low bpp values, which is especially important for TV production.

The encoder consists of multiple convolutional layers (3×3) with downsampling using stride 2, while the decoder uses transposed convolutions for reconstruction. Training is performed using a combined loss function:

$$L = \lambda_1 \cdot MSE + \lambda_2 \cdot \frac{LPIPS}{Perceptual} + \lambda_3 \cdot TV \tag{3}$$

where λ_1, λ_2, λ_3 are weighting coefficients (hyperparameters) that determine the relative importance of each component; LPIPS (Learned Perceptual Image Patch

Block	Operation	Channels	Output size
Enc-1	Conv 3×3, stride 2, ReLU	3→64	800×600×64
Enc-2	Conv 3×3, stride 2, ReLU	64→128	400×300×128
Enc-3	Conv 3×3, stride 2, ReLU	128→192	200×150×192
Latent	Conv 1×1 + quantization	192→N	200×150×N
Dec-1	DeConv 3×3, stride 2, ReLU	N→192	400×300×192
Dec-2	DeConv 3×3, stride 2, ReLU	192→128	800×600×128
Dec-3	DeConv 3×3, stride 2, Tanh	128→3	1600×1200×3

Table 1: Example of CAE architecture (input 1600×1200×3, output 1600×1200×3)

Similarity) is a perceptual similarity measure based on deep neural networks; TV (Total Variation Loss) enforces image smoothness (*cf.* [8, 4, 1, 10]).

Latent quantization and entropy coding are used for bitrate control.

The following hyperparameters were used: Adam (lr=10^{-4}), batch=8, epoha=50, λ_1=1.0, λ_2=0.2, λ_3=0.001. The model was trained on 5000 images of studio scenes (various layouts and light intensities) with augmentations (random crop, flip).

Experimental methodology

The experiment was conducted in a studio measuring 12 × 8 m with a blue background screen, under different combinations of front and back light intensity (*cf.* [5]). To ensure precise control of lighting, a luxmeter and a color temperature meter were used, allowing accurate comparison of results across different lighting conditions. This setup provided conditions close to real TV production.

Additionally, the experiment was designed so that compression was performed under realistic conditions, where only one lighting parameter was varied while all other conditions remained fixed. In this way, bias was avoided and a clear comparison of the effect of individual lighting factors on compression performance was enabled. For each combination of lighting and compression method, 10 test images were generated, and results were presented as average metric values.

The CAE method was implemented in a Python/TensorFlow environment, while SPIHT was realized in MATLAB. This enabled a direct comparison of the classical and AI-based methods under practically identical conditions. Calculating SSIM values additionally contributed to the analysis, since this metric better reflects human perception of image quality than PSNR alone.

Images were compressed using both SPIHT and CAE methods. The tested bitrates were: 0.05, 0.1, 0.2, 0.3, 0.5, 1.0, 1.5, and 3.0 bpp. Front light values: 200 lx, 800 lx, 1400 lx. Back light values: from 200 lx to 1400 lx, in steps of 200 lx.

The experiment was conducted so that results of each method were compared under the same lighting conditions and for identical bpp values, ensuring objective evaluation. Evaluation was performed on a CPU (SPIHT) and a GPU (CAE) to simulate typical systems (edge encoder, GPU server)

Results

Figure 2 and Figure 3 show the graphs of PSNR and SSIM variations, respectively, as functions of bpp when the front light was 800 lx and the back light 1000 lx.

Table 2 and Table 3 present PSNR and SSIM values, respectively, for fixed front light and varying back light. Table 4, Table 5, and Table 6 show PSNR values for

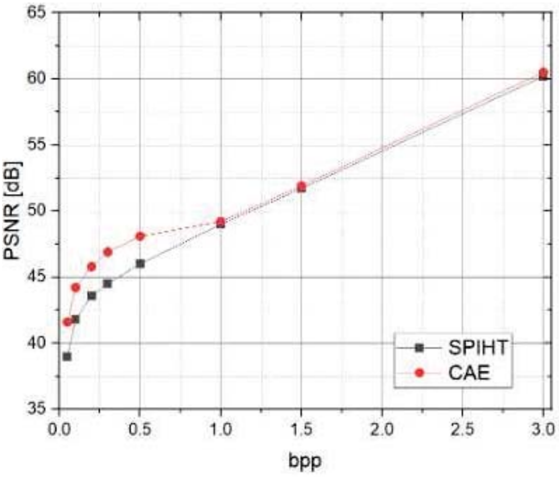

Figure 2: PSNR (dB) as a function of bpp

fixed back light and varying front light. Table 7, Table 8, and Table 9 present SSIM values for fixed back light and varying front light. Table 10 contains latency and memory values.

