The 5th Mediterranean International Conference of Pure & Applied Mathematics and Related Areas (MICOPAM 2022)

Antalya, TURKEY

October 27-30, 2022

PROCEEDINGS BOOK OF MICOPAM 2022

Editors

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TITLE

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EDITION AND YEAR

First Edition, 2022

OPENING CEREMONY TALK of MICOPAM 2022

Dear distinguished participants of the 5th Mediterranean International Conference of Pure & Applied Mathematics and Related Areas (MICOPAM 2022), held at Sherwood Exclusive Lara Hotel (Ultra All-Inclusive) in Antalya, TURKEY on October 27-30, 2022.

After today, we will show you this by attending the MICOPAM conference and presenting it to you with our hospitality and friendships, which will take place in unforgettable conferences in your life. With the help of all our members, the "MICOPAM conference" has taken its place among the prestigious conferences in the world.

Each year, these participating members will increase exponentially and witness developments in line with the spirit of MICOPAM.

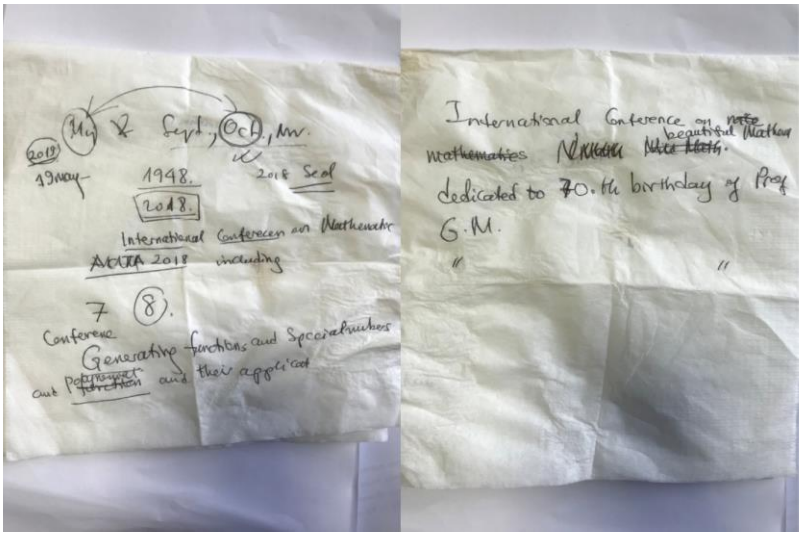
Due to the coronavirus pandemic, this conference was organized as a hybrid event, including both physical and virtual presentations.

Let me start my talk with the following meaningful sentence that will reflect the frightening and excruciating impact of the coronavirus disease, which reminds us of our deepest sorrows:

As the committee-in-chief of the organizing committee, I wish “Healthy Days” to all the people who are suffering due to the coronavirus pandemic in our world. In addition, we wish that God would rest their souls for the people who died due to the coronavirus pandemic, and also, we hope the other effected people will recover as soon as possible.

On behalf of the Scientific and Organizing Committees, I would also like to say "Welcome to our conference” to all physical and virtual participants form all of the world.

The construction of the MICOPAM conference has been firstly appeared in 2017 in Belgrade, Serbia, while speaking with Professor Gradimir V. Milovanović.



Brainstorming for conference name on napkin with Professor Gradimir V. Milovanović in 2017 at Belgrade, Serbia.

A photo taken immediately after the discussion for the organization of MICOPAM

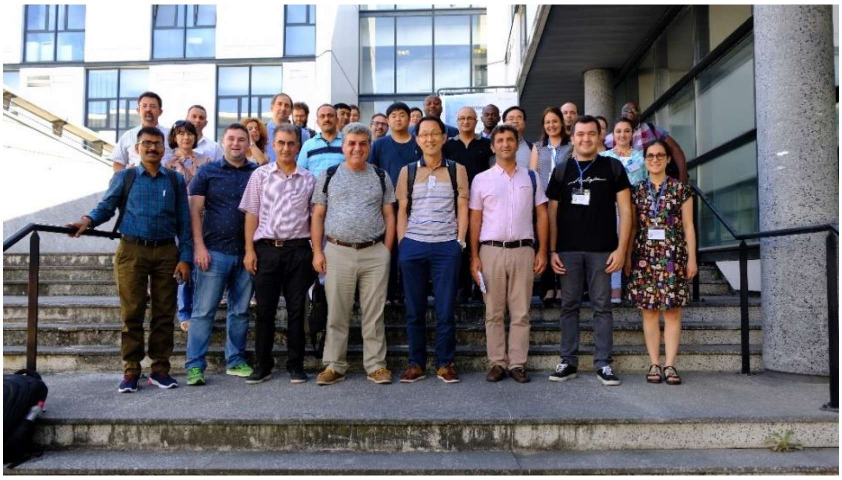
From Left to Right: *Professor Yilmaz Simsek, Professor Walter Gautschi, Professor Gradimir V. Milovanović*

Our dreams came true in 2018 at Antalya and MICOPAM 2018 was successfully held over four days, with presentations made by not only researchers coming from the international communities, but also distinguished keynote speakers.

MICOPAM conference series was started in Antalya-Turkey in 2018. The first conference was dedicated to the 70th birthday of Professor Gradimir V. Milovanović.

Conference Group Photo of MICOPAM 2018

With the same passion, in 2019, this conference series carried out as MICOPAM 2019 at University of d'Evry Val d'Essonne in Paris France by the Professor Abdelmejid Bayad, me, and Professor Mustafa Alkan.



Conference Group Photo of MICOPAM 2019

Due to the big effect of the Coronavirus pandemic, in 2020, MICOPAM 2020 could not be organized and decided to postpone to the year 2021 by combining with MICOPAM 2021. Thus, in 2021, this conference series carried out as MICOPAM 2020-2021, at Faculty of Science Department of Mathematics Antalya, TURKEY on November 11-12, 2021. Due to the coronavirus pandemic, MICOPAM 2020-2021 was organized as a hybrid event with a limited number of physical participants as well as researchers participating or presenting virtually (with a videoconferencing/webinar platform).

Some photos taken during MICOPAM 2020-2021

Today, we are very happy to make the opening ceremony of the 5th of the conference MICOPAM together. Thus, dear distinguished participants, you have given honor to us by attending our conference: MICOPAM 2022.

Conference Group Photo of MICOPAM 2022

I would like to remind you that MICOPAM conference will be held regularly every year. In particular, “The 6th Mediterranean International Conference of Pure & Applied Mathematics and Related Areas” is scheduled to be held in another beautiful, touristic, and historical city of the world to be announced soon on the MICOPAM web site, in 2023!

I would like to thank to the following my colleagues and students who helped me at every stage of the 5th Mediterranean International Conference of Pure & Applied Mathematics and Related Areas (MICOPAM 2022):

My sincerely thanks go to not only all my colleagues including all speakers, who sent an abstract of their talks, in all over the world who supported me without hesitation even during the most difficult days of this pandemic period, and also to the Local Organizing Committee: (including especially Professor Dr. Mustafa Alkan, Associate Professor Dr. Irem Kucukoglu, Associate Professor Dr. Ortaç Öneş, Assistant Professor Dr. Rahime Dere, Assistant Professor Dr. Neslihan Kilar, PhD student Damla Gün, Instructor Büşra Al, PhD student Buket Simsek); my precious family: (my wife Saniye, my daughters Burcin and Buket); and also other friends whose names that I did not mention here.

Thanks also to the Editors (Professor Dr. Yilmaz Simsek, Professor Dr. Mustafa Alkan, Associate Professor Dr. Irem Kucukoglu, and Associate Professor Dr. Ortaç Öneş) for their most valuable contribution on preparing the Proceedings Book of MICOPAM 2022.

I would also like to state few words about Mathematics. Mathematics is not only the common heritage of each people in the world, but also the common language of the world. That is always passed from generation to generation by refreshing.

It would also be appropriate to say the following:

In addition to the poetic and artistic aspect of mathematics, mathematics has such a spiritual, magical, and logical power, all-natural science and social science cannot breathe and survive without mathematics. Mathematics is such a branch of science that other sciences cannot develop without it. Therefore, Mathematics, which is the oldest of science, has contributed fundamentally to the development of our world civilizations. Thus, we can enter into the science and technology centers using the power of mathematics and its branches. So, mathematics and its branches create the possibility of bridgework and communication between the Natural Sciences and the Engineering Sciences as well as the Economic and also Social Science.

As for the aim of the MICOPAM conference, it is to bring together leading scientists of the pure and applied mathematics and related areas to present their research, to exchange new ideas, to discuss challenging issues, to foster future collaborations and to interact with each other. In fact, the main purpose of this conference is to bring to the fore the best of research and applications that will help our world humanity and society. Due to the valuable idea of the MICOPAM, this conference welcomes speakers whose talk or poster contents are mainly related to the following areas: Mathematical Analysis, Algebra and Analytic Number Theory, Combinatorics and Probability, Pure and Applied Mathematics & Statistics, Recent Advances in General Inequalities on Pure & Applied Mathematics and Related Areas, Mathematical Physics, Fractional Calculus and Its Applications, Polynomials and Orthogonal systems, Special numbers and Special functions, Q-theory and Its Applications, Approximation Theory and Optimization, Extremal problems and Inequalities, Integral Transformations, Equations and Operational Calculus, Partial Differential Equations, Geometry and Its Applications, Numerical Methods and Algorithms, Scientific Computation, Mathematical Methods and computation in Engineering, Mathematical Geosciences.

To summarize my speech, this conference has provided a novel opportunity to our distinguished participants to meet each other and share their scientific works and friendships in the above areas.

I am delighted to note that all participants have free and active involvement and meaningful discussion with other participants during the conference physically and virtually by the “Microsoft Teams” platform and other online connections.

By the way, I would like to point out that a special issue of Montes Taurus Journal of Pure and Applied Mathematics ISSN: 2687-4814 (MTJPAM) (https://mtjpamjournal.com/special-issues/) will be allocated to this conference and it will be dedicated to our colleagues who died and are still suffering from the Coronavirus, and to all humanity in the world.

Due to the Coronavirus Pandemic, I wish you and your family healthy days.

I sincerely thank my dear and precious family for their unwavering support throughout the conference;

I sincerely thank our local committee members whose names are written below: "Prof. Dr. Mustafa Alkan", "Associate Prof. Dr. İrem Küçükoğlu", "Associate Professor Dr. Ortaç Öneş", "Assistant Prof. Dr. Neslihan Kılar, Assistant Professor Prof. Ayşe Yılmaz Ceylan", "Instructor Büşra Al", and "PhD student Damla Gün", and my other students;

My heartfelt thanks to the members of the international organizing committee, as well as to all Invited Speakers listed below by surname:

Elvan Akin (USA), Abdelmejid Bayad (France), Ronald N. Goldman (USA), Subuhi Khan (India), Mokhtar Kirane (United Arab Emirates), Helmuth Robert Malonek (Portugal), Gradimir V. Milovanović (Serbia), Manuel López-Pellicer (Spain), Wolfgang Sprößig (Germany), Hari Mohan Srivastava (Canada), Richard Tremblay (Canada), Yusuf Uras (Turkey);

In the meantime, I would like to sincerely thank all our chairmen who scientifically organized all the session programs, our valuable participants who honored us by making face-to-face presentations, and also our other valuable participants who made oral presentations online for their extraordinary support.

On the other hand, I sincerely thank all the staff of Sherwood Exclusive Lara Hotel, Kundu, Antalya, especially the sales manager Tuğba Yılmaz Parscan, for their hospitality and assistance. With the help of "Assistant Professor Dr. Ayşe Yılmaz Ceylan" and our director Tuğba, we are holding our MICOPAM2022 conference in this most magnificent and beautiful hotel.

In conclusion, I would like to extend my sincere thanks to all those who contributed to the realization of MICOPAM2022. We look forward to your support and participation in our MICOPAM 2023 conference next year. Consequently, I send my sincerely thanks to all valuable participants of the conference MICOPAM 2022.

On behalf of the Organizing Committee of MICOPAM 2022

PROF. DR. YILMAZ SIMSEK

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FOREWORD

Why we call the name of the conference as “Mediterranean International Conference of Pure & Applied Mathematics and Related Areas (MICOPAM)”. Because by the Mediterranean Sea, almost all the countries, the sea and the oceans are connected. For this reason, just like Mediterranean Sea, the main aim of MICOPAM is to give connection between many areas of sciences including physical mathematics and engineering, especially all branches of mathematics. In addition, it covers Chemistry mathematics, Engineering mathematics, Social Sciences and Linguistics mathematics, and other fields. A few of these areas can be given as follows: Pure and Computational and Applied Mathematics, Statistics, Mathematical Physics (related to *p*-adic Analysis, Umbral Algebra and Their Applications). Another important purpose of MICOPAM is to bring together leading scientists of the pure and applied mathematics and related areas to present their researches, to exchange new ideas, to discuss challenging issues, to foster future collaborations and to interact with each other in the following areas: Mathematical Analysis, Algebra and Analytic Number Theory, Combinatorics and Probability, Pure and Applied Mathematics & Statistics, Recent Advances in General Inequalities on Pure & Applied Mathematics and Related Areas, Mathematical Physics, Fractional Calculus and Its Applications, Polynomials and Orthogonal systems, Special numbers and Special functions, Q-theory and Its Applications, Approximation Theory and Optimization, Extremal problems and Inequalities, Integral Transformations, Equations and Operational Calculus, Partial Differential Equations, Geometry and Its Applications, Numerical Methods and Algorithms, Scientific Computation, Mathematical Methods and computation in Engineering, Mathematical Geosciences, and also Linguistics mathematics.

A brief description about the contents of “Proceedings Book of the 5th Mediterranean International Conference of Pure & Applied Mathematics and Related Areas (MICOPAM 2022)” is given as follows:

The first section of the Proceedings Book of MICOPAM 2022 includes opening ceremony talk, foreword, and some information about MICOPAM including the meaning of its name and the name of conference committee members. The rest of this book includes all the contributed talks and their related manuscripts.

In this regard, we would like to thank to all speakers and participants for their valuable contributions.

Finally, we express our sincere thanks to all members of the scientific committee and all members of the organizing committee because of their efforts to the success of this conference and this book.

Editors of MICOPAM 2022

Prof. Dr. Yilmaz Simsek
Prof. Dr. Mustafa Alkan
Assoc. Prof. Dr. Irem Kucukoglu
Assoc. Prof. Dr. Ortaç Öneş

ABOUT CONFERENCE

The MICOPAM conference series has been started by organizing it in Antalya-Turkey in 2018, and the latter has been held in Paris, France in 2019, then the third-forth has been held together in Antalya, Turkey in 2022. Over the last four years, this conference series has brought together the researchers, who work on pure & applied mathematics and related areas, from all over the world.

The 5th Mediterranean International Conference of Pure & Applied Mathematics and Related Areas (MICOPAM 2022) has been held at Sherwood Exclusive Lara Hotel, Antalya, TURKEY for four days from October 27 to October 30, 2022.

During four dates of MICOPAM 2022, a great number of excellent oral and poster presentations was made by 85 participants from 24 different countries [Algeria (8), Canada (3), Croatia (2), Finland (1), France (1), Georgia (1), Germany (2), India (3), Indonesia (1), Iran (1), Italy (1), Kuwait (3), Libya (1), North Cyprus (6), Pakistan (1), Russia (3), Serbia (1), South Africa (2), South Korea (1), Spain (1), Turkey (33), United Arab Emirates (6), United Kingdom (1), USA (2)].

Contents of oral presentations are mainly related to the following areas: Mathematical Analysis, Algebra and Analytic Number Theory, Combinatorics and Probability, Pure and Applied Mathematics & Statistics, Recent Advances in General Inequalities on Pure & Applied Mathematics and Related Areas, Mathematical Physics, Fractional Calculus and Its Applications, Polynomials and Orthogonal systems, Special numbers and Special functions, Q-theory and Its Applications, Approximation Theory and Optimization, Extremal problems and Inequalities, Integral Transformations, Equations and Operational Calculus, Partial Differential Equations, Geometry and Its Applications, Numerical Methods and Algorithms, Scientific Computation, Mathematical Methods and computation in Engineering, Mathematical Geosciences.

Further details about MICOPAM 2022 are given as follows:

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- Buket Simsek, (Akdeniz University, Turkey)
- Yilmaz Simsek, (Akdeniz University, Turkey)
- Fusun Yalcin, (Akdeniz University, Turkey)

Invited Speakers of MICOPAM 2022

(Sorted list alphabetically by speaker's last name)

- Elvan Akin, (Missouri University of Science and Technology, USA)
- Abdelmejid Bayad, (Université d'Évry Val d'Essonne, Université Paris-Saclay, France)
- Ronald N. Goldman, (Rice University, USA)
- Subuhi Khan, (Aligarh Muslim University, India)
- Mokhtar Kirane, (Khalifa University, United Arab Emirates)
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- Gradimir V. Milovanović, (Serbian Academy of Sciences and Arts, Serbia)
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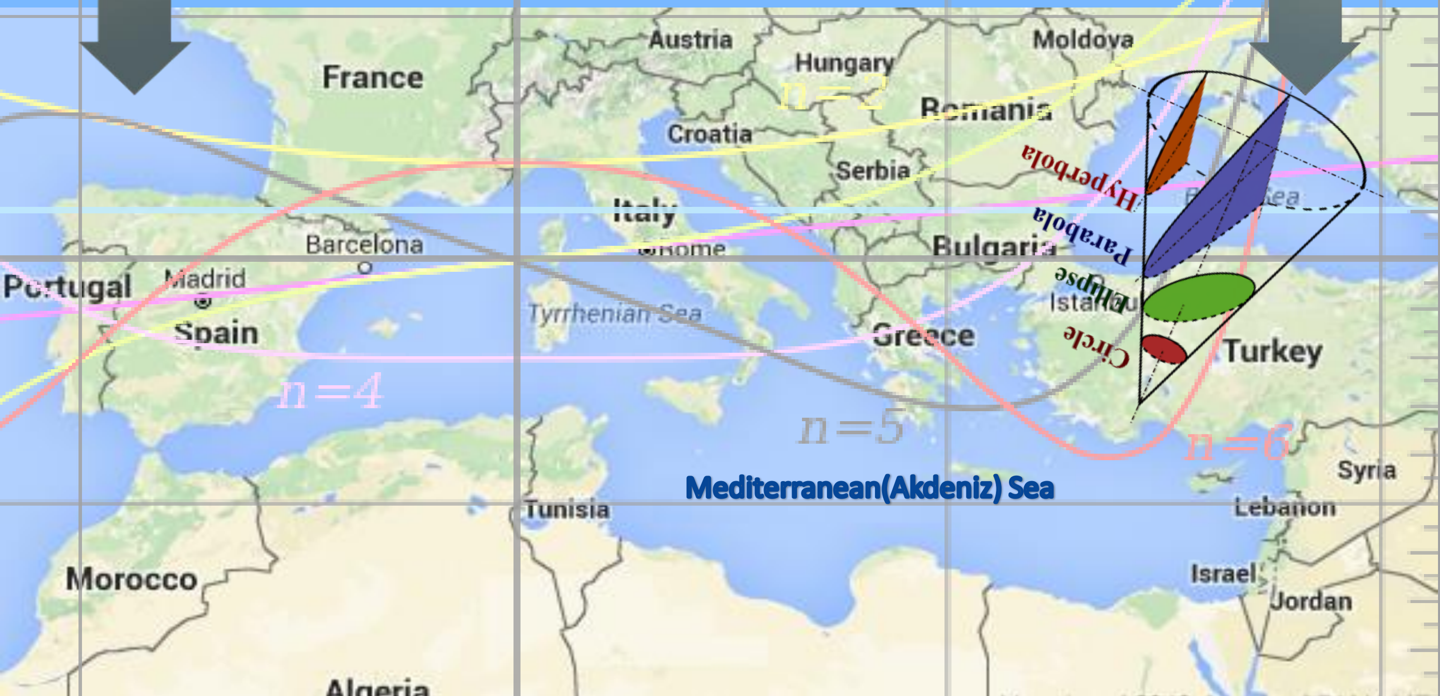
- Wolfgang Sprößig, (TU Bergakademie Freiberg, Institute of Applied Analysis, Germany)
- Hari Mohan Srivastava, (University of Victoria, Canada)
- Richard Tremblay, (Université du Québec à Chicoutimi, Canada)
- Yusuf Uras, (Kahramanmaras Sutcu Imam University, Turkey)

MICOPAM 2022

The 5th Mediterranean International Conference of Pure & Applied Mathematics and Related Areas



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- Yilmaz Simsek, Turkey

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Plenary Speakers

- Elvan Akin, USA
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- Richard Tremblay, Canada
- Subuhi Khan, India

Antalya, TURKEY, October 27–30, 2022

Conference Website: https://micopam.com

Conference Venue: Sherwood Exclusive Lara Hotel-ANTALYA

Proceedings Book of the 5th Mediterranean International Conference of Pure & Applied Mathematics and Related Areas (MICOPAM 2022)

November 23, 2022

Contents

1 INVITED TALKS

Modular forms and Partition functions

Abdelmejid Bayad

In this talk, first we give a survey on the Ramanujan congruence type on partition functions. We then connect this problem to modular forms and Hecke operators. In particular, we investigate the generalized partition functions $p_{[1^{\lambda_0} l^{k\lambda_1}]}(n)$ given by their generating function

$$\sum_{n=0}^{\infty} p_{[1^{\lambda_0} l^{k\lambda_1}]}(n) q^n = \frac{q^{\frac{\lambda_0 + l^k \lambda_1}{24}}}{\eta(\tau)^{\lambda_0} \eta(l^k \tau)^{\lambda_1}},$$

where $\eta(\tau)$ is the Dedekind eta function, $l = 5, 7, 11$, $k$be a positive integer and λ_0, λ_1 arbitrary integers. We prove Ramanujan's type congruences for the partition function $p_{[1^{\lambda_0} l^{k\lambda_1}]}(n)$ modulo powers of l . This study generalizes Atkin, Gordon and others results on the congruences for the partition function. By the way, for $l = 11$, we improve Gordon's results.