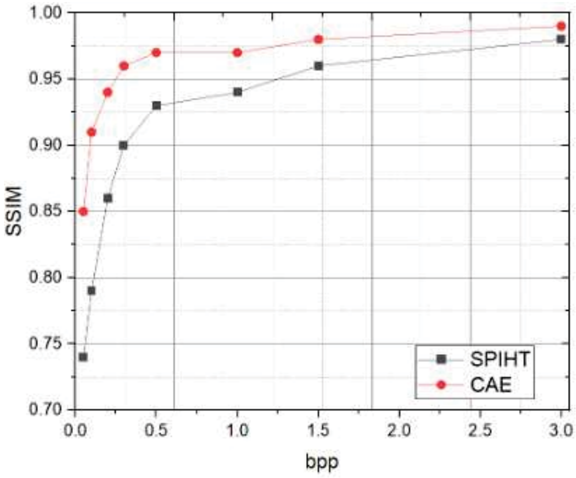

Figure 3: SSIM as a function of bpp

From the obtained results, it can be seen that the gains of the CAE method are most pronounced in the range 0.05–0.3 bpp, where SSIM improves by 0.05–0.15 absolutely compared to SPIHT. At $bpp \geqslant 1.0$, differences in PSNR become marginal, but CAE still maintains better perceptual characteristics (SSIM). The effect of lighting is significant: increasing back light degrades both methods (weaker edge contrast), but CAE suffers less reduction thanks to learned priors.

In terms of resources, SPIHT has much lower latency and memory footprint, making it suitable for real-time pipelines. CAE requires GPU acceleration; however, weight quantization and smaller latent sizes can reduce latency below 10 ms per frame, which is acceptable for many production systems.

Results show that CAE maintains more stable quality under different lighting combinations, especially in cases of weak front and strong back light. SPIHT shows

Front light	200 lx	800 lx	1400 lx
SPIHT	42.1	44.5	45.7
CAE	44.8	46.9	47.3

Table 2: PSNR for back light = 800 lx and different front light values (bpp = 0.3)

Front light	200 lx	800 lx	1400 lx
SPIHT	0.91	0.93	0.89
CAE	0.95	0.97	0.94

Table 3: SSIM for back light = 800 lx and different front light values (bpp = 0.3)

Back light	200 lx	600 lx	1000 lx	1400 lx
SPIHT	41.2	42.0	42.5	41.9
CAE	44.1	45.0	45.5	44.7

Table 4: PSNR for front light = 200 lx and different back light values (bpp = 0.3)

Back light	200 lx	600 lx	1000 lx	1400 lx
SPIHT	45.6	46.0	46.4	45.9
CAE	47.5	48.0	48.5	47.8

Table 5: PSNR for front light = 800 lx and different back light values (bpp = 0.5)

Back light	200 lx	600 lx	1000 lx	1400 lx
SPIHT	48.5	49.0	49.3	48.7
CAE	49.8	50.2	50.5	49.9

Table 6: PSNR for front light = 1400 lx and different back light values (bpp = 1.0)

Back light	200 lx	600 lx	1000 lx	1400 lx
SPIHT	0.76	0.78	0.80	0.77
CAE	0.88	0.90	0.92	0.89

Table 7: SSIM for front light = 200 lx and different back light values (bpp = 0.3)

Back light	200 lx	600 lx	1000 lx	1400 lx
SPIHT	0.90	0.91	0.92	0.90
CAE	0.95	0.96	0.97	0.95

Table 8: SSIM for front light = 800 lx and different back light values (bpp = 0.5)

Back light	200 lx	600 lx	1000 lx	1400 lx
SPIHT	0.93	0.94	0.95	0.93
CAE	0.96	0.97	0.97	0.96

Table 9: SSIM for front light = 1400 lx and different back light values (bpp = 1.0)

	Encode (ms)	Decode (ms)	Mem (MB)	Remark
SPIHT	8.4	6.1	120	CPU-only
CAE	14.7	10.5	980	GPU FP16

Table 10: Latency (ms) and memory (MB) per frame (bpp = 0.3)

greater oscillations in quality, while SSIM for CAE is consistently higher.

Conclusion

SPIHT is simple and fast, making it suitable for real-time TV systems where computational power is limited. CAE requires a GPU and training time, but provides better visual quality and greater robustness to unfavorable lighting. The AI-based method particularly excels at low bpp values, which is important for streaming and bandwidth-constrained applications.

One of the limitations of CAE is the need for large training datasets. In practical TV studio applications, training can be limited to specific shooting scenarios, thereby reducing training time and resources.

The comparison shows that the AI-based method (CAE) outperforms SPIHT at low bitrates and under demanding lighting conditions. However, SPIHT remains a relevant method due to its efficiency and ease of implementation. Future research should focus on hybrid approaches that combine the efficiency of traditional methods with the intelligence of AI models. In this way, it is possible to achieve an optimal balance between speed, quality, and resource requirements.

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A modal analysis of an AFM micro-cantilever considering varying geometric parameters for higher flexural eigenmodes

Cagri Yilmaz [*1] *and Cagrihan Celebi* [2]

In the current work, a modal analysis on an Atomic Force Microscopy (AFM) micro-cantilever is conducted using a finite element approach for the first six eigenmodes. Analytical expressions, Timoshenko, Euler-Bernoulli, and solid models are utilized to obtain eigenfrequencies at different vibrational modes. Changes in the fundamental and higher eigenmode frequencies are noted as the geometrical parameters of the rectangular silicon micro-cantilever such as the length (100 - 300 μm), the width (20 - 50 μm), and the thickness (0.8 - 7 μm) change. Mode shapes of the AFM micro-cantilever are also demonstrated and evaluated for the first and higher vibration modes. The simulation results indicate that the results of the solid model obtained in the COMSOL software environment agree well with the analytical ones. Based on micro-cantilever models, eigenfrequencies have distinct tendencies to variations in values of geometrical parameters. For instance, the eigenfrequency at the sixth eigenmode varies by around 0.56 % in the width range of 20-40 μm based on the solid model. On the other hand, it changes by approximately 323 % in the thickness range of 0.8-6 μm. Therefore, the theoretical results reveal that the solid model can be robustly utilized to obtain more accurate eigenfrequencies at higher eigenmodes, compared with other cantilever beam models. Using solid models brings great opportunities to explore the observable responses of the AFM micro-cantilevers to external forces such as tip-sample interaction forces, and acoustic forces for nanometrology applications.

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Keywords: Modal analysis, finite element method, micro-cantilever, eigenfrequency, higher eigenmode

Introduction

Responsiveness of micro-cantilevers to external forces such as tip-sample interaction forces, and acoustic forces significantly depend on the resonance frequencies. Micro-cantilevers exhibit higher sensitivity to Casimir and Van der Waals forces in monomodal and multimodal AFM operations [1]. Driving the AFM micro-cantilever in multi-frequency operations enables to achieve higher sensitivity of oscillation observables such as amplitude, phase shift, and frequency shift responses. Thus, higher observable sensitivity will provide better high-resolution images of surface samples with higher accuracy. Moreover, it will enable to extract proper surface properties of samples in AFM. Acoustic emissions can also be robustly measured using rectangular micro-cantilevers with different resonance frequencies. Especially, measurements of dynamic acoustic forces strongly depend on the resonance frequencies of the micro-cantilevers [2]. Notable amplitude changes are observed in measurement of acoustic emissions with the frequency, which is near the eigenmode frequency of the micro-cantilevers. Particularly, implementing a single-frequency excitation scheme based on higher eigenmodes enables to quantify the micro-cantilever sensitivity to acoustic forces at higher frequencies. For instance, the frequency band of 440-570 kHz is achieved based on amplitude responses at the second eigenmodes of two AFM micro-cantilevers in bimodal-frequency operations [3]. It is also worth mentioning that phase shift responses can be utilized to obtain the high-sensitivity frequency range, depending on the second eigenmode frequencies. Therefore, the determination of accurate resonance frequencies is quite critical for nanometrological applications. The present study introduces a modal analysis of an AFM micro-cantilever considering the effects of geometrical parameters. Theoretical results of resonance frequencies, determined based on the analytical expressions, solid model, Euler Bernoulli beam model, and Timoshenko beam model of the AFM micro-cantilever are demonstrated in the current work.

Micro-cantilevers are developed using different theoretical models and their dynamic responses are explored considering the effects of several system parameters. In the work [4], Euler-Bernoulli beam theory is used to construct an AFM micro-cantilever model in the investigation of effects of tip mass and cantilever density on dynamic behaviors. This model considers the influence of tip mass. System resonance frequencies are explored based on the Euler-Bernoulli distributed parameter model of the AFM. Additionally, a micro-cantilever is modeled based on Timoshenko beam model to examine oscillatory responses in contact resonance force microscopy [5]. Resonance frequencies determined using Timoshenko model are compared with the ones obtained based on the Euler-Bernoulli model. The results indicate that higher eigenmode responses are more accurately predicted using Timoshenko model, when compared with the Euler-Bernoulli model. Furthermore, the study [6] deals with the extraction of the test cantilever spring constant based on the model of the implied Euler-Bernoulli. A commercial AFM micro-cantilever is utilized to obtain the force-displacement curves. The results suggest that using a cantilever array provides greater precision, rather than repeated measurements on a single cantilever. Moreover, the Euler-Bernoulli beam model is considered to develop an AFM micro-cantilever for studying the influences of material-length-scale parameter and dimensionless thickness on resonance frequencies [7]. Hamilton's principle is used to obtain the Equation of Motion (EOM). According to the results, smaller micro-cantilever sensitivity at the flexural modes is achieved, when compared with the ones obtained using the classical beam theory. An expression for the resonance frequencies of AFM micro-cantilever in liquid is also derived considering Euler-Bernoulli beam theory [8]. Using this expression, the effect of surface contact stiffness on the flexural mode is explored for the operating medium of fluid and the results are compared with the ones for the air. The results reveal that tip mass has no notable effect on resonance frequency of the micro-cantilever, immersed in liquid. In addition, considering the influences of rotary inertia and shear deformation, Timoshenko beam model is used to obtain the modal frequencies of micro-cantilever oscillations at the flexural modes [9]. The influences of quality factors and contact stiffness on the resonance frequencies are examined in the study. It is demonstrated that for low contact stiffness, the damping effect on the oscillation frequency is remarkable for the higher eigenmodes. In another study [10], analytical expressions for the resonance frequency and stiffness are derived utilizing Timoshenko beam theory. AFM experiments are conducted using the prototypes of the micro-cantilevers. It is noted that theoretical and experimental results exhibit good agreement with each other. Based on Timoshenko beam model, the resonance frequency of a V-shaped AFM micro-cantilever is explored considering the effects of contact stiffness [11]. The nonlinear differential EOM is solved using the differential quadrature method and the results exhibit that the resonance frequencies at high order modes remarkably decrease as the thickness of the micro-cantilever increases.