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Mathematical Modeling of Infectious Diseases on Time Scales

Elvan Akın

Higher dimensional dynamic systems come to light in various infectious diseases to understand the dynamics of transmissions of infectious diseases. In this talk, we first introduce the theory of time scales and then show its applications on epidemic models such as HIV (human immunodeficiency virus), swine influenza, and HTLV (human T-cell lymptotropic virus) infection and the development of ATL (adult T-cell leukemia). The advantage of time scale modeling in applications is to obtain a variety of continuous and discrete models simultaneously by choosing a proper domain (time scale). Such a time scale would be a choice of the set of real numbers, the set of integers, the set of numbers separated by a positive number h, i.e., a closed subset of real numbers. In this talk, we present continuous and discrete epidemic models and investigate their stability analyses and numerical results.

2010 Mathematics Subject Classifications: 34A34, 93C15, 39A05, 34N05, 93A30

Keywords: Modeling, Infectious Disease, Differential Equations, Difference Equations, Dynamic Equations, Dynamical Systems, Time Scales

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A family of double integrals stemming from the Boltzmann equation in the kinetic theory of gasses

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The main object of this lecture is to revisit a certain double integral involving Kummer's confluent hypergeometric function ${}_1F_1$, which arose in the study of the collision terms of the celebrated Boltzmann equation in the kinetic theory of gases. We present a systematic investigation of some novel extensions and generalizations of this family of double integrals. We also point out some relevant connections of the results, which are presented here, with other related recent developments in the theory and applications of hypergeometric functions.

2010 Mathematics Subject Classifications: 33C15, 33C20, 33E12, 11M35, 33C60, 76P05, 82B40, 82C40

Keywords: Kinetic theory of gases, Boltzmann equation, Kummer's confluent hypergeometric function, Generalized hypergeometric functions, Fox-Wright function, General Mittag-Leffler type and Hurwitz-Lerch type functions

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Methods for Slowly Convergent Series and Applications to Special Functions

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Methods for summation of slowly convergent series are described and their application to calculating some special functions and mathematical constants are presented.

2010 Mathematics Subject Classifications: 65B10, 65D30, 65D32, 40A25

Keywords: Slowly convergent series, Special functions, Orthogonal polynomials, Gaussian quadrature formula, Weight function, Mathieu series, Riemann zeta function

Introduction

Slowly convergent series appear very often in calculation of some special functions and important constants (e.g., the Euler-Mascheroni constant γ, Apéry's constant $\zeta(3)$, Theodorus constant [3], Erdős-Borwein constant, etc.), but also in many problems in applied and computational sciences.

There are several numerical methods based on linear and nonlinear transformations. In general, starting from the sequence of partial sums $\{S_n\}_{n=1}^{\infty}$ of the slowly convergent series S $(= \lim_{n\to\infty} S_n)$, these transformations give other sequences with faster convergence to the same limit S. In other words, these so-called *accelerating transformations* $\{S_n\}_{n=1}^{\infty} \to \{T_n\}_{n=1}^{\infty}$ must be *limit-preserving*, i.e.,

$$\lim_{n\to\infty} \frac{T_n - S}{S_n - S} = 0.$$

We mention here some well-known transformations as Euler's transformation, Aitken's Δ^2-process, Shanks's transformation, etc. (For more details see [2], [8], [14]).

Some alternative methods of summation of slowly convergent series are based on integral representations of series and an application of the Gaussian quadratures. Such summation/integration procedures for slowly convergent series have been developed in [6] (Laplace transform method), [9, 10] (Contour integration method), and [12] (Modified contour integration method). In [11] we derived a method for fast summation of trigonometric series.

Under certain conditions (see [12]) we can prove that

$$T_m = \sum_{k=m}^{+\infty} f(k) = \frac{\pi}{4} \sum_{\nu=1}^{n} A_\nu^{(n)} \Phi \left(m - \frac{1}{2}, \frac{\sqrt{\xi_\nu^{(n)}}}{2} \right) + R_n(\Phi), \tag{1}$$

where F is an integral of f, $\Phi(x,y) = -\frac{1}{2}\left[F(x+\mathrm{i}y) + F(x-\mathrm{i}y)\right]$, $\left(A_\nu^{(n)}, \xi_\nu^{(n)}\right)$, $\nu = 1,\ldots,n$, are the parameters (weights and nodes) of the n-point Gaussian quadrature rule

$$\int_0^{+\infty} \frac{g(x)}{\sqrt{x}\cosh^2 \dfrac{\pi\sqrt{x}}{2}}\,\mathrm{d}x = \sum_{\nu=1}^{n} A_\nu^{(n)} g\big(\xi_\nu^{(n)}\big) + R_n(g), \tag{2}$$

and $R_n(g)$ is the corresponding error term.

Some Special Functions and Constants

In this section we only give short account on calculation some special functions and mathematical constants defined by slowly convergent series. A rich treasury of significant numbers, mathematical constants, can be found in a two-volume book written by Finch [4, 5]. We use only a few of them.

Mathieu series

We consider the famous infinite functional series so-called Mathieu series of the form [7]

$$S(r) = \sum_{k=1} \frac{2k}{(k^2+r^2)^2}, \quad \widetilde{S}(r) = \sum_{k=1} (-1)^{k-1} \frac{2k}{(k^2+r^2)^2}, \quad r > 0.$$

The last alternating version of Mathieu series was introduced and investigated by Pogány et al. [16, p. 72]. Using the approach from [9], Milovanović and Pogány [13] obtained the integral representations for these series $S(r)$ and $\widetilde{S}(r)$,

$$S(r) = \pi \int_0^\infty \frac{r^2 - x^2 + \frac14}{\left(x^2 - r^2 + \frac14\right)^2 + r^2} \frac{\mathrm{d}x}{\cosh^2(\pi x)},$$

$$\widetilde{S}(r) = \pi \int_0^\infty \frac{x}{\left(x^2 - r^2 + \frac14\right)^2 + r^2} \frac{\sinh(\pi x)\,\mathrm{d}x}{\cosh^2(\pi x)},$$

In a recent joint paper with Parmar and T. K. Pogány [15] we have proved the following series expansions for all $r > 0$,

$$S(r) = \frac{1}{r}\sum_{n=0}^{\infty} s\left\{ \mathrm{e}^{-rs}\Re\big[E_1\big((-r+\tfrac{\mathrm{i}}{2})s\big)\big] - \mathrm{e}^{rs}\Re\big[E_1\big((r+\tfrac{\mathrm{i}}{2})s\big)\big]\right\}\Big|_{s=2\pi(n+1)},$$

$$\widetilde{S}(r) = \frac{1}{r}\sum_{n=0}^{\infty} s\left\{ \mathrm{e}^{rs}\Re\big[E_1\big((r+\tfrac{\mathrm{i}}{2})s\big)\big] - \mathrm{e}^{-rs}\Re\big[E_1\big((-r+\tfrac{\mathrm{i}}{2})s\big)\big]\right\}\Big|_{s=\pi(2n+1)},$$

where $E_1(z) = \int_z^\infty x^{-1}\mathrm{e}^{-x}\,\mathrm{d}x$ ($|\arg(z)| < \pi$) is the exponential integral of the first order [1, p. 228, Eq. 5.1.1] and $\Re[z]$ denotes the real part of $z \in \mathbb{C}$.

For calculating these last slowly convergent series $S(r)$ and $\widetilde{S}(r)$, we used the well-known Euler-Abel transformation very successfully (see [15] for details).

Riemann zeta function

The Riemann zeta function $s \mapsto \zeta(s)$ is defined by $\zeta(s) = \sum_{k=1}^{+\infty} k^{-s}$ for $\Re s > 1$. The series converges for any s with $\Re s > 1$, uniformly, for any fixed $\sigma > 1$, in any subset of $\Re s \geq \sigma$, which establishes that $\zeta(s)$ is an analytic function in $\Re s > 1$. The function $\zeta(s)$ admits analytic continuation to $\mathbb{C}$, where it satisfies the functional equation $\zeta(s) = 2^s\pi^{s-1}\sin\frac{\pi s}{2}\,\Gamma(1-s)\zeta(1-s)$. Thus, by means of analytic continuation,

$\zeta(s)$ is analytic function for any complex s, except for $s = 1$, which is a simple pole of $\zeta(s)$ with residue 1.

Using our approach we get an integral representation of $\zeta(s+1)$. Since $f(z) = 1/z^{s+1}$ and $F(z) = -1/(sz^s)$, using (1) and (2) we obtain

$$\zeta(s+1) = \sum_{k=1}^{m-1} \frac{1}{k^{s+1}} + \frac{\pi}{4s\left(m-\frac{1}{2}\right)^s} \sum_{\nu=1}^{n} A_\nu^{(n)} g\left(\xi_\nu^{(n)}\right) + E_{n,m}(s),$$

where

$$g(t;s) = \exp\left(-\frac{s}{2}\log(1+t^2)\right)\cos\left(s\arctan t\right), \quad c_m = \frac{1}{2m-1},$$

and $E_{n,m}(s)$ is the corresponding error term.

Euler-Mascheroni constant

The Euler-Mascheroni constant γ is defined as

$$\gamma = \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k} - \log n\right) = 0.57721566490153286060651209008 2\ldots\ .$$

This constant can be expressed as the following slowly convergent series (cf. [4, p. 30])

$$\gamma = \sum_{k=1}^{\infty}\left(\frac{1}{k} - \log\left(1+\frac{1}{k}\right)\right).$$

Applying our method to summation of this series we use (1), with

$$f(z) = \frac{1}{z} - \log\left(1+\frac{1}{z}\right) \quad \text{and} \quad F(z) = 1-(z+1)\log\left(\frac{z+1}{z}\right),$$

where $\lim_{z\to\infty} F(z) = 0$.

The corresponding relative errors $\mathrm{err}_{n,m} = \left|\ (Q_{n,m}(f)-\gamma)/\gamma\ \right|$ for number of nodes n in the quadrature formula (2) and $m = 1, 2, 3, 6, 11$ and 16 are presented in Table 1.

n	$m=1$	$m=2$	$m=3$	$m=6$	$m=11$	$m=16$
10	2.73(−4)	2.03(−9)	2.12(−13)	4.50(−22)	2.39(−31)	1.34(−37)
20	5.92(−5)	2.79(−11)	1.90(−16)	1.49(−28)	5.53(−43)	1.32(−53)
30	2.43(−5)	2.26(−12)	3.06(−18)	2.00(−32)	3.53(−50)	6.92(−64)
100	1.77(−6)	1.30(−15)	1.43(−23)	5.23(−44)	4.60(−72)	7.33(−96)

Table 1: Relative errors $\mathrm{err}_{n,m}$ in of Gaussian approximations $Q_{n,m}(f)$ of $T_1(10)$ for $n = 10, 20, 30$ and 100 and for some selected values of m

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Equivalences with the separable quotient problem

S. López-Alfonso [1], *M. López-Pellicer* [*2] *and S. Moll-López* [3]

The centenary of the still open problem of separable quotient Banach space X will be in 2032. We gather several equivalences of this problem and present its positive solution Banach spaces with large or small density character.

2010 MATHEMATICS SUBJECT CLASSIFICATIONS: Banach space, Barrelled space, Basic sequence, Bounding cardinal, Density character, Schauder basis, Separable quotient, Strongly normal sequence

KEYWORDS: 46B28, 46E27, 46E30

Introduction

All Banach spaces X considered in this paper are infinite dimensional and, as usual, X verifies the separable quotient problem if X admits an infinite dimensional separable quotient. The following problem is unsolved; it was proposed by Stanisław Mazur in 1932.

Problem 1. *Does every Banach space X verifies the separable quotient problem?*

It is equivalent withthe following problems:

1. *Is it possible to map every Banach space X onto a separable Banach space*? Equivalence follows by the open mapping theorem [22].
2. *Does every Banach space X admits an infinite dimensional quotient with a Schauder basis*? To get the equivalence remind that every infinite dimensional separable Banach space admits a quotient with a Schauder basis ([9, Theorem IV.1(i)]).
3. *Does every infinite dimensional Banach space X contains a separable closed subspace which has a proper quasi-complement in X*? Equivalence is obvious recalling that if Y is a separable closed subspace of the Banach space X and X/Y has a separable quotient, then Y is quasi-complemented in X ([8, Theorem 2]).

Problem 1 has positive answer, among others, for the weakly compactly generated Banach spaces and, in particular, the reflexive spaces (see [11] and [15]). Moreover, using Ramsey Theory it was proved in [2] that for every separable Banach space X with non-separable dual X^*, the bidual space X^{**} contains an unconditional family of size $|X^{**}|$, that implies Theorem 1, obtained in [2].

Theorem 1 (Argyros-Dodos-Kanellopoulos)**.** *Every dual Banach space X^* has a separable quotient.*

This deep theorem and the fact that X^* is complemented in $\mathcal{L}(X,Y)$ imply that the space $\mathcal{L}(X,Y)$ of bounded linear operators between Banach spaces X and Y equipped with the operator norm has a separable quotient, provided $Y \neq \{\mathbf{0}\}$.

Recall that the *density character* of a Banach space X is the smallest of cardinals of dense subsets of X and the *bounding cardinal* $\mathfrak{b}$ is the minimum of cardinals of unbounded subsets of $(\mathbb{N}^{\mathbb{N}}, \leq^*)$, where $\leq^*$is the *eventual preorder* ($\alpha \leq^* \beta$ means that $\{n \in \mathbb{N} : \alpha(n) > \beta(n)\}$ is finite). $\mathfrak{b}$ is a regular cardinal, $\aleph_1 < \mathfrak{b} \leq \mathfrak{c}$ and Martin's Axiom implies $\mathfrak{b} = \mathfrak{c}$. For 'large' Banach spaces was proved in [21] the following deep result .

Theorem 2 (Todorcevic)**.** *Under Martin's axiom every Banach space of density character $\aleph_1$ has a quotient space with an uncountable monotone Schauder basis, and thus a separable quotient.*

In the following sections firstly we gather the classical Saxon-Wilansky and the Śliwa equivalences with Problem 1, which enable prove that 'small' Banach spaces have a separable quotient (see [19] and [20]), and secondly we present Banach spaces that admit quotients isomophic to ℓ_1 or ℓ_2. Additional related results may be found in [7].

Several equivalences with the separable quotient problem

The following Theorem 3 provides three equivalent conditions to Problem 1. For the equivalence between (2) and (3) below in the class of locally convex spaces we refer the reader to [10], [16] and [17], where interesting applications are provided. The equivalence between (1) and (3) is in [19]. Recall that a locally convex space E is *barrelled* if E verifies the Banach-Steinhaus theorem, or equivalently if every absolutely convex closed and absorbing subset of E is a neighborhood of zero. Additional information and some partial results related with Theorem 3 are in [3] and [9].

Theorem 3 (Saxon-Wilansky)**.** *The following assertions are equivalent for an infinite-dimensional Banach space X.*

1. *X contains a dense non-barrelled linear subspace.*
2. *X admits a strictly increasing sequence of closed subspaces of X whose union is dense in X.*
3. *X^* admits a strictly decreasing sequence of weak*-closed subspaces whose intersection consists only of the zero element.*
4. *X has a separable quotient.*

Proof. We prove only the equivalence between (2) and (4).

(4) $\Rightarrow$ (2) Note that every separable Banach space has the property stated in (2) and this property is preserved under preimages of surjective linear operators (which clearly are open maps).

(2) $\Rightarrow$ (4) Let $\{X_n : n \in \mathbb{N}\}$ be a strictly increasing sequence of closed subspaces of X whose union is dense in X. We may assume that $\dim(X_{n+1}/X_n) \geq n$ for all $n \in \mathbb{N}$. Let $x_1 \in X_2 \setminus X_1$. There exists x_1^* in X^* such that $x_1^* x_1 = 1$ and x_1^* vanishes on X_1. Assume that we have constructed $(x_1, x_1^*), \ldots, (x_n, x_n^*)$ in $X \times X^*$ with $x_j \in X_{j+1}$ such that $x_j^* x_j = 1$ and x_j^* vanishes on X_j for $1 \leq j \leq$

n. Choose $x_{n+1} \in \left\{X_{n+2} \cap (x_1^*)^{-1}(\mathbf{0}) \cap \cdots \cap (x_n^*)^{-1}(\mathbf{0})\right\} \setminus X_{n+1}$. Then there exists $x_{n+1}^* \in X^*$ with $x_{n+1}^* x_{n+1} = 1$ and such that x_{n+1}^* vanish on X_{n+1}. Since $X_{n+1} \subseteq \operatorname{span}\{x_1, x_2, \ldots x_n\} + \bigcap_{k=1}^{\infty} \ker x_k^*, n \in \mathbb{N}$, we conclude that $\operatorname{span}\{x_n : n \in \mathbb{N}\} + \bigcap_n \ker x_n^*$ is dense in X. Then X/Y is separable for $Y := \bigcap_n \ker x_n^*$. □

From Theorem 3 and [1] follow easily that a weakly compact generated Banach space or with separable weak*-dual have separable quotient.

Recall that the Josefson-Nissenzweig theorem states that the dual of any infinite-dimensional Banach space contains a *normal sequence*, i. e., a normalized weak*-null sequence [13], and that (cf. [20]) a sequence $\{y_n^*\}$ in the sphere $S(X^*)$ of X^* is *strongly normal* if the subspace $\{x \in X : \sum_{n=1}^{\infty} |y_n^* x| < \infty\}$ is dense in X. Clearly every strongly normal sequence is normal.

In [20] it is proved that any strongly normal sequence in X^* contains a subsequence $\{y_n\}$ which is a w^*-Schauder basis in its w^*-closed linear span. Hence the conditions (2) and (3) of the next Theorem 4 are equivalent.