The current work introduces a modal analysis of an AFM micro-cantilever utilizing the finite element method considering the effects of geometrical parameters. Resonance frequencies of the rectangular silicon AFM micro-cantilever are obtained for the first six flexural eigenmodes. Theoretical results are determined using the analytical expressions, solid models, Timoshenko, and Euler-Bernoulli beam models of the micro-cantilever. The results are compared and evaluated considering the influences of the length, width, and thickness of the micro-cantilever on the eigenfrequencies. The mode shapes are also revealed for the fundamental and higher eigenmodes in the present study.

Micro-cantilever models

The Euler-Bernoulli equation for the AFM micro-cantilever is given as introduced in [12].

$$EI\frac{d^4u}{dx^4} = F_{ext} \tag{1}$$

Where u is the flexural deflection of the micro-cantilever. E, I, and x are the Young's modulus, the second moment of inertia, and the position along the length (L) of the micro-cantilever respectively. F_{ext} is the external force acting on the micro-cantilever. For this case, the shear deformation, axial effects, and rotary inertia are neglected.

$$I = \frac{wc^3}{12} \tag{2}$$

Where w, and c are the width and thickness of the micro-cantilever respectively. All boundary conditions are taken as zero for numerical computations.

The Timoshenko beam equations for the AFM micro-cantilever can be introduced as two coupled linear partial differential equations as mentioned in [10]:

$$\rho A\frac{\partial^2 u}{\partial^2 t} = \frac{\partial}{\partial x}(AkG(\frac{\partial u}{\partial x} - \theta)) \tag{3}$$

$$\rho I\frac{\partial^2 u}{\partial^2 t} = \frac{\partial}{\partial x}(EI\frac{\partial \theta}{\partial x}) + AkG(\frac{\partial u}{\partial x} - \theta) \tag{4}$$

Where ρ, A, and θ are the material density, the cross-sectional area, and the angular displacement respectively. For our work, the rectangular AFM micro-cantilever is considered to be made of silicon material. G, and k are the shear modulus, and the Timoshenko shear coefficient respectively. The computations using the Euler-Bernoulli beam model and Timoshenko beam model are performed in COMSOL software environment.

A solid model of the AFM micro-cantilever is also developed in COMSOL software environment. Figure 1 depicts the solid model of a tipless rectangular AFM micro-cantilever. A finite element model is constructed using quadratic tetrahedral elements. The number of the elements is equal to 30459. Performing mesh convergence analysis, the maximum mesh element size is chosen as approximately 1/10 of the micro-cantilever width, corresponding to a maximum mesh size of 4.5 μm. Resolving regions with high stress gradients near the fixed boundary, a finer mesh with minimum element size of 0.045 μm is applied at the fixed end.

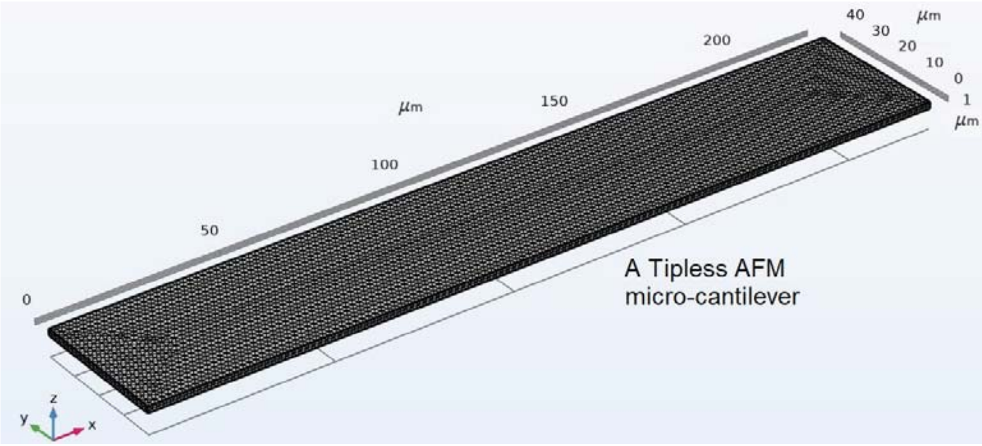

Figure 1: A solid model of a tipless AFM micro-cantilever.

Analytical expressions are also widely utilized to determine the eigenfrequencies of the micro-cantilever. The expression for the eigenfrequency at the fundamental (first) eigenmode is written as follows [13]:

$$f_1 = 0.162\frac{c}{L^2}\sqrt{\frac{E}{\rho}} \tag{5}$$

The eigenfrequency f_i at the higher eigenmode i is computed using the following expression as given in [1]:

$$f_i = f_1(\frac{\beta_i}{\beta_1})^2 \tag{6}$$

Where β_i are the real roots of the characteristic equation of the rectangular micro-cantilever, as given below:

$$1 + cos(\beta_i)cosh(\beta_i) = 0 \tag{7}$$

For the first six flexural eigenmodes, β_1=1.875, β_2=4.694, β_3=7.855, β_4=10.996, β_5=14.136, and β_6=17.278. For investigation of the effects of geometric parameters on the eigenmode frequencies, the length varies in the range of 100 - 300 μm, the width changes in the range of 20 - 50 μm, and the thickness varies in the range of 0.8-7 μm. For the silicon material, the Young's modulus (E) is taken as 165 GPa, and the material density (ρ) is equal to 2320 $\frac{kg}{m^3}$. The analytical solutions are obtained using the MATLAB software environment.

Results and discussions

A good agreement between the analytical and solid model results is achieved owing to the contribution of the effects of shear deformation, axial effects, and rotary inertia to the solid model (Figure 2). Differences among the eigenfrequencies obtained using Euler-Bernoulli and Timoshenko beam models are quite higher for higher eigenmodes.

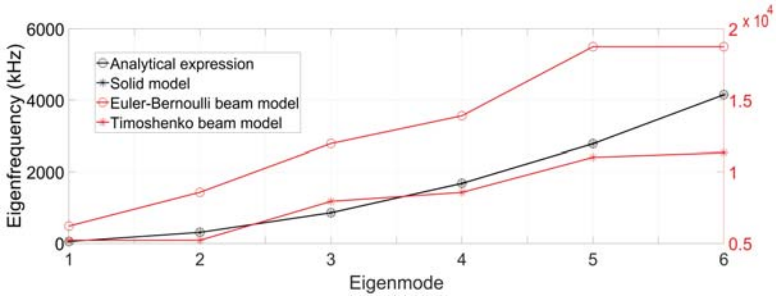

Figure 2: Eigenfrequencies of an AFM micro-cantilever determined using different models for the first six flexural eigenmodes.

For the Euler-Bernoulli beam model, neglecting the shear effect results in higher rigidity. Correspondingly, eigenfrequencies obtained using Euler-Bernoulli beam models are quite higher than the ones for the Timoshenko beam models due to the higher rigidity. Even for the lower eigenmodes, remarkable deviations among the eigenfrequencies are observed.

The eigenfrequencies of all beam models at the first sixth flexural eigenmodes exhibit similar decreasing tendencies as the length increases from 100 μm to 300 μm (Figure 3). The eigenfrequencies decrease owing to the higher micro-cantilever flexibility, increasing as the micro-cantilever length rises. As the width of the micro-cantilever varies from 20 μm to 50 μm, the eigenfrequencies at the vibrational modes demonstrate different behaviors (Figure 4). Furthermore, as the thickness of the micro-cantilever increases from 0.8 μm to 7 μm, the eigenfrequencies determined utilizing Euler-Bernoulli and Timoshenko beam models remain almost the same (Figure 5). On the other hand, except for the fifth flexural eigenmode (Figure 5(e)), the eigenfrequencies obtained using the solid model monotonically rise with increasing thickness. As the width or thickness increases, the rigidity of the micro-cantilever ascends. It is worth mentioning that the effects of increases in the cross-sectional area on the eigenfrequencies are notably higher for the higher eigenmodes.

Figure 6 depicts the mode shapes of a micro-cantilever at the first, second, fourth, and sixth flexural eigenmodes. The displacements of around 0-250 pm are noted for the selected vibrational modes. The sensitivity of the micro-cantilevers to external forces such as tip-sample interaction forces and acoustic forces strongly depends on dynamic responses at higher flexural eigenmodes. In dynamic AFM operation modes, observable responses for the separation distance below around 400 nm to Casimir forces exhibit notable sensitivity at higher flexural eigenmodes [1]. Furthermore, driving the micro-cantilevers at higher resonance frequencies enables to obtain notable amplitude and phase shift responses in measurement of acoustic forces within the kHz band [3].

Obtaining the mode shapes, a solid model of the micro-cantilever is constructed based on the assumption of linear elastic material. The isotropic property of the micro-cantilever material considerably affects the results of the solid models. Additionally, the accuracy of the results strongly depends on the higher stress levels at the fixed end. It is clear that decreasing the element size leads to remarkable changes in the eigenfrequencies.