Theorem 4 (Śliwa)**.** *Let X be an infinite-dimensional Banach space. The following conditions are equivalent:*

1. *X has an infinite dimensional separable quotient.*
2. *X^* has a strongly normal sequence.*
3. *X^* has a basic sequence in the weak* topology.*
4. *X^* has a pseudobounded sequence, i. e., a sequence $\{x_n^*\}$ in X^* that is pointwise bounded on a dense subspace of X and $\sup_n \|x_n^*\| = \infty$.*

Proof. (1) ⇒ (2) By Theorem 3 the space X contains a dense non-barrelled subspace. Hence there exists a closed absolutely convex set D in X such that $H := \operatorname{span}(D)$ is a proper dense subspace of X. For each $n \in \mathbb{N}$ choose $x_n \in X$ so that $\|x_n\| \leq n^{-2}$ and $x_n \notin D$. Then select x_n^* in X^* such that $x_n^* x_n > 1$ and $|x_n^* x| \leq 1$ for all $x \in D$. Set $y_n^* := \|x_n^*\|^{-1} x_n^*$ for all $n \in \mathbb{N}$. Since $\|x_n^*\| \geq n^2$, $y_n^* \in S(X^*)$ and $\sum_{n=1}^{\infty} |y_n^* x| < \infty$ for all $x \in D$, the sequence $\{y_n^*\}$ is as required.

(2) ⇒ (1) Assume that X^* contains a strongly normal sequence $\{y_n^*\}$. By [20, Theorem 1] (see also [9, Theorem III.1 and Remark III.1]) there exists a subsequence of $\{y_n^*\}$ which is a weak*-basic sequence in X^*. This implies that X admits a strictly increasing sequence of closed subspaces whose union is dense in X. Indeed, since (X^*, w^*) contains a basic sequence, and hence there exists a strictly decreasing sequence $\{U_n : n \in \mathbb{N}\}$ of closed subspaces in (X^*, w^*) with $\bigcap_{n=1}^{\infty} U_n = \{\mathbf{0}\}$, the space X has a sequence as required. This provides a biorthogonal sequence as in the proof of (2) ⇒ (3), Theorem 3.

(4) ⇒ (1) Let $\{y_n^*\}$ be a pseudobounded sequence in X^*. Set $Y := \{x \in X : \sup_{n \in \mathbb{N}} |y_n^* x| < \infty\}$. The Banach-Steinhaus theorem applies to deduce that Y is a proper and dense subspace of X. Note that Y is not barrelled since $V := \{x \in X : \sup_{n=1}^{\infty} |y_n^* x| \leq 1\}$ is a barrel in H which is not a neighborhood of zero in H. Now apply Theorem 3.

(1) ⇒ (4) Assume X contains a dense non-barrelled subspace Y. Let W be a barrel in Y which is not a neighborhood of zero in Y. If V is the closure of W in X, the linear span H of V is a dense proper subspace of X. So, for every $n \in \mathbb{N}$ there is $x_n \in X \setminus V$ with $\|x_n\| \leq n^{-2}$. Choose $z_n^* \in X^*$ so that $|z_n^* x_n| > 1$ and $|z_n^* x| = 1$ for all $x \in V$ and $n \in \mathbb{N}$. Then $\|z_n^*\| \geq n^2$ and $\sup_n |z_n^* x| < \infty$ for $x \in H$. □

Corollary 5 follows from Theorem 3 in [18, Theorem 3]. Here is obtained from Theorem 4.

Corollary 5 (Saxon–Sánchez Ruiz)**.** *If the density character $d(X)$ of a Banach space X satisfies that $\aleph_0 \le d(X) < \mathfrak{b}$ then X has an infinite dimensional separable quotient.*

Proof. Let $\{y_n^*\}$ be a normalized weak*-null sequence in the dual Banach space X^* and, by hypothesis, let D be a dense subset of X such that the cardinality of D is less than the bounding cardinal $\mathfrak{b}$. For $x \in D$ choose $\alpha_x \in \mathbb{N}^{\mathbb{N}}$ such that for each $n \in \mathbb{N}$ and every $k \ge \alpha_x(n)$ one has $|y_k^* x| < 2^{-n}$. Then $\sum_n |y_{\beta(n)}^* x| < \infty$ if $\alpha_x \le^* \beta$. The inequality $|D| < \mathfrak{b}$ implies that $\{\alpha_x : x \in D\}$ is a bounded subset of $(\mathbb{N}^{\mathbb{N}}, \le^*)$, hence there exist $\gamma \in \mathbb{N}^{\mathbb{N}}$ with $\alpha_x \le^* \gamma$ for each $x \in D$. Hence the sequence $\{y_{\gamma(n)}^*\}$ is strongly normal sequence in $X^*$and Theorem 4 implies that X^* has an infinite dimensional separable quotient. □

Quotients isomophic to ℓ_1 or ℓ_2

By Theorem 4 a Banach space X has a separable quotient if and only if X^* contains a strongly normal sequence. A stronger property is characerized in Proposition 6.

Proposition 6. *A Banach space X has a quotient isomorphic to ℓ_1 if and only if X^* contains a normal sequence $\{y_n^*\}$ such that $\sum_n |y_n^* x| < \infty$ for all $x \in X$.*

Proof. If there is a bounded linear operator Q from X onto ℓ_1, its adjoint map fixes a sequence $\{x_n^*\}$ in X^* such that the formal series $\sum_{n=1}^{\infty} x_n^*$ is weakly unconditionally Cauchy and $\inf_{n\in\mathbb{N}} \|x_n^*\| > 0$. Setting $y_n^* := \|x_n^*\|^{-1} x_n^*$ for each $n \in \mathbb{N}$, the sequence $\{y_n^*\}$ is as required. Conversely, if there is a normal sequence $\{y_n^*\}$ like that of the statement, it defines a weak* Cauchy series in X^*. Since the series $\sum_{n=1}^{\infty} y_n^*$ does not converge in X^*, according to [4, Chapter V, Corollary 11] the space X^* must contain a copy of ℓ_∞. Consequently, X has a complemented copy of ℓ_1 by [4, Chapter V, Theorem 10]. □

The next Theorem 8 (see [12]) provides a quotient space isomorphic to ℓ_2. It follows from Lemma 7 (see [15, Corollary 1.6, Proposition 1.2]).

Lemma 7. *Let X be a Banach space such that X^* contains an infinite-dimensional reflexive subspace Y. Then X has a quotient isomorphic to Y^*. Consequently X has a separable quotient.*

Theorem 8 (Mujica)**.** *If X is a Banach space that contains an isomorphic copy of ℓ_1, then X has a quotient isomorphic to ℓ_2.*

Proof. If X contains a copy of ℓ_1, the dual space X^* contains a copy of $L_1[0,1]$, see [4]. It is well known that the space $L_1[0,1]$ contains a copy of ℓ_2. As the dual of ℓ_2 is isometricaly isomorphic to ℓ_2, then this theorem follows from Lemma 7. □

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Reaction-Diffusion systems that I liked

Mokhtar Kirane

I will present some results concerning global solvability and large time behavior for reactiondiffusion systems without or with fractional in time and space derivative, with a balance law.

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Some examples of application of the operator ${}_zO_\beta^\alpha$ to special functions, in particular to the Christoffel-Darboux identity for orthogonal polynomials

Richard Tremblay

In the field of special functions, the theory relating to sequences of functions of orthogonal polynomials and of multiple orthogonal polynomials is fundamental. There are several formulas and many useful applications in mathematical physics, numerical analysis, statistics and probability and in many other disciplines. For example, the well-known identity of Christoffel Darboux has generated a large number of research articles.

Several summation formulas involving the classical orthogonal polynomials are obtained using the well-balanced fractional operator defined in terms of the fractional derivative, ${}_zO_\beta^\alpha \equiv \frac{\Gamma(\beta)}{\Gamma(\alpha)} z^{1-\beta} D_z^{\alpha-\beta} z^{\alpha-1}$. This operator has several operational properties and it has already been used in several articles involving special functions, for example obtaining several new higher-order transformations of the Gaussian hypergeometric function (R. Tremblay, New quadratic transformations of hypergeometric functions and associated summation formulas obtained with the Well-Poised fractional calculus operator, Montes Taurus J. Pure Appl. Math. 2 (1), 3662, 2020). Numerous examples unequivocally demonstrate the effectiveness of the fractional operator ${}_zO_\beta^\alpha$ to generate new relations involving the special functions of one or more variables. New formulas involving the generalized hypergeometric function and an extension of the generalized Bernoulli polynomials are obtained. By applying the operator ${}_zO_\beta^\alpha$ to the classical Christoffel-Darboux identity, two general summation formulas for orthogonal polynomials are deduced and are applied to the main orthogonal polynomials.

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Bernstein Bases and Blossoming

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Originally introduced to provide a constructive proof of the Weierstrass approximation theorem for uniform polynomial approximation of continuous functions over a prescribed interval, the Bernstein basis functions today lie at the very foundation of Computer Aided Geometric Design. A powerful technique, *blossoming*, is used to study properties of the Bernstein basis functions and the corresponding Bzier curves and surfaces with these blending functions. We aim to review four different variants of the blossoming scheme associated with four different forms of the Bernstein bases: *the classical Bernstein bases*, *the q-Bernstein bases*, *the negative degree Bernstein bases*, and *the negative degree q-Bernstein bases*. We compare several fundamental identities involving these bases, including the Marsden identities, the partition of unity properties, the reparametrization formulas, and the formulas for representing monomials. We also show that the dual functional properties corresponding to these four types of blossom play a fundamental role in the derivations of these identities.

2010 Mathematics Subject Classifications: 65Dxx, 65D17, 41Axx, 05A30

Keywords: Bernstein basis, q-Bernstein basis, Dual functionals, Blossom, q-blossom, Multirational blossom, Multirational q-blossom, Marsden identity

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Certain new families related to q-Bessel polynomials and q-Bessel functions

Subuhi Khan

In this work, a generalized version of the q-Bessel polynomials is considered. Generating equation, series expansion, determinant form and other properties for these polynomials are established. As an example, the continuous $2D$ q-Hermite-Bessel polynomials are considered. Further, the q-Fubini polynomials are combined with q-Bessel functions in order to introduce the q-Fubini-Bessel functions. Certain results for these q-hybrid functions are derived. Their relations with the classical Fubini polynomials and Bessel functions are established and q-Fubini-Bessel polynomials are explored. The graphs and plots depicting the behavior of the continuous $2D$ Hermite polynomials, continuous $2D$ Hermite-Bessel polynomials and continuous $2D$ q-Hermite-Bessel polynomials are drawn. The distribution of zeros for these polynomials is presented. The graphical representations of the q-Fubini polynomials and q-Fubini-Bessel polynomials for particular values of indices and variables are also considered.

2010 Mathematics Subject Classifications: 11B83, 05A30, 33E99, 33D99, 11B73, 11B65

Keywords: q-Bessel polynomials, Generalized $2D$ q-Bessel polynomials, Continuous $2D$ q-Hermite-Bessel polynomials, Determinant expressions, q-Fubini polynomials, q-Bessel functions, q-Bessel polynomials

As known, q-calculus emerged as a link between mathematics and physics. Applications of quantum calculus arise in several branches of mathematics, like fractional calculus, theory of relativity, combinatorics etc. The evolution of q-calculus also led to the introduction of q-analogues of several important special functions. Recently, Mumtaz Riyasat and Subuhi Khan [8] studied the q-Bessel polynomials $p_{n,q}(u)$. The q-Bessel polynomials $p_{n,q}(u)$ have the following generating equation

$$e_q(u(1-\sqrt{1-2\alpha})) = \sum_{n=0}^{\infty} p_{n,q}(u)\frac{\alpha^n}{[n]_q!}. \tag{1}$$

The following relationship between the polynomials $p_{n,q}(u)$ and an other form of the q-Bessel polynomials $\theta_{n,q}(u)$ holds:

$$\theta_{n,q}(u) = \left(\frac{1}{u}\right)\ p_{n+1,q}(u). \tag{2}$$

The q-Bessel polynomials $\theta_{n,q}(u)$ are defined by the following generating function [8]:

$$\frac{1}{\sqrt{1-2\alpha}}\ \ e_q(u(1-\sqrt{1-2\alpha})) = \sum_{n=0}^{\infty} \theta_{n,q}(u)\frac{\alpha^n}{[n]_q!} \tag{3}$$

and possess the following series expansion:

$$\theta_{n,q}(u) = \left(\frac{1}{u}\right)\ p_{n+1,q}(u) = \sum_{k=0}^{n} \frac{[n+k]_q!}{[n-k]_q!\ [k]_q!}\frac{u^{n-k}}{2^k}. \tag{4}$$

The special polynomials of two variables play an important role in the derivation of results in a straightforward manner. Certain classes of functional and differential equations are analyzed by using the 2-variable forms of the generalized and extended polynomials. We explore the q-analogue of the generalized 2-variable polynomials $p_n(u,v)$ [9].

The $2D$ q-Bessel functions $J_n^{p,q}(u,v)$ are defined by the generating function

$$\exp(i(u\sin p\alpha + v\sin q\alpha)) = \sum_{n=0}^{\infty} J_n^{p,q}(u,v)e^{in\alpha}. \tag{5}$$

The Bessel functions in two dimensions have applications in several areas such as in periodic structures of quantum dynamics [7].

The generalized $2D$ q-Bessel polynomials are defined by means of the following generating function:

$$e_q(u(1-\sqrt{1-2\alpha}))\ \varphi_q(v,1-\sqrt{1-2\alpha}) = \sum_{n=0}^{\infty} {}_g p_{n,q}(u,v)\frac{\alpha^n}{[n]_q!}, \tag{6}$$

where

$$\varphi_q(v,1-\sqrt{1-2\alpha}) = \sum_{n=0}^{\infty}\varphi_{n,q}(v)\frac{(1-\sqrt{1-2\alpha})^n}{[n]_q!}.$$

For the generalized $2D$ q-Bessel polynomials ${}_g p_{n,q}(u,v)$, the following explicit series expansion holds:

$${}_g p_{n,q}(u,v) = \sum_{k=0}^{n-1}\frac{[n-1+k]_q!}{2^k\ [n-1-k]_q!\ [k]_q!}g_{n-k,q}(u,v). \tag{7}$$

The determinant representations of the special polynomials are helpful in solving the linear interpolation problems and are also useful in numerical computations. The determinant form of the generalized $2D$ q-Bessel polynomials is established in this work.

An example is considered by taking a particular member of the generalized $2D$ q- Bessel polynomials. By choosing the function $\varphi_q(v,1-\sqrt{1-2\alpha})$ of the continuous q-Hermite polynomials $H_{n,q}^{(s)}(v)$ [2] in the generating equation of the generalized $2D$ q-Bessel polynomials, we get the continuous $2D$ q-Hermite-Bessel polynomials ${}_H p_{n,q}^{(s)}(u,v)$. These polynomials possess the following generating function:

$$e_q(u(1-\sqrt{1-2\alpha}))\ E_q(v(1-\sqrt{1-2\alpha}))\ e_q\left(\frac{-s(1-\sqrt{1-2\alpha})^2}{1+q}\right) = \sum_{n=0}^{\infty} {}_H p_{n,q}^{(s)}(u,v)\frac{\alpha^n}{[n]_q!}. \tag{8}$$

The series expansion, determinant form, recurrence formula, and summation formulae for the continuous $2D$ q-Hermite-Bessel polynomials ${}_H p_{n,q}^{(s)}(u,v)$ are also obtained.

The orthogonal polynomials have connections with certain problems arising in approximation theory and other mathematical branches. These polynomials are also useful in theoretical physics and chemistry. The orthogonality of the q-Bessel polynomials can be established by deriving a three-term recurrence relation for these polynomials. This fact makes these polynomials useful in the field of wavelet analysis. The following recurrence formula for the continuous $2D$ q-Hermite-Bessel polynomials ${}_Hp^{(s)}_{n,q}(u,v)$ is derived:

$$
\begin{aligned}
{}_Hp^{(s)}_{n+1,q}(u,v) &= \left(\frac{2s}{1+q}\right) {}_Hp^{(s)}_{n,q}(qu,qv) \\
+\sum_{k=0}^{n} \frac{\left(\frac{-1}{2}\right)_k \; 2^k \; [n]_q!}{[n-k]_q!k!} &\left[\left(v-\frac{2s}{1+q}\right) {}_Hp^{(s)}_{n-k,q}(qu,qv) + u_H p^{(s)}_{n-k,q}(u,v)\right].
\end{aligned} \tag{9}
$$

For other suitable choices of $\varphi_q(v,(1-\sqrt{1-2\alpha}))$, certain new polynomials belonging to the generalized $2D$ q-Bessel family are obtained. The graphs and plots depicting the behaviour of the continuous $2D$ Hermite polynomials, continuous $2D$ Hermite-Bessel polynomials, and continuous $2D$ q-Hermite-Bessel polynomials are drawn. The distribution of zeros for these polynomials is also presented.

The q-Bessel functions (or basic Bessel functions) [6] are the q-analogues of the Bessel functions. The q-Bessel functions $J_\nu(w;q)$ and the 2-dimensional q-Bessel functions $J_\nu(u,v;q)$ are defined by means of the following generating functions [3]:

$$
e_q\left(\frac{wt}{2}\right) e_q\left(-\frac{w}{2t}\right) = \sum_{\nu=-\infty}^{\infty} J_\nu(w:q)t^\nu, \quad t \neq 0, \quad |w| < \infty, \tag{10}
$$

$$
\begin{aligned}
&e_q\left(\frac{ut}{2}\right) E_q\left(\frac{vt}{2}\right) e_q\left(-\frac{u}{2t}\right) E_q\left(-\frac{v}{2t}\right) \\
&= \sum_{\nu=-\infty}^{\infty} J_\nu(u,v;q)t^\nu, \quad t \neq 0, \quad |u| < \infty.
\end{aligned} \tag{11}
$$

For $q \to 1^-$, the q-Bessel functions reduce to the Bessel functions $J_\nu(u)$ [1]:

$$
\exp\left(\frac{u}{2}\left(t-\frac{1}{t}\right)\right) = \sum_{\nu=-\infty}^{\infty} J_\nu(u)t^\nu, \quad t \neq 0, \quad |u| < \infty. \tag{12}
$$

The Fubini polynomials play an important role in the theory and applications of mathematics. These polynomials appear in combinatorial mathematics, thus attracted an appreciable amount of interest of number theory and combinatorics experts. Fubini polynomials possess the generating equation

$$
\frac{1}{1-w(e^t-1)} = \sum_{r=0}^{\infty} F_r(w)\frac{t^r}{r!}, \tag{13}
$$

The 2-variable Fubini polynomials are defined by the generating function

$$
\frac{e^{ut}}{1-w(e^t-1)} = \sum_{r=0}^{\infty} F_r(u,w)\frac{t^r}{r!}, \tag{14}
$$

A q-analogue of the Fubini polynomials has recently been introduced [5]. The q-Fubini polynomials of one and three variables are defined by means of the generating functions

$$
\frac{1}{1-w(e_q(t)-1)} = \sum_{r=0}^{\infty} F_{r,q}(w)\frac{t^r}{[r]_q!} \tag{15}
$$

and

$$\frac{1}{1-w(e_q(t)-1)}e_q(ut)E_q(vt)=\sum_{r=0}^{\infty}F_{r,q}(u,v;w)\frac{t^r}{[r]_q!}, \tag{16}$$

respectively.

For $q \to 1^-$, the q-Fubini polynomials $F_{r,q}(w)$ and $F_{r,q}(u,v;w)$ reduce to the corresponding classical Fubini polynomials $F_r(w)$ and $F_r(u,v;w)$ respectively.

The q-Fubini-Bessel functions ${}_FJ_r(w;q)$ are introduced by means of the following generating equation and series expansion:

$$\frac{1}{\left[1-w\left(e_q\left(\frac{t}{2}\right)-1\right)\right]\left[1-w\left(e_q\left(\frac{-1}{2t}\right)-1\right)\right]}=\sum_{r=-\infty}^{\infty}{}_FJ_r(w;q)t^r \tag{17}$$

and

$${}_FJ_r(w;q)=\sum_{k=0}^{\infty}\sum_{m=0}^{r+k}\sum_{n=0}^{k}\frac{(-1)^k[m]_q![n]_q!S_{2,q}(r+k,m)S_{2,q}(k,n)w^{m+n}}{[r+k]_q![k]_q!2^{r+2k}}, \tag{18}$$

respectively. Here, $S_{2,q}(r,k)$ denotes q-Stirling numbers of second kind [4].