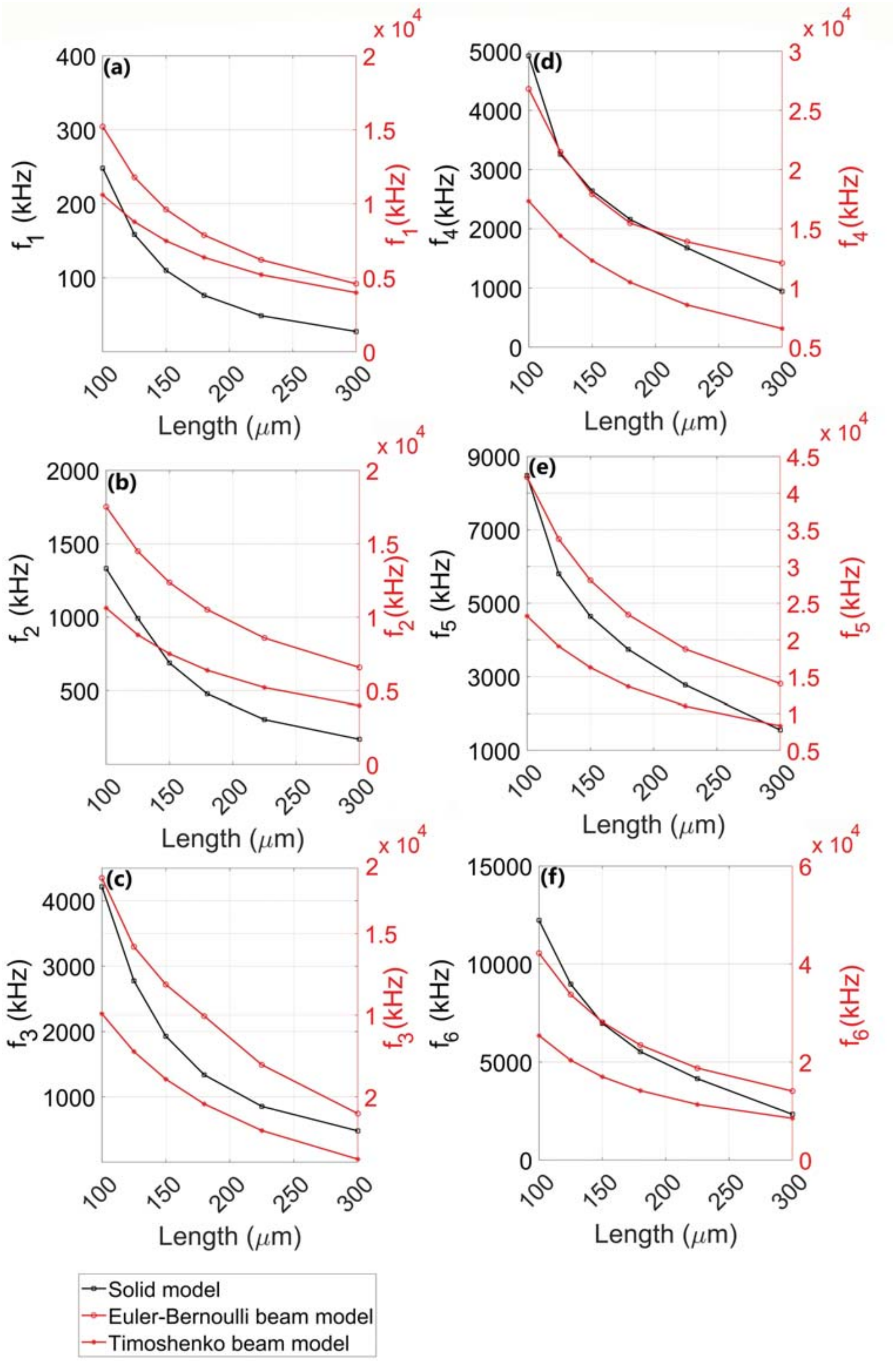

Figure 3: Eigenfrequencies at the first six flexural eigenmodes obtained utilizing different micro-cantilever beam models for varying length.

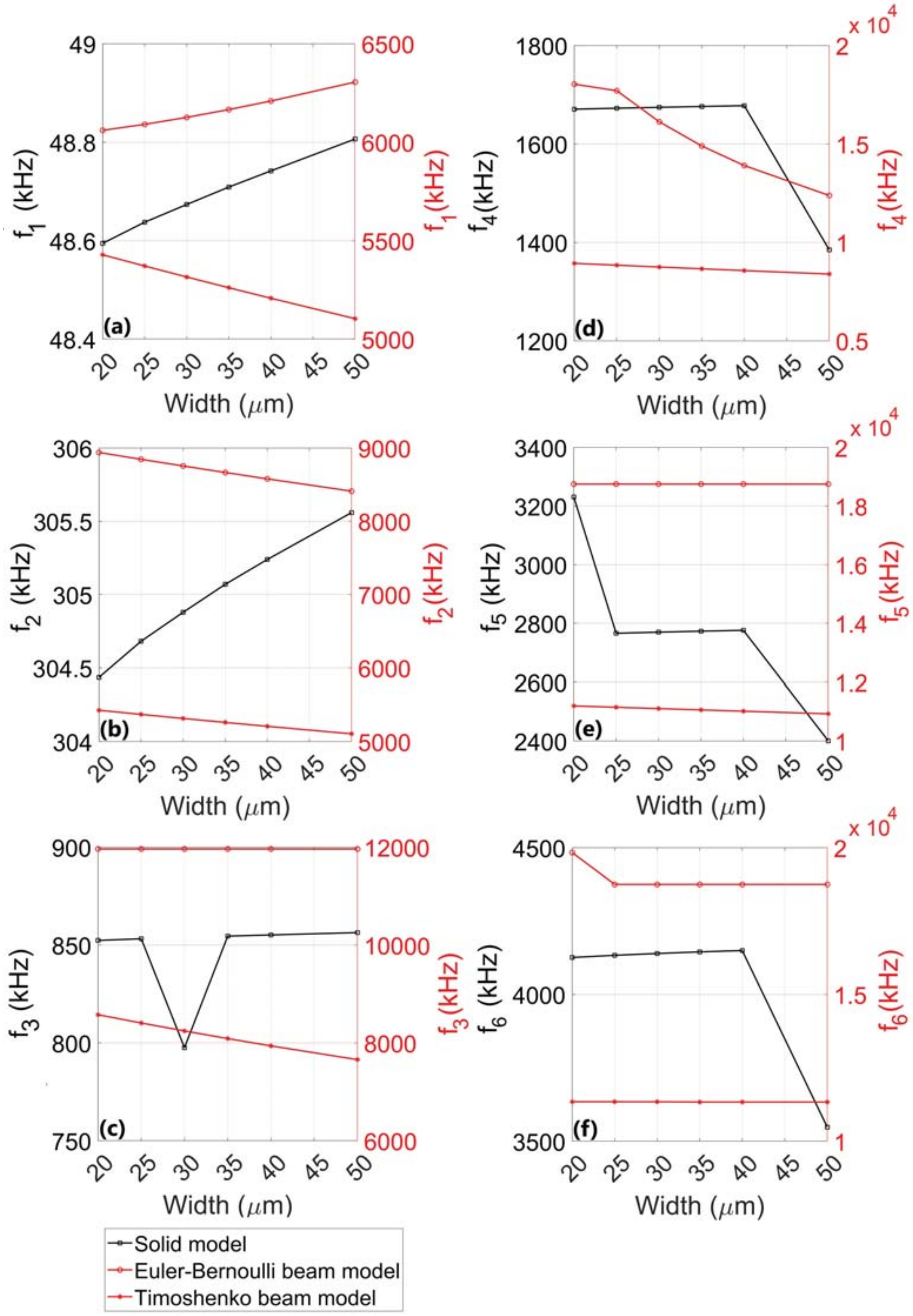

Figure 4: Eigenfrequencies at the first six flexural eigenmodes obtained utilizing different micro-cantilever beam models for varying width.