The q-analogues of multi-variable special functions were suggested by problems arising in physical phenomena. In order to further stress the importance of multi-variable q-special functions, the q-Fubini-Bessel functions of three variables are also considered. The 3-variable q-Fubini-Bessel functions ${}_FJ_r(u,v,w;q)$ are introduced by means of the following generating relation:

$$\frac{e_q\left(\frac{ut}{2}\right)E_q\left(\frac{vt}{2}\right)e_q\left(\frac{-u}{2t}\right)E_q\left(\frac{-v}{2t}\right)}{\left(1-w\left(e_q\left(\frac{t}{2}\right)-1\right)\right)\left(1-w\left(e_q\left(\frac{-1}{2t}\right)-1\right)\right)}=\sum_{r=-\infty}^{\infty}{}_FJ_r(u,v,w;q)t^r. \tag{19}$$

Further, the explicit summation formulae and q-differential recurrence relations for the 3-variable q-Fubini-Bessel functions ${}_FJ_r(u,v,w;q)$ are derived.

By adapting the procedure used above, we get the generating function for the q-Fubini-Bessel polynomials ${}_Fp_r(w;q)$ and 3-variable q-Fubini-Bessel polynomials ${}_Fp_r(u,v,w;q)$:

$$\frac{1}{1-w(e_q(1-\sqrt{1-2t})-1)}=\sum_{r=0}^{\infty}{}_Fp_r(w;q)\frac{t^r}{[r]_q!} \tag{20}$$

and

$$\frac{e_q(u(1-\sqrt{1-2t}))E_q(v(1-\sqrt{1-2t}))}{1-w(e_q(1-\sqrt{1-2t})-1)}=\sum_{r=0}^{\infty}{}_Fp_r(u,v,w;q)\frac{t^r}{[r]_q!}, \tag{21}$$

respectively. The series representation for the q-Fubini-Bessel polynomials ${}_Fp_r(w;q)$ and 3-variable q-Fubini-Bessel polynomials ${}_Fp_r(u,v,w;q)$ are also obtained. The graphical representations of the q-Fubini polynomials and q-Fubini-Bessel polynomials for particular values of indices and variables are also considered.

Acknowledgments

The contents are based on the joint work with Mumtaz Riyasat (Department of Applied Mathematics, Aligarh Muslim University, Aligarh, India) and Mehnaz Haneef (Department of Mathematics, Aligarh Muslim University, Aligarh, India).

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On the usefulness of quaternion analysis

Wolfgang Sproessig

We start with the introduction of Hamilton's quaternions. We give an overview on further algebras with four units. Then we explain Fueter's theory for the construction of so-called quaternionic -holomorphic functions and study their properties. A quaternionic operator calculus is introduced. Applications to some boundary value problems are discussed. Higherdimensional Pi-operators are considered. Among others factorisation principles are described. In this connection a Riccati type problem (Guerlebeck, Bernstein) is presented.
The idea of this talk is to make the audience familiar with basic concepts in order to adapt these methods to their own work.

2010 Mathematics Subject Classifications: 30G35, 76W05

Keywords: quaternion analysis, operator calculus

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The importance of drinking water in terms of medical geology; Hidden danger in waters: (Ba, Ca, I, Mg, etc.)

Yusuf Uras

It is important to proceed by synthesizing geology and medical sciences and to target scientific studies by examining the scarcity and abundance of elements from nature that cause people to get sick without being aware of this synthesis. As a science that deals with mother nature, it is important that medical geology always undertakes this task, so that our people can feel that our most important asset is health and that it comes first and foremost, by touching their lives at least a little. It is important that it aims to embroider by revealing an inseparable whole. Does water, which is the indispensable companion of our body, guide the life it offers us correctly? The study areas are Krehir and Kahramanmara and the villages around them, where there are intense diseases and physical problems, no external factors, where water and rock come into contact with each other, did not affect the water, and traces from the geography formed the study area. The scarcity and excess of elements in the waters, which have led to interesting scientific studies such as triode disease caused by the drinking water of the communities living in the villages, hearing loss, fluoridation in the teeth and short stature affected by even physical appearances, have created hidden dangers in drinking water for us. The abundance of fluorine element in the waters Intensely, the fluorine element in the drinking water of the village people living in the village of Krehir's phrenk affects the development of children. Another danger in drinking water in the Derebogazi town of Kahramanmara is the absence of any element in the waters. The fact that the physical appearance of the village people is short, the problem here may of course be genetic, and the analysis of the waters, considering that this genetic triggering element may be water, and the absence of any element in the waters as a result of intense work has once again presented the importance of medical geology. It is really interesting and exciting that the water is purified and cleaned in a wonderful cycle underground, leaving all the minerals in it and coming to the surface again and consuming this beautiful water. The absence of any mineral in the waters is the most important factor that may be the reason for the village people to go to the hospital because of their short stature and if they have problems in their bodies due to the lack of calcium in postmenopausal women. Of course, these studies are carried out together with medical doctors, but they find the solution to the disease, but scientists who are interested in medical geology should be needed to speculate on the cause. Medical geologists are scientists who aim to eradicate the problem. Anzorey, also known as Yukarkargabk town of Afin municipality in Kahramanmara, is a very interesting case where most of the villagers are deaf after the age of 50 and they walk around with hearing aids in their ears. It is an interesting result that studies with the support of an otolaryngologist and a scientist interested in audiometry caused an old water source used by the public and as a result, excess barium in the water source caused deafness. The audiometry specialist has proven with tests that it is not age related, not caused by working in noisy places, etc. Medical geologists assumed that it was our duty here and believed that they should use other sources by proving the damage that the waters cause to the public.

Keywords: Drinking Water, Hydrogeochemistry, Isotope Geochemistry, Medical Geology, Kahramanmaras

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2 CONTRIBUTED TALKS

Interval valued PCA-based approach for fault detection in dynamic systems

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Dynamic principal component analysis (DPCA) is an expansion of Principal component analysis (PCA) and has been widely applied to monitor dynamic process. In this paper, we propose a new diagnosis procedure for uncertain process using an extension of DPCA for interval data, namely, an interval-valued DPCA (IV-DPCA). IVDPCA is a repeated DPCA application on data obtained under the same system conditions where the model is built based mainly on the extracted interval eigenvalues, their corresponding interval eigenvectors, and the interval fault detection indices thresholds. A simulation example is used to compare between IV-DPCA, DPCA and PCA in terms of good detections. The collected results demonstrate the superiority of the developed method over the well-established PCA,DPCA techniques.

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Nonlinear process monitoring based on interval-valued PCA approach

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Kernel principal component analysis (KPCA) is a knowledgeable, data-driven method based on fault detection and diagnosis. It has attracted a lot of interest because it can monitor nonlinear systems. However, KPCA is subject to system uncertainty and inaccurate measurements. These have a significant impact on decision-making regarding the operational status of a process. This task transforms the data collected from various sensors from a single value to an interval value, which quantifies measurement uncertainties. A new interval fault detection named Interval-Valued (IV-KPCA) is proposed. As a result, process modeling based on the KPCA method was performed on the interval-valued (IV-KPCA). As a result, the interval value representation preserves various fault detection statistics such as T^2 and Q as well as Φ. The proposed method is proven by the cement rotary kiln process. Its performance concerning false and missed alarms and detection delays is compared to other techniques through unintentional system faults and various other types of sensor faults. The results show that there is a high advantage in detecting clear faults quickly and accurately in stochastic environments, where there are unknown and uncontrolled uncertainties.

KEYWORDS: Fault detection, Kernel principal component analysis, Interval-valued kernel principal component analysis (IV-KPCA), Quantification of the uncertainties, Cement rotary kiln

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Robust simultaneous estimation of actuator and sensor faults for Uncertain Lipschitz nonlinear systems

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In this paper, an Unknown Input Observer for an augmented system is designed to robustly estimate the actuator/sensor faults. This augmented system was created by introducing a new vector state composed from system states and the desired faults. To attenuate the effect of disturbances, the observer gains are obtained by solving LMI (linear matrix inequality) using H_∞ optimization. LMI regions was applied in this study for pole placement. A simulation example is provided to demonstrate the effectiveness of this approach.

KEYWORDS: Robust Fault estimation, Unknown Input Observer (UIO), LMI regions, H_∞ optimization, Lyapunov stability

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Root-elements, root-subgroups and the maximal parabolic subgroup P_6 in $E_6(K)$ for fields K of characteristic two

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The purpose of this article is to give an elementary and explicit description of the root-elements of the Chevalley group E of type $E_6(K)$ in fields K of characteristic two. Furthermore, we show that there is a bijection between the root subgroups of E_6 and the family V_6, that is the family of all 6-dimensional submodules of the 27-dimensional module E_6 over fields K of characteristic two, on one hand and give a full description of the long roots of the group ${}^2E_6(K)$, on the other hand. A new notion of root-subgroups of $E_6(K)$ is introduced and we show that the two notions coincide. The stabilizer of a 6-dimensional Tits subspace in V_6, which is a Borel subgroup of $E_6(K)$ or the maximal parabolic subgroup P_6 in $E_6(K)$, is constructed. This construction is new.

2010 Mathematics Subject Classifications: 17A74, 17A45

Keywords: Root-elements, Root-subgroups, Chevalley group, Borel subgroup, Tits subspace, Generalized quadrangle

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Main and Non-Main eigenvalues of Connected Chain Graphs

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An eigenvalue of a chain graph is said to be main eigenvalue if it has an eigenvector not orthogonal to the all 1-vector. Otherwise, it is called non-main eigenvalue. In this presentation, we consider connected chain graphs and study their main and non-main eigenvalues. We present some examples where upper and lower bounds on the number of main eigenvalues in connected chain graphs are attainable. Moreover, we show that there exists a chain graph with k main eigenvalues for any positive integer k. We also prove that there is no connected controllable chain graph.

2010 Mathematics Subject Classifications: 05C50, 05C35

Keywords: Chain graphs, controllable graphs, main/non-main eigenvalues.

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Classifications of C^*-algebras Using Unitary Groups

Ahmed Al-Rawashdeh

In operator algebras, one of the main tools which is used in the classification is the K-theory. The group of unitaries is also a good tool. We shall review main results regarding the unitary groups of until C^*-algebras. In the case of von Neumann algebras, H. Dye proved that the discrete unitary group in a factor determines the algebraic type of the factor. Afterwards, for a large class of simple unital C^*-algebras, Al-Rawashdeh, Booth and Giordano proved that the algebras are $*$-isomorphic if and only if their unitary groups are isomomorphic as abstract groups. The isomorphism between the unitary groups induces a bijection between the projections, which is modified to be an orthoisomorphism (a bijective preserving orthogonality). Then we establish isomorphisms on the K-theory levels. Hence using main results of classifications which were deduced by Elliott-Gong-Dadarlat and Kirchberg, we obtain an isomorphism between the C^*-algebras. The simplicity of the algebra is a necessary condition, as we give a counter example in the non-simple case. Indeed, we give two C^*-algebras with isomorphic unitary groups but the algebras themselves are not $*$-isomorphic. We also give more resent results on the unitary groups and projections of until C^*-algebras.

2010 Mathematics Subject Classifications: MSC 46L05 , MSC 46L35

Keywords: C*-algebras, Classification with unitary groups

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Solving Various Types of Schlömilch's Integral Equations

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The Schlömilch's integral equation has an important place in many ionospheric problems and it is an important and useful equation in terrestrial physics. In this work, we propose a reliable method that based on the regularization and Laguerre polynomials for solving Schlömilch's integral equations and Schlömilch-type equations.

2010 Mathematics Subject Classifications: 45B05, 33C45, 47A52

Keywords: Integral equations; Orthogonal polynomials, Regularization

Introduction

The standard *linear Schlömilch's integral equation* has the following form:

$$f(x) = \frac{2}{\pi}\int_0^{\pi/2} u(x\sin\theta)\,d\theta, \quad -\pi \le x \le \pi. \tag{1}$$

It's known that this equation has a solution which admits the following form:

$$u(x) = f(0) + x\int_0^{\pi/2} f'(x\sin\theta)\,d\theta,$$

where the derivative is taken with respect to the argument $\eta = x\sin\theta$.

In addition to standard linear Schlömilch's integral equation, there are also three other forms that we focus on.

The first one is the *generalized Schlömilch's integral equation* and it admits the following form

$$f(x) = \frac{2}{\pi}\int_0^{\pi/2} u(x\sin^n\theta)\,d\theta, \quad n \ge 1.$$

The second one is the *nonlinear Schlömilch's integral equation* and it takes the form

$$f(x) = \frac{2}{\pi}\int_0^{\pi/2} F\Big(u(x\sin\theta)\Big)\,d\theta,$$

where $F\big(u(x\sin\theta)\big)$ is a nonlinear function of $u(x\sin\theta)$ and f is a continuous differential function on $-\pi \le x \le \pi$.
The third one is the *linear Schlömilch-type integral equation* which has the following form:

$$f(x) = \frac{2}{\pi}\int_0^{\pi/2} u(x\cos\theta)\,d\theta, \quad x \in \Omega,$$

where Ω is a closed and bounded domain of x [2, 4, 5, 6, 7, 8].

Main results

In this presentation, for the sake of brevity, we consider only linear Schlömilch's integral equation given in (1).
Our main tools are the regularization method proposed by Lavrentiev [3] and the Laguerre polynomials. Employing the regularization method, one gets

$$u_\alpha(x) = \frac{1}{\alpha} f(x) - \frac{2}{\alpha\pi} \int_0^{\pi/2} u_\alpha(x \sin\theta)\, d\theta \tag{2}$$

If the data function $f(x)$ and $u_\alpha(x)$ are approximated by truncated Laguerre series, with appropriate notation, a matrix equation is obtained.

$$\begin{aligned} f(x) &= \sum_{i=0}^{n} f_i L_i(x) \\ u_\alpha(x) &= \sum_{i=0}^{n} c_i L_i(x) \end{aligned} \tag{3}$$

$$(\mathbf{I} + \mathbf{A})\mathbf{C} = \mathbf{F},$$

where **I** is the identity matrix, **C** is the vector consisting of coefficients of the regularized solution in terms of truncated Laguerre polynomials, **F** is the vector formed by coefficients of the data function, and the matrix **A** is obtained by arranging terms in (2).
Solving the last equation for **C** and substituting the coefficients c_i back into the equation (3), we get the regularized solution $u_\alpha(x)$. The desired solution $u(x)$ is obtained by taking the limit of $u_\alpha(x)$ as $\alpha \to 0$.

We want to emphasize that many examples in the literature include the data function f in (1) as a polynomial function. The following theorem will be useful for the examples considered here.

Theorem 1. *[1] The data function f in (1) is a polynomial function of degree n if and only if the solution function u of (1) is a polynomial function of the same degree.*

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On the solution of Volterra integral equation with a weakly singular kernel: A special case

Ahmet Altürk

In this work, we study Volterra integral equation of the second kind with a weakly singular kernel.

$$\phi(t) = f(t) + \int_0^t K(t,x)\phi(x)\,dx,$$

where $K(t,x) = (\frac{x}{t})^{\mu}\frac{1}{x}$ for some $\mu > 0$. We consider the interesting case where $0 < \mu < 1$. Application of the homotopy perturbation method constructed by a convex homotopy or the Adomian decomposition method introduces a geometric series with a constant ratio $r = \frac{1}{\mu}$ between successive terms. This produces a non-convergent series under the assumption $0 < \mu < 1$, which is not desired. In this study, we propose an approach to overcome this issue.

2010 Mathematics Subject Classifications: 44A15, 45D05, 65H20C45

Keywords: Volterra Integral equations; Homotopy, Special kernel

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Bezier forms with dual Bernstein bases

Ayşe Yılmaz Ceylan

The aim of this talk is to construct a Bezier curve with dual Bernstein polynomials which has many applications in computer graphics and related areas. Especially, the properties of this new kind of Bezier curve are investigated. Moreover, examples are given.

2010 Mathematics Subject Classifications: 14H50, 65D17

Keywords: Bezier curve, Dual Bernstein bases

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Odd Compositions and Odd Partitions on Positive Integers

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In this paper, composition and partition definitions were made in positive integers and odd partitions and compositions were examined. Odd composition sets of odd integers and even integers are examined separately. We obtained the restricted set by constraining the smallest of the summands of its odd compositions.

2010 Mathematics Subject Classifications: 05A15, 05A17, 05A18, 11B39, 11B99

Keywords: Compositions, Partitions, Odd composition, Odd partition, Restricted composition, Restricted partition

Introduction

All people, from past to present, have been very interested in the history and partition of numbers and have done a lot of research. The first question that raised the partition theory is Leibniz's question. In 1674, Leibniz's question of how many different ways a positive integer can be written as the sum of positive integers awaked great interest and many studies have been done on this subject. Recently, many researcher have written many publication on the theory of partition of a number because the rich history of partition of a number has been gone back to not only to very famous mathematicians Leonard Euler, but also Jacobi and also Indian mathematicians S. Ramanujan and English mathematicians G. H. Hardy ([4], [6], [7], [8], [12]). For a positive integer n, the partition function to be studied is the number of ways n can be written as a sum of positive integers n. The number of these parts will be denoted by $p(n)$.

Partition and Composition

Partition are divided into compositions and partitions. The displacement of summands in commutative partitions are not important, but the displacement of summands in non commutative partitions are important.

Let n be a positive integer and we define the set

$$P_n = \left\{(a_1, a_2, ..., a_t) : \ a_1 + a_2 + ... + a_t = n, \quad a_i, t \in \mathbb{Z}^+\right\}.$$

In fact, the element of P_n is a partition of integer n and so P_n is the set of a partition of an integer n. The number of elements of the set $P(n)$ is denoted by $p(n)$ ([4]).

Let n be a positive integer and we define the set

$$P_n = \left\{(a_1, a_2, ..., a_t) : \ a_1 + a_2 + ... + a_t = n, \quad a_i, t \in \mathbb{Z}^+\right\},$$

With $a_1 = (x+y)$ and $a_2 = (y+x)$, $a_1 \neq a_2$ in this set. In fact, the element of P_n is a partition of integer n and so P_n is the set of a partition of an integer n. The number of elements of the set $P(n)$ is denoted by $|P(n)|$.

The odd combinations set of an integer

Now we focus on the combinations of an integer whose each part is odd. We will examine odd compositions with set theory. Let us use the notion

$$O_n = \{(2a_1+1, ..., 2a_t+1) : \ 2a_1+1+...+2a_t+1 = n \text{ and } \ a_i \text{ positive integer}\}$$

and we call the set as an odd combination set O_n set of an integer n.

Theorem 1. *[2] For a positive integer n, we get the decomposition of an odd combination of an integer n as a disjoint union of subset of odd combinations set of integers;*

$$O_{2n+1} = \{(2n+1)\} \cup \bigcup_{i=0}^{n-1} \left((2i+1) \odot O_{2(n-i)}\right) \tag{1}$$

$$O_{2n} = \bigcup_{i=0}^{n-1} \left((2i+1) \odot O_{2(n-i)-1}\right). \tag{2}$$

Proof. It is enough to show one inclusion for the odd number $n = 2k+1$ where k an integer. Let $x = (2a_1+1, ..., 2a_t+1)$ and assume that t is different from 1. Then $n - 2a_1 - 1 = 2m$ for an integer even and so the element $b = (2a_2+1, ...2a_t+1)$ is O_{2m}. Therefore $x = (2a_1+1) \odot O_{2n-2a_2}$ and this complete the proof. □

Corollary 2. *The number k_n of element of the odd combination set of an integer is the n.th Fibonacci number.*

Restricted Partitions

In the literature, the restricted partitions are substantial as unrestricted partition of an integer ([3], [5], [9], [10], [11]).

Theorem 3. *The generating function of the number of partitions of an integer n into odd part is*

$$\prod_{n=1}^{\infty} \frac{1}{1-x^{2n-1}} = \sum_{n=0}^{\infty} Q(n)x^n.$$

Example 4. *The set of partition of 6 is*

$$\begin{aligned} P_6 = \ & \{6; (\mathbf{5}+\mathbf{1}); (4+2); (4+1+1); (\mathbf{3}+\mathbf{3}); (3+2+1); (\mathbf{3}+\mathbf{1}+\mathbf{1}+\mathbf{1}); \\ & (2+2+2); (2+2+1+1); (2+1+1+1+1); (\mathbf{1}+\mathbf{1}+\mathbf{1}+\mathbf{1}+\mathbf{1}+\mathbf{1})\}. \end{aligned}$$

Since the partitions indicated in bold in the above set are the odd partitions of the number 6, the number 6 has 4 into odd parts. Thus $Q(6) = 4$.

From [4, page 309], The number of partitions of k into parts not exceeding m is denoted by $p_m(k)$ for integers m, k. Then $p_m(k) = p(k)$ for $m \geq k$. It is clear that $p_m(k)$ is less than $p(k)$ and the computation of $p_m(k)$ is simpler for integers m, k. The generating function for the number of partitions of k into parts not exceeding m is defined as

$$F_m(x) = \prod_{i=1}^{m} \frac{1}{1-x^i} = 1 + \sum_{i=1}^{\infty} p_m(i)x^i.$$

Example 5. *The set of partition of* 6 *is*

$$\begin{aligned} P_6 &= \{\mathbf{6}; (\mathbf{5}+\mathbf{1}); (\mathbf{4}+\mathbf{2}); (\mathbf{4}+\mathbf{1}+\mathbf{1}); (\mathbf{3}+\mathbf{3}); (\mathbf{3}+\mathbf{2}+\mathbf{1}); (3+1+1+1); \\ &\quad (\mathbf{2}+\mathbf{2}+\mathbf{2}); (2+2+1+1); (2+1+1+1+1); (1+1+1+1+1+1)\}. \end{aligned}$$

The partitions indicated in bold in the above set are the number of partitions of 6 *into parts not exceeding* 3. *Thus* $p_3(6) = 7$.

The composition of an integer n whose summands less than the fix integer $m = 2t+1$. For positive integer n,t; we use the notation the subset $O_{n,2t+1}$ of O_n for the biggest summand $2t+1$ of odd composition of n i.e.

$$O_{n,m} = \left\{(2a_1+1, ..., 2a_k+1) : 2(a_1 + ... + a_k) + k = n,\ a_i \leq m \text{ for all } i, k \in \mathbb{Z}^+.\right\}$$

When $n \leq m$, it is clear that $O_{n,m} = O_n$. And again, if $m = 2l$ by definition, $O_{n,m} = O_{n,2t}$.

We will now define a new notation to derive the recurrence relation, which gives parts of the odd compositions of the positive integer n, the largest summand of which is m.

First we define notation with the a composition $b = (2b_1+1, 2b_2+1, ..., 2b_k+1)$ of integer n, i;

$$(i \odot b) = (i, 2b_1+1, 2b_2+1, ..., 2b_k+1)\,,$$

Then $i \odot b \in O_{n+i}$ and sowe also use the notaion $i \odot O_n$ for the set of new type elements, i,e;

$$i \odot O_n = \{i \odot b : b \in O_n\}. \tag{3}$$

Lemma 6. *[2]For positive integers* t, n,

$$O_{n,m} = \bigcup_{i=0}^{t} \left((2i+1) \odot O_{n-2i-1,m}\right). \tag{4}$$

Proof. Because of the definition of $O_{n,m}$, summands of the compositions in the set must be odd positive integers. When obtaining $O_{n,m}$, the largest summand must be m. The sum of the remaining summands of the composition whose first summand is m must be $n-m$. And again, since these summands will be at largest m, the expression becomes $m \odot O_{n-m,m}$. The proof is completed. □

Example 7. *For* $n = 7$ *and* $t = 2$,

$$O_{7,5} = \bigcup_{i=0}^{2} \left((2i+1) \odot O_{6-2i,5}\right) = (1 \odot O_{6,5}) \cup (3 \odot O_{4,5}) \cup (5 \odot O_{2,5})$$

considering the

$$O_{6,5} = \{(5,1); (3,3); (3,1,1,1); (1,5); (1,3,1,1); (1,1,3,1); (1,1,1,3), (1,1,1,1,1,1)\}\,,$$

$$O_{4,5} = O_4 = \{(3,1); (1,3); (1,1,1,1)\} \text{ and } O_{2,5} = O_2 = \{(1,1)\}$$

sets results in

$$O_{7,5} = \left\{ \begin{array}{c} (5,1,1); (3,3,1); (3,1,3); (3,1,1,1,1); (1,5,1); (1,1,5); (1,3,3); \\ (1,3,1,1,1), (1,1,3,1,1); (1,1,1,3,1); (1,1,1,1,3); (1,1,1,1,1,1,1) \end{array} \right\}.$$

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Study of the arithmetic properties of Bernoulli numbers of higher orders

Chouaib Khattou

Let r be any positive integer. In this work, we revisite explicit and recurrence formulas satisfied by the Bernoulli numbers $B_n^{(r)}$ of higher order r. By using the unsigned Stirling numbers, we give them new forms and a clearer aspect. The essential part of this study, consists in generalize the arithmetic properties satisfied by the classical Bernoulli numbers to the numbers $B_n^{(r)}$. Among other things, we establish Kummer congruences for the numbers $\frac{B_n^{(r)}}{r\binom{n}{r}}$ and we construct a new family of Eisenstein series whose constant term is $\frac{(-1)^r}{2}\frac{B_k^{(r)}}{r\binom{k}{r}}$ and which verifies Kummer congruences. We give explicit formulas for the denominators of the numbers $(r-1)!B_n^{(r)}$, as well as a formula analogous to that of Clausen-von Staudt. We finish our work by constructing several multiples of the numerators of the numbers $\frac{B_n^{(r)}}{r\binom{n}{r}}$ and $(r-1)!B_n^{(r)}$.

KEYWORDS: Stirling number, Bernoulli number of higher order, Kummer congruence, p-integrality, Clausen and von Staudt formula, numerator of Bernoulli number of higher order, Eisenstein series

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Identities derived from Dirichlet convolution and regular convolutions

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In this study, the value of the convolution sums of various divisor functions is presented using Dirichlet convolution sums.

2010 Mathematics Subject Classifications: 11R27, 11D09

Keywords: Arithmetic functions, convolution sums

Introduction

It is well known that arithmetic functions are rings for addition and Dirichlet convolution sums [6], [7], [10]. It is also known that more general properties can be found using a regular regular convolution rather than a Dirichlet convolution sum.

The Dirichlet convolution sum $h_1 * h_2$ of h_1 and h_2 is defined by $h_1 * h_2(m) = \sum_{d|m} h_1(d)h_2(\frac{m}{d})$ for all m. Define the function δ by $\delta(1) = 1$ and $\delta(n) = 0$ for $n > 1$. Then $h_1 * \delta = \delta * h_1 = h_1$. In fact, the set of arithmetical functions, together with the binary operation of addition and convolution, is a commutative ring $\mathfrak{A}$. Let $h_1 \in \mathfrak{A}$. An arithmetical function h_1^{-1} is called an inverse of h_1 if $h_1 * h_1^{-1} = h_1^{-1} * h_1 = \delta$. Let $U(\mathfrak{A})$ be the set of units of $\mathfrak{A}$. In fact, $h_1 \in U(\mathfrak{A})$ if and only if $h_1(1) \neq 0$. In detail,

$$h_1^{-1}(1) = \frac{1}{h_1(1)} \tag{1}$$

and

$$h_1^{-1}(m) = -\frac{1}{h_1(1)} \sum_{\substack{d|m \\ d>1}} h_1(d)h_1^{-1}(\frac{m}{d}) \tag{2}$$

for all $m > 1$.

On the other hand, the theory of convolution sum of divisors is also very well known [11]. Identities related to various convolution sums have been found using q-seires, partition theory, Liouville identities, theta series, special functions and modular forms, etc [1], [2], [3], [4], [5], [8]. In this study, the value of the convolution sums of various divisor functions is presented using Dirichlet convolution sums.

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Some special series involving two variable Fibonacci type polynomials and combinatorial numbers

Damla Gun *[1] *and Yilmaz Simsek* [2]

The aim of this presentation is to study on some special series including the Fibonacci-type polynomials, the Humbert polynomials, and the combinatorial numbers. The motivation for this presentation is associated with the results of the paper which was given by Simsek [14]. By the aid of some families of special numbers and polynomials, we give some infinite series representations involving the combinatorial numbers and polynomials, the Fibonacci-type polynomials, and the Humbert polynomials.

2010 Mathematics Subject Classifications: 11B83, 05A15, 12D10, 11B68, 26C05

Keywords: Combinatorial numbers and polynomials, Changhee numbers, Daehee numbers and polynomials, Humbert polynomials, Generating function

Introduction

Italian mathematician Fibonacci (c. 1170 – c. 1240–50), who is also known as Leonardo Bonacci, Leonardo of Pisa, or Leonardo Bigollo Pisano), was the famous mathematician of the Middle Ages. He discovered new sequences of numbers one of which is today commonly called Fibonacci numbers.

These numbers have also represented by the generating functions.

After that, these generating functions were revised in order to define Fibonacci polynomials. Recently two variables, and also multivariable variable Fibonacci polynomials have been defined.

The aim of this presentation,using two variable Fibonacci polynomials and other special combinatorial numbers, we give exact values of some infinite series representations.

In this section, we give some definitions and notations to be used in this talk. Let $\mathbb{N}$ denotes the set of natural numbers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Throughout of this talk, we assume that λ is a real or complex numbers with $\lambda \neq 0, 1$.

In [13], Simsek defined the following combinatorial numbers $Y_n(\lambda)$ and polynomials $Y_n(\lambda, x)$, respectively:

$$\frac{2}{\lambda^2 t + \lambda - 1} = \sum_{n=0}^{\infty} Y_n(\lambda) \frac{t^n}{n!} \tag{1}$$

and

$$\frac{2(1+\lambda t)^x}{\lambda^2 t + \lambda - 1} = \sum_{n=0}^{\infty} Y_n(x;\lambda) \frac{t^n}{n!}.$$

By using (1), we have

$$Y_n(\lambda) = 2(-1)^n \frac{n!}{\lambda - 1}\left(\frac{\lambda^2}{\lambda - 1}\right)^n \tag{2}$$

(*cf.* [13]; see also [2]-[4] [15]).

The Changhee numbers Ch_n are defined by

$$F_{Ch}(t) = \frac{2}{t+2} = \sum_{n=0}^{\infty} Ch_n \frac{t^n}{n!} \tag{3}$$

and

$$Ch_n = \frac{(-1)^n n!}{2^n} \tag{4}$$

(*cf.* [7]).

The numbers $\mathfrak{Ch}_n(\lambda)$ is given by the following explicit fomula:

$$\mathfrak{Ch}_n(\lambda) = \frac{2(-1)^n}{\lambda + 1}\left(\frac{\lambda^2}{\lambda + 1}\right)^n n!. \tag{5}$$

(*cf.* [13]). We note that the generalization of the numbers $\mathfrak{Ch}_n(\lambda)$ is called the generalized Apostol-Changhee numbers.

The Daehee numbers D_n are defined by

$$F_D(t) = \frac{\log(1+t)}{t} = \sum_{n=0}^{\infty} D_n \frac{t^n}{n!} \tag{6}$$

and

$$D_n = \frac{(-1)^n n!}{n+1} \tag{7}$$

(*cf.* [1], [6]).

The Humbert polynomials $\Pi_{n,m}^{(\alpha)}(x)$ are defined by

$$\frac{1}{(1 - mxt + t^m)^{\alpha}} = \sum_{n=0}^{\infty} \Pi_{n,m}^{(\alpha)}(x)\, t^n \tag{8}$$

(*cf.* [5]).

The Fibonacci-type polynomials in two variables $\mathcal{G}_n(x, y, k, m, j)$ are defined by the following generating function:

$$\frac{1}{1 - x^k t - y^m t^{m+j}} = \sum_{n=0}^{\infty} \mathcal{G}_n(x, y, k, m, j)\, t^n, \tag{9}$$

where $k, m, j \in \mathbb{N}_0$ (*cf.* [9]).

The series product rule is given as follows

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} A(n, m) = \sum_{m=0}^{\infty}\sum_{n=0}^{\left[\frac{m}{2}\right]} A(n, m - 2n), \tag{10}$$

where $[x]$ denotes the greatest integer function (*cf.* [10, p. 57, Lemma 11, Eq-(7)]).

Main results

In this section, we give some infinite series representations including the combinatorial numbers and polynomials, the Changhee numbers, the Daehee numbers and polynomials, and the Humbert polynomials with the help of some algebraic operations methods in infinite series.

Theorem 1. *We assume that* $\left|\frac{\lambda^2}{\lambda-1}\right| < 1$. *Then we have*

$$\sum_{n=0}^{\infty} \frac{Y_n(\lambda)\,\mathfrak{Ch}_n(\lambda)}{(n!)^2} = -4\sum_{n=0}^{\infty} \Pi_{n,2}\left(\frac{1}{2}\right)\lambda^{2n}.$$

Proof. By using (2) and (5), we get

$$\sum_{n=0}^{\infty} \frac{Y_n(\lambda)\,\mathfrak{Ch}_n(\lambda)}{(n!)^2} = 4\sum_{n=0}^{\infty} (-1)^{n+1} \frac{\lambda^{4n}}{(1-\lambda^2)^{n+1}}. \tag{11}$$

By using geometric series in the above equation, we have

$$\sum_{n=0}^{\infty} \frac{Y_n(\lambda)\,\mathfrak{Ch}_n(\lambda)}{(n!)^2} = \frac{-4}{1-\lambda^2+\lambda^4}.$$

On the other hand substituting $x = \frac{1}{2}, \alpha = 1, m = 2$ and $t = \lambda^2$ into (8), then combining above equation, we obtain desired result. □

Corollary 2. *We assume that* $\left|\frac{\lambda^4}{\lambda^2-1}\right| < 1$. *Then we have*

$$\sum_{n=0}^{\left[\frac{m}{2}\right]} (-1)^n \binom{m-n}{m-2n} = \Pi_{m,2}\left(\frac{1}{2}\right) = \mathcal{G}_m(1,-1,1,1,1).$$

Proof. Combining (11) with the following binomial series,

$$\frac{1}{(1-\lambda^2)^{n+1}} = \sum_{m=0}^{\infty} \binom{m+n}{m} \lambda^{2m} \tag{12}$$

we have

$$\sum_{n=0}^{\infty} \frac{Y_n(\lambda)\,\mathfrak{Ch}_n(\lambda)}{(n!)^2} = 4\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} (-1)^{n+1} \binom{m+n}{m} \lambda^{2m+4n}.$$

Using the above equation and (10), we obtain

$$4\sum_{m=0}^{\infty}\sum_{n=0}^{\left[\frac{m}{2}\right]} (-1)^{n+1} \binom{m-n}{m-2n} \lambda^{2m} = -4\sum_{m=0}^{\infty} \Pi_{m,2}\left(\frac{1}{2}\right)\lambda^{2m}.$$

Comparing the coefficients of λ^{2m} on both sides of the above equation, we get the desired result. □

Corollary 3. *We assume that* $\left|\frac{\lambda^2}{\lambda-1}\right| < 1$. *Then we have*

$$\sum_{n=0}^{\left[\frac{m}{2}\right]} (-2)^n \binom{m-n}{m-2n} = \Pi_{m,2}\left(\frac{1}{2\sqrt{2}}\right)\left(\sqrt{2}\right)^m = \mathcal{G}_m(1,-2,0,1,1).$$

Proof. By using (2), (4) and (12), we get

$$\sum_{n=0}^{\infty}\frac{Y_n\left(\lambda\right)}{Ch_n}=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\left(-2\right)^{n+1}\binom{m+n}{m}\lambda^{2n+m}.$$

By applying (10) in the above equation, we have

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\left[\frac{m}{2}\right]}\left(-2\right)^{n+1}\binom{m-n}{m-2n}\lambda^{m}=-2\sum_{m=0}^{\infty}\Pi_{m,2}\left(\frac{1}{2\sqrt{2}}\right)\left(\sqrt{2}\lambda\right)^{m}.$$

Comparing the coefficients of λ^m on both sides of the above equation, we get the desired result. □

Corollary 4.

$$\mathcal{G}_n\left(1,-1,1,1,1\right)=\Pi_{m,2}\left(\frac{1}{2}\right)=\sum_{n=0}^{\left[\frac{m}{2}\right]}\left(-1\right)^{n}\binom{m-n}{m-2n}.$$

Proof. By using (2), (7) and (12), we get

$$\sum_{n=0}^{\infty}\frac{Y_n\left(\lambda\right)\left(n+1\right)D_n}{\left(n!\right)^2}=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\left(-1\right)^{n+1}2\binom{m+n}{m}\lambda^{2n+m}.$$

Using the above equation and (10) we have

$$-2\sum_{n=0}^{\infty}\Pi_{m,2}\left(\frac{1}{2}\right)\lambda^{m}=\sum_{m=0}^{\infty}\sum_{n=0}^{\left[\frac{m}{2}\right]}\left(-1\right)^{n+1}2\binom{m-n}{m-2n}\lambda^{m}.$$

Comparing the coefficients of λ^m on both sides of the above equation, we arrive at the desired result. □

By using (4), (7) and geometric series, we get the following corollary:

Corollary 5.

$$\sum_{n=0}^{\infty}\frac{\left(n+1\right)Ch_nD_n}{\left(n!\right)^2}=2.$$

Theorem 6. *We assume that* $\left|\frac{\lambda^2}{\lambda+1}\right|<1$. *Then we have*

$$\sum_{n=0}^{\infty}\frac{\mathfrak{Ch}_n(\lambda)}{\left(n+1\right)D_n}=2\sum_{n=0}^{\infty}\Pi_{n,2}\left(\frac{i}{2}\right)\left(i\lambda\right)^{n}=2\sum_{n=0}^{\infty}\mathcal{G}_n\left(-1,1,1,1,1\right)\lambda^{n}.$$

Proof. By using (5) and (7), we get

$$\sum_{n=0}^{\infty}\frac{\mathfrak{Ch}_n(\lambda)}{\left(n+1\right)D_n}=\sum_{n=0}^{\infty}\frac{2\lambda^{n}}{\left(\lambda+1\right)^{n+1}}. \tag{13}$$

Combining (13) with (8) and (9), we arrive at the Theorem 6. □

Corollary 7.

$$\mathcal{G}_m\left(-1,1,1,1,1\right)=\Pi_{m,2}\left(\frac{i}{2}\right)i^{m}=\sum_{n=0}^{\left[\frac{m}{2}\right]}\left(-1\right)^{m}\binom{m-n}{m-2n}.$$

Proof. Combining (13) with (12), we have

$$\sum_{n=0}^{\infty} \frac{\mathfrak{Ch}_n(\lambda)}{(n+1)\,D_n} = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} (-1)^m\, 2\binom{m+n}{m}\lambda^{2n+m}.$$

By applying (10) in the above equation, we have

$$\sum_{n=0}^{\infty} \frac{\mathfrak{Ch}_n(\lambda)}{(n+1)\,D_n} = 2\sum_{m=0}^{\infty}\sum_{n=0}^{\left[\frac{m}{2}\right]} (-1)^{m-2n}\binom{m-n}{m-2n}\lambda^m.$$

Combining the above equation with the Theorem 6, then comparing the coefficients of λ^m on both sides of the above equation, we get the desired result. □

Theorem 8.

$$\sum_{n=0}^{\infty} \frac{(n+1)\,\mathfrak{Ch}_n\ (\lambda) D_n}{(n!)^2} = 2\sum_{n=0}^{\infty} \Pi_{n,2}\left(\frac{i}{2}\right)(i\lambda)^n = 2\sum_{n=0}^{\infty} \mathcal{G}_n\left(-1,1,1,1,1\right)\lambda^n.$$

Proof. By using (5), (7) and geometric series, we get

$$\sum_{n=0}^{\infty} \frac{(n+1)\,\mathfrak{Ch}_n(\lambda) D_n}{(n!)^2} = \frac{2}{1+\lambda-\lambda^2}.$$

Substituting $x = \frac{i}{2}, \alpha = 1, m = 2$ and $t = i\lambda$ into (8), then combining above equation, we arrive at the desired result. □

Theorem 9. *We assume that* $\left|\frac{\lambda^2}{\lambda-1}\right| < 1$. *Then we have*

$$\sum_{n=0}^{\infty} \frac{Y_n\left(\lambda\right) Ch_n}{(n!)^2} = -2\sum_{n=0}^{\infty} \mathcal{G}_n\left(-1,-\frac{1}{2},0,1,1\right)\lambda^n$$

Proof. By using (2), (4) and geometric series, we get

$$\sum_{n=0}^{\infty} \frac{Y_n\left(\lambda\right) Ch_n}{(n!)^2} = \frac{-2}{1-\lambda+\frac{\lambda^2}{2}}$$

Substituting $x = -1, y = -\frac{1}{2}, k = 0, m = 1$ and $j = 1$ into (9), then combining above equation, we arrive at the desired result. □

Remark 1. *By using* (2) *and* (4), *we get*

$$\sum_{n=0}^{\infty} \frac{\mathfrak{Ch}_n(\lambda)}{Y_n\left(\lambda\right)} = \frac{\lambda-1}{2}.$$

Conclusion

By using Fibonacci-type polynomials, the Humbert polynomials, and some certain families of the combinatorial numbers, some infinite series representations were given. By blending the results of this presentation with other infinite series, it is planned to give formulas on the convergence of infinite series containing new definite number families in the near future.