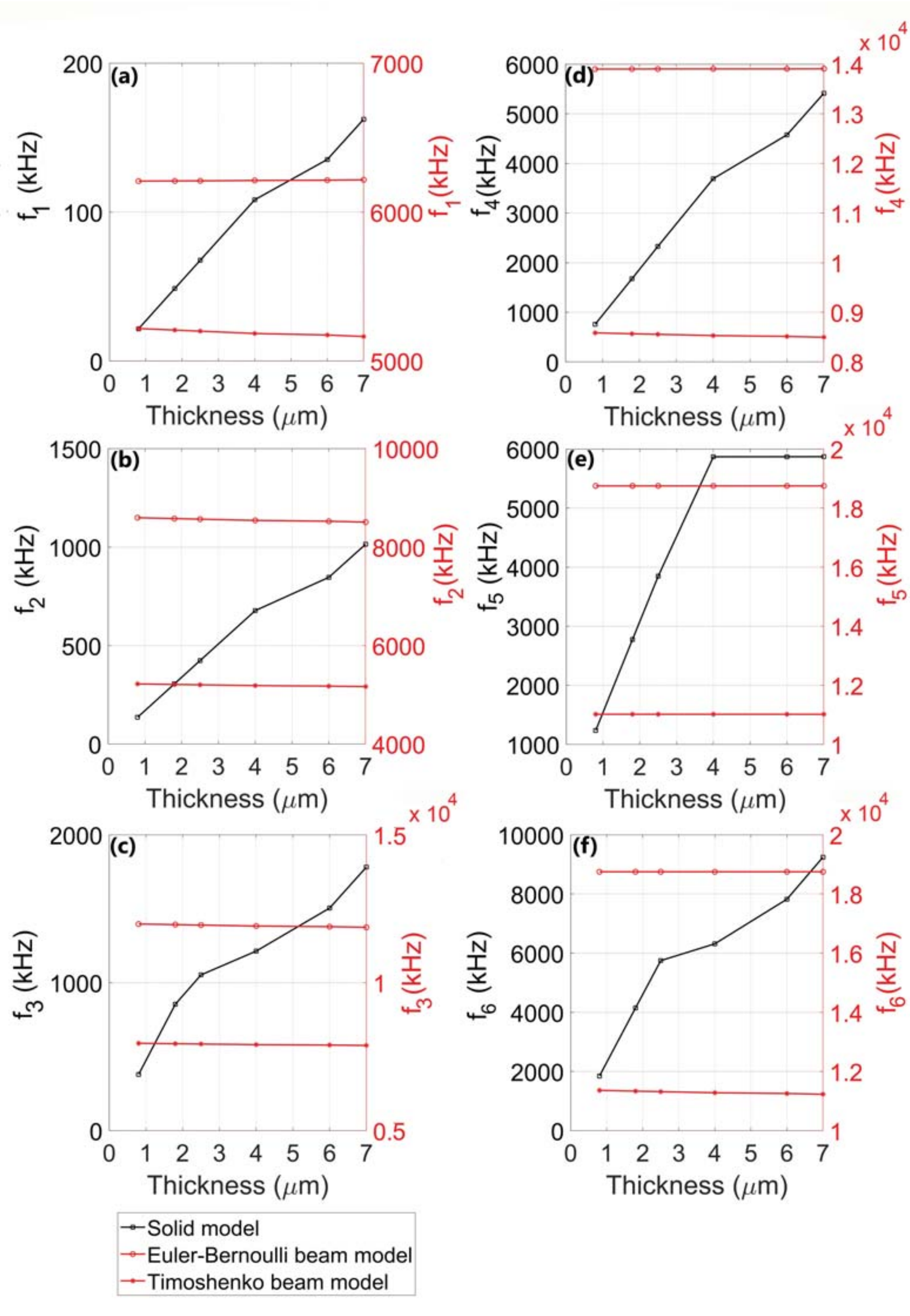

Figure 5: Eigenfrequencies at the first six flexural eigenmodes obtained utilizing different micro-cantilever beam models for varying thickness.

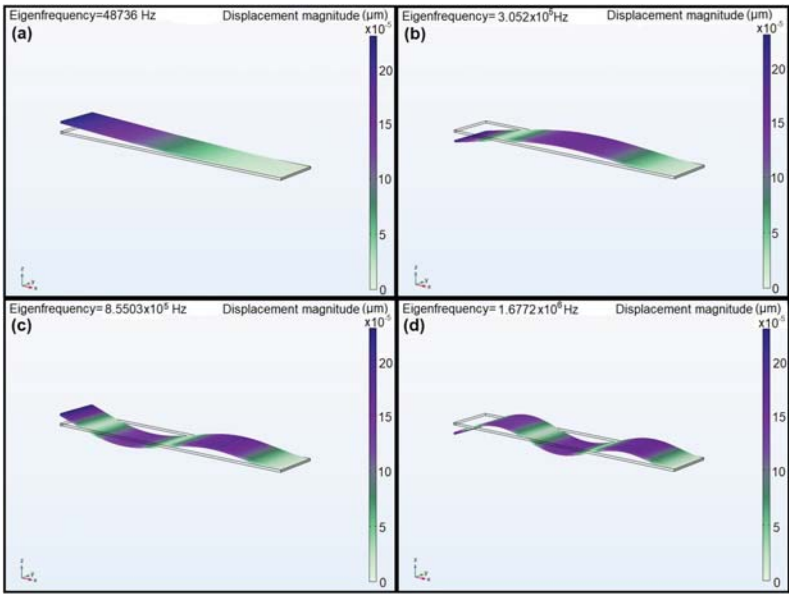

Figure 6: The mode shape of an AFM micro-cantilever for (a) the first flexural eigenmode. (b) the second flexural eigenmode. (c) the fourth flexural eigenmode. (d) the sixth flexural eigenmode.

Conclusions

In the present study, eigenfrequencies of an AFM micro-cantilever at the first six flexural eigenmodes are theoretically determined. Analytical expressions, solid model, Euler-Bernoulli, and Timoshenko beam models are utilized to observe resonance behaviors in the modal analysis. The results of solid models match well with the ones obtained utilizing analytical expressions. This result proves that considering the effects of shear deformation, axial effects, and rotary inertia in the solid model increases the accuracy of the results, when compared with other beam models. More significantly, eigenfrequencies computed using the Euler-Bernoulli beam model are remarkably higher than the ones for the Timoshenko beam model due to the higher rigidity. Note that ignoring the shear effect leads to the higher rigidity of the micro-cantilever. Irrespective of beam models used in numerical computations, the eigenfrequencies decrease with increasing length for the fundamental and higher eigenmodes. More interestingly, the eigenfrequencies at the higher vibrational modes based on different beam models might exhibit different behaviors in response to changes in width, and length. Therefore, these results suggest that optimization of micro-cantilever design using solid models can bring great benefits for nanometrological applications. In measurement of acoustic emissions, the values of length, width and thickness of the rectangular micro-cantilever are to be selected according to the target frequency of acoustic forces for enhanced quantification of sensitivity. The sensitivity of tip-sample interaction forces such as Van der Waals forces and Casimir forces can also be explored using the micro-cantilever with the optimized resonance frequencies. In this regard, developing solid models of the AFM micro-cantilevers enables to determine more accurate eigenfrequencies at the fundamental and higher eigenmodes.

Acknowledgments

This paper is dedicated to Professor Manuel López-Pellicer on the occasion of his 81st birth anniversary.

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