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A Korovkin type approximation for (p,q)-Bernstein operators via power series statistical convergence

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In this paper, we investigate some Korovkin type approximation properties of the (p,q)-Bernstein operators via power series statistical convergence. We also compute the rate of convergence by modulus of continuity.

2010 Mathematics Subject Classifications: 40A35, 40G10, 41A36

Keywords: (p,q)- Bernstein operators, power series statistical convergence, rate of convergence

Introduction

Recently, post-quantum ((p,q))- calculus has been studied intensively. Since great variety applications to mathematics and other fields, researchers have been encouraged to define further concepts of (p,q)- calculus and to study it in different theories (see; [7]). In approximation theory, (p,q) -calculus have been used by defining a new sequence of linear and positive operators in [4], firstly.

Now, let us recall some notations from (p,q)- calculus([4], [7]):
For any fixed real number $0<q<p\leq 1$, the (p,q)-integer $[j]$ is defined by $[j]_{p,q}=\frac{p^j-q^j}{p-q}$, $[j]_{p,q}=1$, if $j=0$ where j is a positive integer. The (p,q)- factorial $[j]!$ of $[j]$ is given with

$$[j]!:=\begin{cases}\prod_{k=1}^{j}[k] & , \quad j=1,2,... \\ 1 & , \quad j=0.\end{cases}.$$

For integers $j\geq r\geq 0$, the (p,q)- binomial coefficient is defined by

$$\left[\begin{array}{c} j \\ r \end{array}\right]_{p,q}=\frac{[j]_{p,q}!}{[r]_{p,q}!\,[j-r]_{p,q}!}$$

In [4], Mursaleen et.al. constructed generalized Bernstein operators via (p,q)- calculus as follows

$$B_j^{p,q}(f;x)=\frac{1}{p^{\frac{j(j-1)}{2}}}\sum_{k=0}^{j}p^{\frac{k(k-1)}{2}}f\left(\frac{[k]}{[j]\,p^{k-j}}\right)\left[\begin{array}{c} j \\ r \end{array}\right]_{p,q}x^k\prod_{m=0}^{j-k-1}\left(p^m-q^m x\right), \qquad (1)$$

for $f\in C[0,1]$, $x\in[0,1]$, $j\in\mathbb{N}$ and $0<q<p\leq 1$. Uniform convergence results of these operators and some generalizations of them have been proved under the following conditions

$$\lim_{j\to\infty}q_j=1 \text{ and } \lim_{j\to\infty}p_j=1, \lim q_j^j=a, \ \lim p_j^j=b \ (0<q<p\leq 1) \qquad (2)$$

where $0 \leq a < b < 1$ hold ([5], [6], [8])). Furthermore, some approximation results of the operators constructed (p, q)- calculus have been obtained by using some summability methods ([2], [3]).

In the present paper, we give power series statistical convergence properties of the operators (1) and calculate rate of the convergence.
Now, let us define power series statistical convergence which is introduced in [10]. It is defined with the help of the concept of density with respect to the power series methods.

Definition 1. *[10] Let P be a regular power series method (see;[1]) and let $E \subset \mathbb{N}_0$. If*

$$\delta_P(E) := \lim_{0<t\to R^-} \frac{1}{s(t)} \sum_{j\in E}^{\infty} s_j t^j$$

exists then $\delta_P(E)$ is called the P-density of E.

Definition 2. *[10] Let $x = (x_j)$ be a sequence and let P_p be a regular power series method. Then x is said to be P-statistically convergent to L if for any $\varepsilon > 0$*

$$\lim_{0<t\to R^-} \frac{1}{s(t)} \sum_{|x_j - L| \geq \varepsilon} s_j t^j = 0,$$

i.e., $\delta_P(\{j \in \mathbb{N}_0 : |x_j - L| \geq \varepsilon\}) = 0$. In this case we write $st_P - \lim x = L$.

Main results

In this section, using power series statistical convergence, we obtain Korovkin type approximation of the operators $\left(B_j^{p,q}\right)$ defined with (1). Throughout this section, we deal with the sequence (q_j), (p_j) such that $0 < q_j < p_j \leq 1$ satisfies

$$st_p\text{-}\lim_{j\to\infty} p_j = 1, st_p\text{-}\lim_{j\to\infty} q_j = 1, st_p\text{-}\lim_{j\to\infty} q_j^j = a, st_p\text{-}\lim_{j\to\infty} p_j^j = b \tag{3}$$

where $0 \leq a < b < 1$, $q_0 = 0$, $p_0 = 1$ and we assume that $B_0^q f = 0$ for any $f \in C[0,1]$. Furthermore, we use the norm of the Banach space $B[0,1]$ defined for any $f \in C[0,1]$ by $\|f\| := \sup_{0\leq x\leq 1} |f(x)|$ where $B[0,1]$ is the space of all bounded real functions defined over $[0,1]$.

We need the following known lemmas in our proofs:

Lemma 3. *[4] The following hold for the operators (1) :*

$$\begin{aligned} B_j^{p,q}(e_0; x) &= 1, \\ B_j^{p,q}(e_1; x) &= x, \\ B_j^{p,q}(e_2; x) &= \frac{p^{j-1}}{[j]} x + \frac{q[j-1]}{[j]} x^2 \end{aligned} \tag{4}$$

where $e_i(x) = x^i$ for $i = 0, 1, 2$.

Now, we recall the following Korovkin type P-statistical approximation theorem which is given in [10].

Theorem 4. *Let P be regular power series method and let (L_j) be a sequence of linear positive operators on $C[0,1]$ such that for $i = 0, 1, 2$*

$$st_p\text{-}\lim \|L_j(e_i) - e_i\| = 0, \tag{5}$$

then for any $f \in C[0,1]$ we have

$$st_p\text{-}\lim \|L_j f - f\| = 0.$$

We are ready to prove the following Korovkin type P-statistical approximation theorem:

Theorem 5. *Assume that P is a regular power series method. If (q_j), (p_j) satisfies the condition (3), then for each $f \in C[0,1]$ we have*

$$st_P\text{-}\lim_{j\to\infty} \left\|B_j^{p,q}(f) - f\right\| = 0.$$

Proof. From Theorem 4, it is enough to demonstrate that (5) holds for $(B_j^{p,q})$. Now, considering Lemma 3, we get for $i = 0,1$ that

$$st_P\text{-}\lim_{j\to\infty} \left\|B_j^{p,q}(e_i) - e_i\right\| = 0.$$

Moreover, using (4), we have that

$$\begin{aligned} 0 \le \left|B_j^{p,q}(e_2;x) - x^2\right| &= \frac{p^{j-1}}{[j]}x + \frac{q[j-1]}{[j]}x^2 - x^2 \\ &= \frac{p^{j-1}}{[j]}x + \frac{q[j-1]-[j]}{[j]}x^2 \end{aligned}$$

Here, let us define the following sets for any $\varepsilon > 0$,

$$N := \left\{j \in \mathbb{N} : \left|B_j^{p,q}(e_2;x) - x^2\right| \ge \varepsilon\right\}$$

$$N_1 := \left\{j \in \mathbb{N} : \frac{p^{j-1}}{[j]} \ge \varepsilon\right\}, N_2 := \left\{j \in \mathbb{N} : \left|\frac{q[j-1]-[j]}{[j]}\right| \ge \varepsilon\right\}.$$

It is obvious that $N \subset N_1 \cup N_2$ which implies with the hypothesis that

$$\begin{aligned} 0 &\le \delta_p\left(\left\{j \in \mathbb{N} : \left\|B_j^{p,q}e_2 - e_2\right\|_{C_B} \ge \varepsilon\right\}\right) \\ &\le \delta_p\left(\left\{j \in \mathbb{N} : \frac{p^{j-1}}{[j]} \ge \varepsilon\right\}\right) + \delta_p\left\{j \in \mathbb{N} : \left|\frac{q[j-1]-[j]}{[j]}\right| \ge \varepsilon\right\} = 0. \end{aligned}$$

Hence, we obtain

$$st_P\text{-}\lim_{j\to\infty} \left\|B_j^{p,q}(e_2) - e_2\right\|_{C_B} = 0.$$

So, the proof is completed. □

The following remark shows that the conditions of Theorem 5 are weaker than the classical conditions (2):

Remark 2. *We remark that if we take classic conditions (2), Theorem 5 works. Conversely, we assume that P is the power series method with (s_j)*

$$s_j := \begin{cases} 0 & , \quad j = 2k \\ 1 & , \quad j = 2k+1 \end{cases}$$

and we consider the sequence $(p_j), (q_j)$

$$q_j := \begin{cases} 0 & , \quad j = 2k \\ 1 - \frac{2}{j} & , \quad j = 2k+1 \end{cases}, p_j := \begin{cases} 0 & , \quad j = 2k \\ 1 - \frac{1}{j} & , \quad j = 2k+1 \end{cases}.$$

Note that if we take (q_j) and (p_j), then approximation theorem given in [4] doesn't work.

In order to compute rate of the convergence we recall the modulus of continuity of $\omega(f,\delta)$ is defined by

$$\omega(f,\delta)=\sup_{\substack{|x-y|\leq\delta\\ x,y\in[0,1]}}|f(x)-f(y)|.$$

From the Theorem 2 in [9], we obtain the rate of P-statistical convergence by means of modulus of continuity for the operators $\left(B_j^{p,q}\right)$ is proved in the following:

Theorem 6. *If (q_j), (p_j) satisfies the condition (3), then for any $f\in C[0,1]$ we have*

$$\left\|B_j^{p,q}(f)-f\right\|\leq 2\omega(f,\alpha_j),$$

where $\alpha_j=\left\{\sup_{0\leq x\leq 1}B_j^{p,q}(t-x)^2\right\}^{\frac{1}{2}}=\left(\frac{p^{j-1}}{2[j]}\right)^{\frac{1}{2}}$.

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Lagrangian formulation and exact solutions for potential Kadomtsev-Petviashvili equation with $p-$power nonlinearity

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This paper aims to study the potential Kadomtsev-Petviashvili equation with $p-$power nonlinearity (PKPp), which arises in a number of significant nonlinear problems of physics and applied mathematics. We carry out Noether symmetry classification on PKPp equation. Four cases arise depending on the values of p and consequently we construct conservation laws for these cases with respect to the second-order Lagrangian. In addition, exact solutions for the PKPp equation are obtained using the Lie group analysis together with the Kudryashov method and direct integration.

2010 Mathematics Subject Classifications: 34C14, MSC 35Q53

Keywords: potential Kadomtsev-Petviashvili equation with $p-$power nonlinearity, Lie group analysis, Conservation laws, Kudryashov method

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Effects of yield stress on the hydrodynamic stability of parallel shear flows of Herschel-Bulkley fluid

Dip Mukherjee *[1] *and Bikash Sahoo* [2]

The purpose of the research is to examine the linear stability of Herschel-Bulkley fluid, a non-Newtonian fluid, in parallel channel flow. The governing modified Orr-Sommerfeld equation for a non-Newtonian fluid has been developed and quantitatively solved numerically using the Chebyshev spectral collocation method to compute the spectrum of eigenvalues. Also, a comparison between the velocity profiles of the non-Newtonian fluid with the Newtonian case has been shown. The spectrum of the non-Newtonian fluid calculated by solving the modified Orr-Sommerfeld equation reveals that the value of the critical Reynolds number for the instability onset increases with the yield stress value. So, the onset point of instability in the flow becomes higher with respect to large value of yield stress. Additionally, the findings demonstrate that when yield stress increases, fluid viscosity also rises. These results will be used as a benchmark for our future studies on Herschel-Bulkley fluid.

2010 Mathematics Subject Classifications: 76D05, 76E05

Keywords: Yield stress, Non-Newtonian fluid, Hydrodynamic stability, Herschel-Bulkley fluid, Chebyshev spectral collocation method, Spectrum

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Application of the method of compositae to obtain an explicit formula for a certain generating function in three variables

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In this paper, we consider the problem of obtaining explicit formulas for the coefficients of multivariate generating functions. To do this, it is proposed to use the method of compositae that is based on the properties of the powers of generating functions. As an example, we have applied this method to one generating function in three variables and have obtained an explicit formula for its coefficients.

2010 Mathematics Subject Classifications: 05A15, 11B83

Keywords: Multivariate generating function, Aztec diamond, Explicit formula, Composita

Introduction

Generating functions are a widely used and powerful tool for solving problems in combinatorics, mathematical analysis, statistics, etc. Generating functions allow obtaining a compact representation of various numerical sequences, as well as for special numbers and polynomials [6, 7]. In addition, the study of generating functions allows to obtain expressions for their coefficients.

To solve the task of obtaining expressions for the coefficients of generating functions, we propose to use the method of compositae [2, 3, 4], that is based on the properties of the powers of generating functions. The effectiveness of applying this method is shown in obtaining explicit formulas for many generating functions. In this paper, we consider the case of working with a generating function in three variables. We also present an example of applying the method of compositae to obtain an explicit formula for a certain generating function in three variables.

Main results

The composita $F^{\Delta}(n,m,l,k)$ of a generating function in three variables

$$F(x,y,z)=\sum_{n\geq 0}\sum_{m\geq 0}\sum_{l\geq 0} f(n,m,l)x^n y^m z^l,$$

where $ord(F(x,y,z))=\min\{n+m+l : f(n,m,l)\neq 0\}\geq 1$, is a coefficients function of the k-th power of $F(x,y,z)$:

$$F(x,y,z)^k=\sum_{n\geq 0}\sum_{m\geq 0}\sum_{l\geq 0} F^{\Delta}(n,m,l,k)\, x^n y^m z^l, \quad F(x,y,\ldots,z)^0=1.$$

For generating functions $A(x,y,z)$, $B(x,y,z)$, $C(x,y,z)$ and $H(x,y,z)$ with their compositae $A^{\Delta}(n,m,l,k)$, $B^{\Delta}(n,m,l,k)$, $C^{\Delta}(n,m,l,k)$ and $H^{\Delta}(n,m,l,k)$, we can find the composita $G^{\Delta}(n,m,l,k)$ of the generating function $G(x,y,z)$ for the following cases:

- addition of generating functions:
$$G(x,y,z) = A(x,y,z) + B(x,y,z);$$
- multiplication of generating functions:
$$G(x,y,z) = A(x,y,z) \cdot B(x,y,z);$$
- composition of generating functions:
$$G(x,y,z) = H(A(x,y,z), B(x,y,z), C(x,y,z));$$
- reciprocation of generating functions:
$$G(x,y,z) \cdot H(x,y,z) = 1;$$
- compositional inversion of generating functions:
$$H(G(x,y,z),y,z) = x, \quad H(x,G(x,y,z),z) = y,$$
$$H(x,y,G(x,y,z)) = z.$$

In addition, having an explicit formula for the composita $G^{\Delta}(n,m,l,k)$ of the generating function $G(x,y,z)$, it is easy to obtain an explicit formula for the coefficients $g(n,m,l)$ of $G(x,y,z)$ by using $g(n,m,l) = G^{\Delta}(n,m,l,1)$.

Hence, the method of compositae allows obtaining explicit formulas for the coefficients of compositions of generating functions. Next, we consider one example of obtaining an explicit formula for a certain generating function in three variables.

Let consider the following generating function in three variables:

$$G(x,y,z) = \frac{\frac{z}{2}}{(1-yz)(1-(x+\frac{1}{x}+y+\frac{1}{y})\frac{z}{2}+z^2)} = \tag{1}$$

$$= \sum_{n>0} \sum_{i=-n}^{n} \sum_{j=-n}^{n} \rho(i,j,n)\, x^i y^j z^n,$$

where $\rho(i,j,n)$ is the north-going edge probability for the cell centered at (i,j) in an Aztec diamond of order n [1, 5].

Having decomposed the generating function (1) and applied the method of compositae, we have obtained a new explicit formula for the values of $\rho(i,j,n)$. The obtained result is shown in the following theorem:

Theorem 1. *The north-going edge probability for the cell centered at (i,j) in an Aztec diamond of order n can be calculated by using the following formula:*

$$\rho(i,j,n) = \frac{1}{2} \sum_{n_i=1}^{n+i} r(n+i-n_i, n+i-2n_i+1, n-n_i),$$

where

$$r(n,m,l) = \sum_{k=0}^{n+m+l} \binom{k}{l-k} \binom{2k-l}{\frac{n-m-l+2k}{2}} \binom{2k-l}{\frac{n+m-3l+2k}{2}} \frac{(-1)^{n+m+k}+(-1)^{l+k}}{2^{2k-l+1}}.$$

Conclusion

The method of compositae is an effective way to obtain explicit formulas for the coefficients of generating functions, including generating functions in several variables. For example, using this method, a new explicit formula for calculating the north-going edge probability for the cell centered at (i, j) in an Aztec diamond of order n has been obtained. The main requirement for applying this method is the need to decompose the original expression of a generating function into simpler functions with known expressions for their compositae and their coefficients.

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New generalization of the Hardy type inequality

Dora Pokaz

In this talk we present how to extend Hardy type inequalities to convex functions of the higher order. The Abel-Gontscharoff interpolation for two points and the remainder in the integral form allow us to refine the Hardy type inequality. Using the Chebyshev functional, we discuss the upper bounds for obtained inequalities and get some examples involving Green function.

2010 Mathematics Subject Classifications: 26D10, 26D15

Keywords: inequalities, convex function, kernel, upper bounds, Hardy type inequality, AbelGontscharoff interpolating polynomial, Green function, Chebyshev functional

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Twice iterated Big q-Appell polynomials and their properties

Duygu Malyalı [1] *and Mehmet Ali Özarslan* [2]

In the present paper, we introduce a family of twice iterated big q-Appell polynomials and obtain their pure recurence relation, lowering, rasing operators and determinantal representation. Some equivalence theorems for the definition are also obtained. Main results can be applied to wide range of polynomials families such as big q-Bernoulli polynomials and big q-Euler polynomials.

2010 Mathematics Subject Classifications: 11B68, 33C05

Keywords: The big q-Appell polynomials, Twice iterated big q-Appell polynomials

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Bright Solitary Shear Horizontal Waves in a Heterogeneous Elastic Layer over Heterogeneous Semi-Space

Ekin Deliktas Ozdemir

In this study, propagation of bright solitary shear horizontal (SH) waves in a nonlinear, elastic, heterogeneous semi-space covered by a heterogeneous layer of uniform thickness is studied. It is assumed that linear and nonlinear material properties change as an exponential function of the depth in the media where both the layer and the semi-space are taken to be vertically heterogeneous. The self-interaction of weakly nonlinear SH waves is described by a nonlinear Schrdinger equation. The effect of heterogeneous nonlinear material properties of the media on the propagation of bright solitary waves is investigated numerically.

2010 Mathematics Subject Classifications: 35Q55, 35B20, 74B20, 74E05, 74J35

Keywords: Heterogeneous layered semi-space, solitary SH waves, nonlinear elasticity

Introduction

Effects of heterogeneity properties of the media consisting of layered semi-space with different material characteristics on the elastic waves, specially Love waves, are investigated by various researchers due to extensive application areas such as seismology, metallurgy, materials science [1, 2]. Since rigidity and density of the Earth's crust and mantle are considered as the function of depth variable, different types of vertically heterogeneity of the media are studied in detail in [2]. Recently, considering the nonlinear material characteristics as well as heterogeneity of the media, studies on the bright solitary shear horizontal wave propagation in a plate [3] and in a double-layered plate [4] have been carried out.

In this study, propagation of bright solitary Love waves in a heterogeneous, nonlinear, elastic layered half-space is investigated. It is assumed that the rigidities and densities of both the layer and half-space vary exponentially in the direction vertical to the wave propagation. A nonlinear Schrdinger equation which asymptotically governs nonlinear self modulation of waves is derived by a perturbation method. Effects of nonlinear material constants and also nonlinear heterogeneity parameters of both the layer and half-space on the presence of bright solitary waves are examined numerically.

Formulation

Let (x, y, z) and (X, Y, Z) represent the spatial and material coordinates of a point in three-dimensional space with respect to the same perpendicular Cartesian coordinate system, respectively. Heterogeneous and nonlinear medium consisting of a layer overlying a half space that occupy the following regions in the reference frame is considered;

$$\mathcal{P}_1 = \{(X,Y,Z) \mid 0 \le Y \le h, -\infty < X < \infty, -\infty < Z < \infty\}, \tag{1}$$

$$\mathcal{P}_2 = \{(X,Y,Z) \mid -\infty \le Y \le 0, -\infty < X < \infty, -\infty < Z < \infty\} \tag{2}$$

in which h stands for the layer's constant thickness. It is assumed that the traction on the free surface $Y = h$ vanishes, the displacements and stresses are continuous along the interface $Y = 0$, and the displacement in half-space satisfies the radiation condition.

The shear waves are described by

$$x = X, \qquad y = Y, \qquad z = Z + u^{(\alpha)}(X,Y,t), \quad \alpha = 1,2 \tag{3}$$

where t is the time, and $u^{(\alpha)}$ represents the particle's displacement in $\mathcal{P}_\alpha$ in the Z-direction. It is assumed that layered half space consists of incompressible, heterogeneous, isotropic and different elastic materials. The heterogeneity properties of the media are represented by the exponential functions of depth variable. Furthermore, it is assumed that strain energy functions depend only on the first invariant of the Green's deformation tensor and depth variable. For such a media, approximate equations of motion can be written as

$$\frac{\partial^2 u^{(\alpha)}}{\partial t^2} - c_\alpha^2 \left(\frac{\partial^2 u^{(\alpha)}}{\partial X^2} + \frac{\partial^2 u^{(\alpha)}}{\partial Y^2} \right) - \frac{1}{\rho_\alpha} \left(\frac{d\left(\rho_\alpha c_\alpha^2\right)}{dY} \frac{\partial u^{(\alpha)}}{\partial Y} \right) =$$

$$n_\alpha \left[\frac{\partial}{\partial X} \left(\frac{\partial u^{(\alpha)}}{\partial X} Q(u^{(\alpha)}) \right) + \frac{\partial}{\partial Y} \left(\frac{\partial u^{(\alpha)}}{\partial Y} Q(u^{(\alpha)}) \right) \right] + \frac{Q(u^{(\alpha)})}{\rho_\alpha} \left(\frac{d\left(\rho_\alpha n_\alpha\right)}{dY} \frac{\partial u^{(\alpha)}}{\partial Y} \right) \text{ in } \mathcal{P}_\alpha,$$

where $Q(\psi) = \left(\frac{\partial \psi}{\partial X}\right)^2 + \left(\frac{\partial \psi}{\partial Y}\right)^2$, the subscripts α refer the regions, c_α are linear shear velocities such that $c_\alpha^2 = \frac{\mu_\alpha}{\rho_\alpha}$ where $\mu_\alpha = 2\frac{d\Sigma_\alpha(3,Y)}{dI}$ are linear shear modulus. ρ_α represent densities and $n_\alpha = 2\frac{d^2\Sigma_\alpha(3,Y)}{\rho_\alpha dI^2}$ are nonlinear material functions.

Modulation of nonlinear waves

Using the method of multiple scales with the following new independent variables, we examine the propagation of small but finite amplitude Love waves

$$x_i = \epsilon^i X, \quad t_i = \epsilon^i t, \quad y = Y, \quad i = 0,1,2 \tag{4}$$

Then, u^α is expanded in the following asymptotic series

$$u^\alpha = \sum_{n=1}^{\infty} \varepsilon^n u_n^\alpha(x_0, x_1, x_2, y, t_0, t_1, t_2) \tag{5}$$

Using this expansion in the equations of motion gives a hierarchy of problems from which u_n^α can be found, successively. In this study, linear shear modulus and nonlinear material functions are taken to be exponential functions of y as in [1];

$$\mu_\alpha(y) = \mu_{0\alpha} e^{\beta_\alpha y}, \quad \rho_\alpha(y) = \rho_{0\alpha} e^{\beta_\alpha y}, \quad n_\alpha(y) = n_{0\alpha} e^{\lambda_\alpha y}, \quad \alpha = 1,2 \tag{6}$$

where $\mu_{0\alpha}$, $\rho_{0\alpha}$ and $n_{0\alpha}$ are constants and β_α and λ_α represent linear and nonlinear heterogeneity parameters, respectively. Under this choice nonlinear material functions, θ_α, are obtained as

$$\theta_\alpha = \theta_{0\alpha} e^{\lambda_\alpha y}, \quad \text{where} \quad \theta_{0\alpha} = \frac{n_{0\alpha}}{c_\alpha^2}, \quad \alpha = 1, 2. \tag{7}$$

Hence, displacement functions in the first order problem can be obtained to be

$$\begin{aligned} u_1^{(1)} &= \frac{\mathcal{A}_1(x_1, x_2, t_1, t_2)}{\sqrt{\mu_{0_1} e^{\beta_1 y}}} (R_1 e^{ikpy} + R_2 e^{-ikpy}) e^{i\phi} + c.c., \\ u_1^{(2)} &= \frac{\mathcal{A}_1(x_1, x_2, t_1, t_2)}{\sqrt{\mu_{02} e^{\beta_2 y}}} R_3 e^{kvy} e^{i\theta} + c.c. \end{aligned} \tag{8}$$

$$p = \left(\frac{c^2}{c_{01}^2} - \frac{\beta_1^2}{4k^2} - 1 \right)^{1/2}, \quad v = \left(-\frac{c^2}{c_{02}^2} + \frac{\beta_2^2}{4k^2} + 1 \right)^{1/2}, \tag{9}$$

Here, $\mathcal{A}_1$ is a complex function that denotes the first order slowly varying amplitude of the nonlinear self modulation. $\mathbf{R} = (R_1, R_2, R_3)^T$ is a column vector. To obtain the first order solution completely, $\mathcal{A}_1$ must be determined via higher order problems. Solvability condition of third order problem gives the following nonlinear Shrdinger equation;

$$i\frac{\partial \mathcal{A}}{\partial \tau} + \Gamma \frac{\partial^2 \mathcal{A}}{\partial \xi^2} + \Delta |\mathcal{A}|^2 A = 0 \tag{10}$$

where $\tau = \omega t_2, \quad \xi = k\varepsilon^{-1}(x_2 - V_g t_2) = k(x_1 - V_g t_1), \quad \mathcal{A} = k\mathcal{A}_1$.

It is known that how a given initial data will evolve for long times for the asymptotic wave field characterized by the NLS equation relies on the sign of $\Gamma\Delta$. Bright solitary wave solution exists when $\Gamma\Delta$ is positive [4].

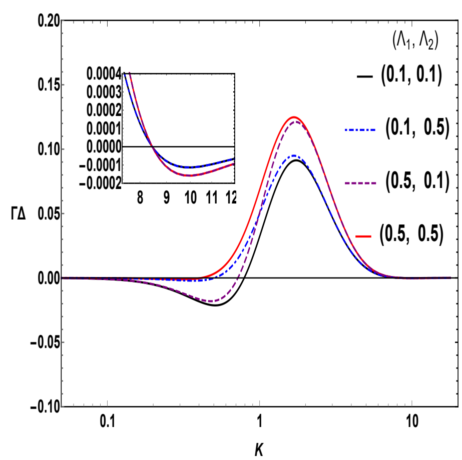

Figure 1: $\Gamma\Delta$ vs. K for various nonlinear heterogeneity parameters (Λ_1, Λ_2) of the hardening half space covered by hardening layer for $\theta_{01} = \theta_{02} = 2$, $B_1 = B_2 = 0.5$.

Conclusion

To investigate the influence of nonlinear heterogeneity on the presence of bright solitary waves, variation of $\Gamma\Delta$ sign with wave number is evaluated. In the numerical

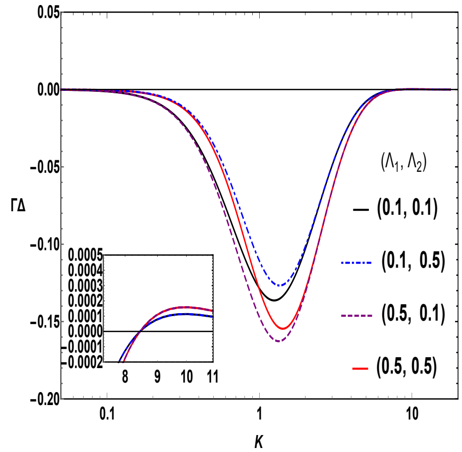

Figure 2: $\Gamma\Delta$ vs. K for various nonlinear heterogeneity parameters (Λ_1, Λ_2) of the hardening half space covered by softening layer for $\theta_{01} = -2$ $\theta_{02} = 2$, $B_1 = B_2 = 0.5$.

calculations, dimensionless linear heterogeneity parameters $B_1 = \beta_1/k$ and $B_2 = \beta_2/k$ are kept fixed as $B_1 = B_2 = 0.5$ and $\Gamma\Delta$ curves versus $K = kh$ are calculated for different choices of dimensionless nonlinear heterogeneity parameters $\Lambda_1 = \lambda_1 h$ and $\Lambda_2 = \lambda_2 h$. Shear hardening half space covered by a hardening and softening layer is considered in Fig. 1 and Fig. 2, respectively. It is observed that the intervals in which $\Gamma\Delta$ is positive change with variation of (Λ_1, Λ_2) for hardening layer, whereas existence of bright solitary waves does not affected by the variation in (Λ_1, Λ_2) for softening layer. It is also observed that curves having same Λ_2 approach each other for small wave numbers whereas curves with the same Λ_1 approach each other for short waves. Hence, it is concluded that nonlinear heterogeneity parameter of the layer dominates the modulation when $K \gg 1$ while that of half space is effective for long waves.

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A New Finite Special Sum Associated With the Hardy Sums and Fibonacci Numbers

Elif Cetin

Some remarkable properties of the new finite sum $C(h,k;1)$ are investigated, and two of useful identities of this sum was given in [12]. The sum $C(h,k;1)$ has many relations with the Hardy type sums and Dedekind sums. In this paper, the reciprocity law for the sum $C(h,k;1)$ is given. Besides, new relations between this sum and other well known finite sums, such as Hardy sums, Dedekind sums, $Y(h,k)$ sums, $B_1(h,k)$ sums, are also given. And finally another reciprocity law of the sum $C(h,k;1)$ is explored when h and k are special Fibonacci numbers.

2010 Mathematics Subject Classifications: 11F20, 11C08

Keywords: Special Finite Sums, Fibonacci Numbers, Hardy Sums, Dedekind Sums

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Remarks on generalization Szâsz-Mirakjan-Baskakov operator and their applications

Elif Şükrüoğlu [*1] *and Yılmaz Şimşek* [2]

In this article, some properties of the Szâsz-Schurer-Baskakov type operator are studied. We study the relation of this operator with the Laguerre polynomial. Correlations between Lah numbers and the numbers $y_6(n;a,b,v)$ will be established. The general purpose of this presentation is to study some properties of the $A_{n,\alpha,\beta_n}(t^r;x)$ operator. By considering the relations between this operator and Laguerre polynomials, connections between Lah numbers and y_6 numbers will be created. It will be planned to obtain new results by applying limit operations for the special cases of this operator β_n and x.

2010 Mathematics Subject Classifications: 11B83, 41A36, 33C45

Keywords: Szâsz-Mirakjan-Baskakov operator, Lah numbers, the numbers $y_6(n;a,b,v)$, Modified Szâsz-Schurer-Baskakov operator

Introduction

In 1995 [1], Gupta and Srivastava studied the convergence properties of the derivatives of the Szâsz-Mirakjan-Baskakov type operators defined as follows;

$$M_n(f;x) = (n-1)\sum_{k=0}^{\infty} e^{-nx}\frac{(nx)^k}{k!}\frac{(n+k-1)!}{k!(n-1)!}\int_0^{\infty}\frac{t^k}{(1+t)^{n+k}}f(t)dt, x\in[0,\infty) \quad (1)$$

For $f\in C[0,\infty)$,we define the Schurer type generalization of the Szâsz-Mirakjan-Baskakov type operator given in (1) as follows;

$$A_{n,\alpha,\beta_n}(f;x) = (n+\alpha-1)\sum_{n=0}^{\infty} e^{-nx}\frac{(nx)^k}{k!}\frac{(n+k-1)!}{k!(n-1)!}\int_0^{\infty}\frac{(n+\alpha+k-1)!}{k!(n+\alpha-1)!}\frac{t^k}{(1+t)^{n+k}}f(t)dt; \quad (2)$$

$$\alpha\in\mathbb{N}_0, x\in[0,\infty)$$

where β_n is a sequence of positive numbers such that

$$\lim_{n\to\infty}\beta_n = 0.$$

Y.Şimşek defined the as following generating functions for three-variable polynomials [2];

$$G(t,x,y,z;a,b,\upsilon)=(b+f(t,a))^z e^{xt+yh(t\upsilon)}=\sum_{n=0}^{\infty} y_6(n;x,y,z;a,b,\upsilon)\frac{t^n}{n!} \tag{3}$$

where $f(t,a)$ is a member of family of analytic functions or meromorfic functions, a and b are any real numbers, υ is positive integer. Setting $f(t,a)=\frac{1}{(1-t)^{a+1}}, h(t,1)=\frac{t}{t-1}$ also $x=b=0$ and $z=1$ into (3) , genetaring function for the Laguerre polynomials as follows [2];

$$G(t,0,y,z;a,0,1)=\frac{1}{(1-t)^{a+1}}e^{\frac{xt}{t-1}}=\sum_{n=0}^{\infty} y_6(n;0,y,1;a,0,1)\frac{t^n}{n!},$$

$$L_n^{(\alpha)}(x)=y_6(n;0,y,1;a,0,1).$$

Setting $f(t,a)=\frac{1}{a!}(\frac{t}{1-t})^a$,also $x=b=0$ and $z=1$ into (3) , Y.Şimşek generating functions for the Lah numbers, $L(n,a)$ as follows [2];

$$G(t,0,0,k;a,0,1)=\frac{1}{a!}(\frac{t}{1-t})^a=\sum_{n=0}^{\infty} y_6(n;0,0,1;a,0,\upsilon)\frac{t^n}{n!}$$

where $a\in\mathbb{N}_0$.
Thus,

$$L(n,a)=y_6(n;0,0,1;a,0,\upsilon).$$

Lah number defined by

$$\left(\frac{t}{1-t}\right)^k=k!\sum_{n=0}^{\infty} L(n,k)\frac{t^n}{n!}.$$

Lemma 1. *[3] For all* $r\in N_0$,

$$A_{n,\alpha,\beta_n}(t^r;x)\frac{r!(n+\alpha-r-2)!}{(n+\alpha-2)!}\sum_{j=0}^{r}\binom{r}{j}\frac{[(n+\beta_n)x]^j}{j!}$$

is a polynomial in x of degree exactly r.
Since $A_{n,\alpha,\beta_n}(t^r;x)$ is related with generalized Laguerre polynomial with the relation

$$A_{n,\alpha,\beta_n}(t^r;x)\frac{r!(n+\alpha-r-2)!}{(n+\alpha-2)!}L_r(-(n+\beta_n)x).$$

Conclusion

We not only surveyed and studied some properties of the Szâsz-Schurer-Baskakov type operator, but also used the relation of this operator with the Laguerre polynomial. Many relationships between the Lah numbers and the numbers $y_6(n;a,b,v)$ were given. In future investigation, we look for novel properties of the $A_{n,\alpha,\beta_n}(t^r;x)$ operator in order to find some relations among the Laguerre polynomials, the Lah numbers and and the numbers $y_6(n;a,b,v)$. We will also plan to investigate new results by applying limit operations for the special cases of this operator β_n and x.

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The regression analysis of the compositions of Antalya Gulf Macroalgae (Turkiye)

Emine Şükran Okudan

The aim of the study is to analyze the compositions of macroalgae seen in Antalya Gulf, located in the south of Turkey, by multivariate statistical methods. According to the regression analysis obtained from multivariate statistical analysis, $R2 = 100\%$ was calculated. According to the results of this analysis, it was understood that the data used in multivariate statistical analysis was sufficient.

2010 Mathematics Subject Classifications: 62H10, 62H86, 62H25, 62H30

Keywords: Regression analysis, Data analysis, Macroalgae, Antalya Gulf

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Utilization of Repulsion Weights Application as an Optimizing Parameter

Emre Demir [*1] *and Niyazi Ugur Kockal* [2]

Mathematical models and methodologies have been crucial for many decades to contribute to the understanding of innumerable engineering processes. In transportation optimization, the researchers use the weights of the nodes in network topology in a positive direction (in terms of attractiveness or demand). However, the applications of the weights in terms of the opposite of attractiveness (i.e., repulsion) are lacking. In this manner, they could be introduced as repellency indicators as they are supposed to be in a negative direction. Such needs can be encountered daily, especially for locating a facility in a specific industry. Therefore, an application in terms of repellency has been proposed in this study. Furthermore, limitations were considered to introduce repulsion weights. As a result, keeping the undesired weighted network nodes away as much as possible makes more compatible to real-life implementations.

KEYWORDS: Location analysis, Optimization, Repellency, Repulsion weight

Introduction

The significance of this study is as follows. Although many researchers examine the weights of particular nodes on a network topology regarding attractiveness or demanding weights (w_a), the importance is given less to the opposite of attractiveness. In other words, repulsion weights (w_r) have not been found much in the literature. Instead, the attractiveness of a location is used in most cases and studies. However, the number of parameters used for attractiveness weights (n_a) may be more than the number of repulsion weights (n_r). Accordingly, an increase in n_a can make analyses unnecessarily complicated and difficult. Nevertheless, handling analyses in terms of w_r can simplify the work. Therefore, our study aims to shed light on future studies considering the types of w_r, such as impulsiveness or repulsion weights.

Methodology

Many optimization studies dealt with the weights of network nodes in terms of attractiveness and its variations. Fundamental equations or inequalities are widely used. For instance, many researchers [1, 2, 3, 4, 5, 6, 7, 8] applied w_a criteria as coefficients in Eqs. (1-3).

$$\min \sum_{i=1}^{n} k a_i d_c \tag{1}$$

$$\max \sum S_i j \tag{2}$$

$$\min \sum m \sum l \tag{3}$$

In Eqs. (1-3), i, j, and n are the origin node, destination node, and the number of nodes, respectively. k is the coefficient of a cost, a_i is the attractiveness coefficient, d_c is the distance cost, and S is the interaction between the origin and destination nodes. Additionally, m is the gravitational force of attraction on an object, and l is the length cost of an object.

Results

It can be suggested that the results will be more accurate if some w_r parameters can be introduced to the techniques mentioned in the methodology section and many other methods. In doing so, the models and the methodologies will be closer to real-life implementations. For instance, during pandemic periods (e.g., the COVID-19 pandemic) and in such cases, using some w_r parameters gives more accurate results than using only w_a parameters. Parameters such as preferring more rural areas for accommodation, staying away from public places, crowded areas, buildings such as hospitals and public transportation facilities can be used as w_r parameters in the pandemic. Moreover, analyses should be done by considering w_r parameters rather than w_a parameters for installing base stations and cell towers. Furthermore, it may be possible to use w_r parameters in the future construction and real estate sectors. What the customers do not prefer rather than what they demand can also be included in the construction and real estate industry analyses. It will be possible to present such buildings and structures to the clients by implementing w_r parameters.

Conclusion

Studies including both w_a and w_r parameters are encountered in the literature, although they are rare. The more the equations and relations obtained as a result of analyzes and models can be moved away from complexity and reduced to simplicity, of course, without reducing precision and accuracy, the applicability will increase accordingly. Having fewer w_r parameters than w_a parameters will reduce the complexity of the relations in models and make them more applicable. Thus, in the future, it will be possible to reach the desired targets more easily with such analyzes and modeling in both private and legal fields, including the construction industry and many sectors.

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The rate of convergence on Kantorovich-Szász operators involving Fubini-type polynomials

Erkan Agyuz

In this study, we give Kantorovich-Szász type operators which involve the generating functions of Fubini type polynomials. By using generating function method we present moment and central moments functions to examine uniformly convergence of our operators. And then, we investigate the convergence properties of our operators such as Korovkins theorem and modulus of continuity.

2010 Mathematics Subject Classifications: MSC 05A15, 47A58, 47B65

Keywords: Generating functions, Positive linear operators, Fubini-type polynomials

Acknowledgements

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Artificial neural network method for solving Black-Scholes-Merton partial differential equation

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One of the most popular machine learning methods today is the Neural Network method. The Neural Network or Artificial Neural Network (ANN) method is a method that imitates the workings of the brain. The neurons received the signals and manipulated them mathematically, then transmitted them to more neurons. We can also use the Neural Network method to find solutions to the Ordinary Differential Equation (ODE) or Partial Differential Equation (PDE) with certain initial and boundary conditions. In contrast to other methods, we can automate all computation processes, and the Neural Network method also learns from the solutions obtained. This paper uses the Neural Network method to determine the option price through the Black-Scholes-Merton Partial Differential Equation.

2010 Mathematics Subject Classifications: 35K05, 68T07, 91G20 35K05, 68T07, 91G20

Keywords: Neural Network, Black-Scholes-Merton PDE, Option Price

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The relation β^*_{ss}

Fatih Gömleksiz [*1] *and Burcu Nişancı Türkmen* [2]

In this study, it has been shown that the relation β^*_{ss}, which is defined in the form $X\beta^*_{ss}Y$, which provides the $(X+Y)/X \subseteq (Soc_s(M)+X)/X$ and $(X+Y)/Y \subseteq (Soc_s(M)+Y)/Y$ conditions for X and Y submodules taken in the set of submodules of an R-module M in the $R-Mod$ category, is an equivalence relations. The main features provided by this relations are examined. It was shown that the epimorphism $f: M \longrightarrow N$ provided the β^*_{ss} relation under certain conditions and the relation β^*_{ss}, was expressed in the maximal submodules. In addition, we obtain the various properties of β^*_{ss}-supplemented modules H_{ss}-supplemented modules, P^*_{ss}-modules using the relation β^*_{ss}.

2020 Mathematics Subject Classifications: 16D99

Keywords: β^*_{ss}, socle of a module

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Approximating fixed points of generalized (α, β) Non-Expansive mapping via KF iteration process

Fayyaz Ahmad

In this paper, we use the KF iteration process for finding fixed points of generalized (α, β)-nonexpansive mappings. We establish weak and strong convergence results in a uniformly convex Banach space setting. A new example is used to support the main results. The presented research work compliment and extend several results of the literature.

2010 Mathematics Subject Classifications: 47H09, 47H10

Keywords: Fixed point, (α, β)-Nonexpansive mapping, KF iteration process, Strong convergence, Weak convergence, Uniformly Convex Banach space

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On the Consequences of a Specific Subdomain Extension Problem

Federico Giovanni Infusino

In this talk, based on recent researches carried out in [1, 2], we discuss some fundamental properties of the *extensible subdomains*, arising from the attempt of suitably extend subdomains of a given integral domain U to specific subdomains of the field of quotients K_U. More in detail, we first characterize extensible subdomains in terms of a *finitary simplicial complex*, whose main properties become a source for a general investigation of finitary simplicial complexes on arbitrary ground sets; next, starting from a specific counterexample determined through idempotent endomorphisms, we exhibit a whole series of techniques useful to obtain informations on the polynomial ring in one variable with coefficients in an extensible subdomain.

2010 Mathematics Subject Classifications: 06A15; 08A02; 16D10

Keywords: Extensible subdomains, Primal Subdomains, Finitary Simplicial Complexes

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Existence results of a Hadamard fractional boundary value problem with a Hadamard fractional derivative condition for $2 < p \leq 3$

Fulya Yoruk Deren [*1] *and Tugba Senlik Cerdik* [2]

In this work, we investigate the following Hadamard type fractional boundary value problem on an infinite interval

$$^{H}D^{p}_{1+}u(t) + w(t)f(t, u(t)) = 0, 2 < p \leq 3, t \in (1, +\infty),$$

$$u(1) = u'(1) = 0, \qquad ^{H}D^{p-1}_{1+}u(\infty) = \sum_{i=1}^{k} a_i {}^{H}D^{r}_{1+}u(n_i),$$

in which $^{H}D^{p}_{1+}$ is the Hadamard-type fractional derivative of order p, $r \in [0, 2]$, $a_i \geq 0$ $(i = 1, 2, ..., k)$, $1 < n_1 < n_2 < ... < n_k < +\infty$, $f \in \mathcal{C}([1, \infty) \times [0, \infty), (0, \infty))$, $f(t, 0) \not\equiv 0$ on any subinterval of $[1, \infty)$, $w : [1, \infty) \to [0, \infty)$ is not identically zero on any closed subinterval of $[1, \infty)$ and $0 < \int_1^\infty w(s)\frac{ds}{s} < \infty$.

Here, positive solutions are established for a Hadamard fractional boundary value problem on infinite domain. Sufficient conditions will be presented in order to obtain the existence of positive solutions. Also, an example is shown to illustrate the applicability and correctness of our main result. For some definitions and lemmas about the fractional calculus, see [1, 2].

2010 Mathematics Subject Classifications: 35040

Keywords: Existence of positive solutions, Hadamard type fractional differential equations

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An Application on the Selection of Weight Matrices in Spatial Regression Models

Fusun Yalcin

Spatial analysis has been preferred by researchers in recent years in data groups where spatial relationship is considered important. Especially the selection of the weight matrix is very important for the model created. Weight matrices that are frequently used in this study will be briefly mentioned.

2010 Mathematics Subject Classifications: 62H11, 91B72, 65F50

Keywords: Applied Statistics, Spatial Regression Models, Weight Matrices

Introduction

There are three elements in determining the spatial relationship. First, a relationship is visually determined through mapping. Then, research-based spatial data analysis calculations are looked at (eg Moran's I). Finally, modeling studies are carried out. The research is concluded by choosing the best model (*cf.*[1]).

In studies in applied earth sciences, sequential data are obtained in the area or in area and time for observations. In such cases, observations can be compiled using a coordinate system.Or, spatial observation values can be created by adding relative positions based on a distance measure to the observed values (*cf.*[1, 2]). A spatial regression model can be created by adding weight matrices in which the position relationship is defined to the models studied with the data created in this way.

Material and Method

The first element to determine the existence of the spatial effect is to create a clustering map. For example; When the cluster map of the air quality index (AQI) of Turkey for 2004 is examined, it is seen that the density of the provinces close to each other is similar.

When we examine the literature of linear spatial regression models, we can see a large number of models containing one or more weight matrices. (*cf.*[6]).

Definition 1. *The matrix created depending on the proximity or other selected properties of the observed values* $(w_{ij} : i, j = 1, ..., n), (nxn)$ *dimensional matrix*

$$W = \begin{bmatrix} w_{11} & w_{12} & \cdots & w_{1n} \\ w_{21} & w_{22} & \cdots & w_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ w_{n1} & w_{n2} & \cdots & w_{nn} \end{bmatrix}$$

where n *is the number of locations or objects. The* w_{ij}*'s in this matrix indicate whether there is a spatial relationship according to the states of the elements in the row and column (cf.[1]).*

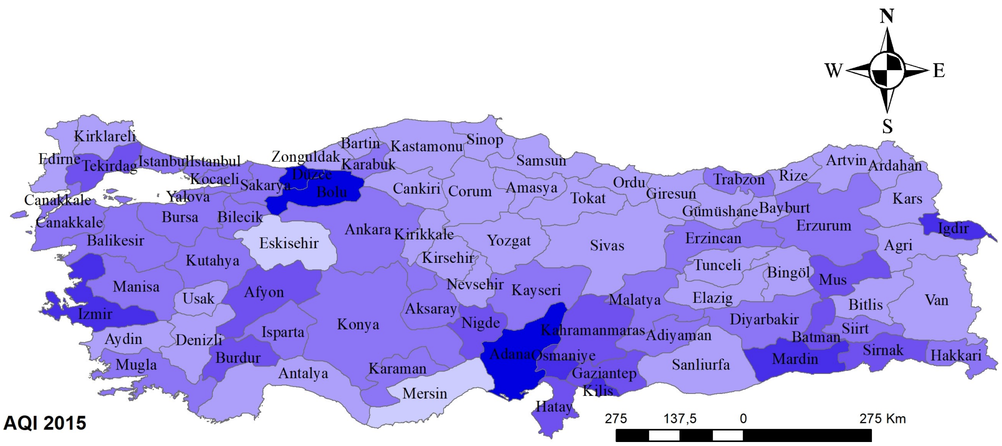

Figure 1: Turkey's air quality index (AQI) cluster distribution map. (*cf.*[4])

In spatial regression models, the elements of the weight matrix can be constructed based on the map, based on socio-economic relations, and based on the distance between locations (*cf.*[1]). Map-based weight matrices are created according to whether there is a border neighborhood or not. In general, weights of 1 and 0 are taken for the binary neighborhood structure,

$$w_{ij} = \{ \begin{array}{ll} 1, & If\ i\ and\ j\ are\ neighbors \\ 0, & other\ situation \end{array}$$

Distance Dependent Weight Matrices are created with the idea that as the distance between observations increases, the spatial cascade dependence between them decreases. These matrices weight the observations according to their proximity to each other. Here, a neighborhood matrix is created according to the distances of the neighborhoods. By taking a critical value, the positions up to this value are formatted as neighbors, not neighbors (*cf.*[1]). Where d is the threshold value

$$w_{ij} = \{ \begin{array}{ll} 1, & 0 < d_{ij} < d \\ 0, & other \end{array}$$

Another weight matrix can be created by considering the socio-economic relations of the observation values. Two boundary observation values may not be socio-economically related, or geographically distant observations may be socio-economically related. Weight matrices can be created by considering these relationships.

In general, linear spatial regression models can be summarized with the algorithm in J. P. Elhorst's article (*cf.*[3]).

Conclusion

In spatial models, the choice of the weight matrix may affect the statistical test values of the model. It can also modify the results of analyzes that allow the spatial effect to be determined (for example Moran's I). For this reason, it is recommended that researchers work on it by creating different weight matrices while creating a

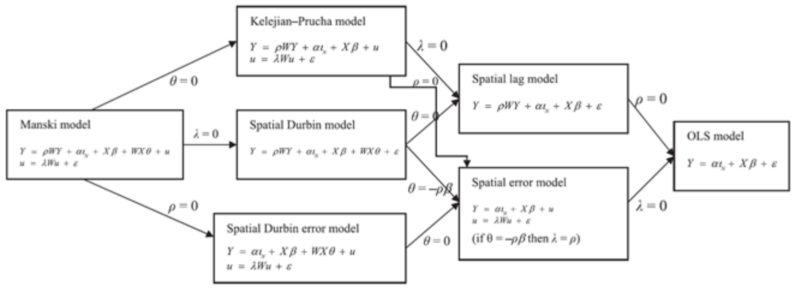

Figure 2: Algorithm of some spatial regression models given for cross-sectional data (*cf.*[3])

spatial model. A weight matrix that is externally included in the model can lead to erroneous inferences if it is formed incorrectly.

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An Application on Examining the Use of Regression Analysis in Time Series

Fusun Yalcin *[1] *and Yagmur Karaca Ulkutanir* [2]

The aim of this study is to examine regression models in time dependent data sets. For this, the annual data set created with the number of midwives in Turkey between the years 1928-2020 was used. While applying time series regression, 3 different regression models were applied to this data set and the best regression model was selected. For this regression model, the number of midwives in Turkey was estimated for the years 2021 and 2022.

2010 Mathematics Subject Classifications: 62-06, 97K80, 62M10, 62J02

Keywords: Applied Statistics, Regression Models, Time Series

Introduction

Statistical analysis that help us determine whether two events affect each other are Regression and Correlation analysis. While correlation analysis tells us how two variables act together, regression analysis explains how much these two variables affect each other numerically. Regression analysis is basically based on a mathematical model. This model may be linear or non-linear. The researcher can decide this by looking at the scatterplot of the data. When the data set is in the form of a time series, ”time series regression models” are frequently used to make predictions.

Time series are used in positive sciences such as mathematics, statistics, physics, in data sets that show increase, decrease, growth, shrinkage depending on time, as well as in social sciences such as finance, banking and econometrics, they are used under many time-related subheadings such as population and stock market. Time series analysis is important in terms of examining and analyzing time-dependent data sets and making predictions for the future by examining the data taken in certain time periods (*cf.*[1, 3]).

Material and Method

In this study, the annual data set created with the number of midwives in Turkey between the years 1928-2020 was used for time series data. This data set is taken from TUIK, which gives open access permission (*cf.*[2]).

A linear regression model to show $x_{i,t}$ the independent variables (($t = 1, 2, ..., n$) ve ($i = 1, 2, ..., p$)) and Y_t the dependent variable is as follows.

$$Y_t = \beta_0 + \beta_1 x_{1,t} + \beta_2 x_{2,t} + ... + \beta_p x_{1p,t} + e_t$$

For this equation to be a simple linear regression equation, it must be p=1 (*cf.*[1, 3, 4]).

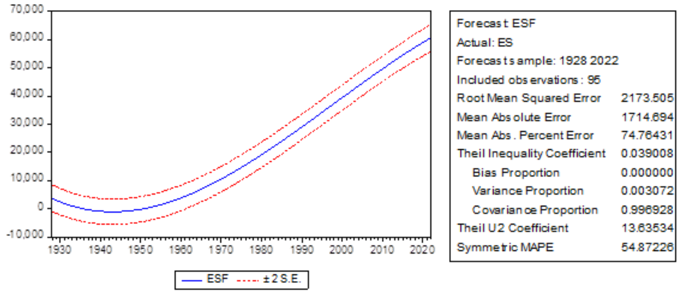

Figure 1: Some statistics of the cubic regression model for the number of midwives in Turkey

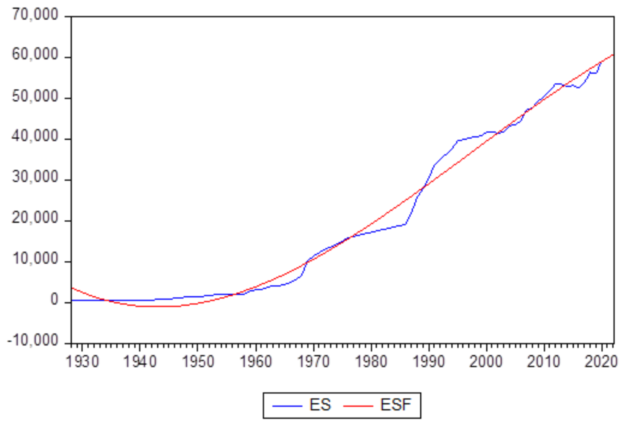

Figure 2: Estimation of the number of midwives in Turkey (cubic regression model).

$$
\begin{aligned}
RMSE &= \sqrt{\frac{\sum_{t=1}^{T} e_t^2}{T}}, \\
MAE &= \frac{\sum_{t=1}^{T} |e_t|}{T}, \\
MAPE &= \frac{\sum_{t=1}^{T} \frac{|e_t|}{y_t}}{T}
\end{aligned}
$$

where y_t is the time series values, e_t is the error terms and T is the number of observations.

Model	RMSE	MAE	MAPE
Simple linear regression model	6028,264	5390,656	352,0316
Quadratic regression model	2814,259	2096,612	81,29627
Cubic regression model	2173,505	1714,694	74,76431

Table 1: Error values of studied time series regression models

Since the model with the smallest error value among the three models created is the cubic regression model, it was chosen as the most appropriate model.

Conclusion

In regression analysis, when the data is a time series, the mathematical model can be examined visually according to its compatibility with the graph. Also, with some error formulas, the best model can be selected. In this study, only three models and three error calculations were examined and it was decided that the most suitable model was the cubic model with the smallest error values.

Acknowledgments

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Fourier Extensions for Solving Boundary Value Problems in Arbitrary Domain

Ghadeer Mohummed Surrati *[1] *and Jitse Niesen* [2]

Solving differential equations is of fundamental importance in many areas of science. Fourier spectral method can solve periodic problems. In the case of non-periodic problems, the lack of periodicity changes the situation due to the Gibbs phenomenon which destroys the rapid convergence. A new class of spectral methods for solving linear boundary value problems (BVPs) is presented in this paper. The advantages of the spectral methods lie on the remarkable accuracy and the use of the Fast Fourier Transform (FFT).

The proposed method is based on Fourier extensions technique which can be used for approximating non-periodic problems. The principle of Fourier extension is to approximate a non-periodic function that is defined on $[-1, 1]$ by a Fourier series that is periodic on a larger domain $[-T, T], T > 1$.

In two dimensions (2D), the domain of the problem is an important complication for the process of finding the solution. Our goal is to provide a spectral method using the Fourier extension approach to solve non- periodic BVPs in arbitrary domain in 2D, because standard Fourier and Chebyshev methods can not be applied. The construction of the method and some numerical examples are presented.

2010 Mathematics Subject Classifications: 65N35, 65D15

Keywords: Fourier extension, numerical methods, Boundary value problems , Gibbs phenomenon, spectral methods

Introduction to Fourier Extension (FE)

Fourier series result in an exponentially convergent approximation of a smooth periodic function. This approximation can be computed numerically via the FFT. However, Fourier approximation lose their fast convergence when the function is smooth but non-periodic because of the Gibbs phenomenon. This is illustrated in Figure 1. The principle of FE is to approximate a non-periodic function that is defined on $[-1, 1]$ by a Fourier series that is periodic on $[-T, T], T > 1$, [1, 2]. By this way, the superior properties of Fourier series are involved in the small interval $[-1, 1]$. Figure 2 shows that the approximation converges to f in the original interval $[-1, 1]$ and Gibbs phenomenon does not appear on the boundaries near $x = \pm 1$. We want to use FE to solve BVPs. We focus our investigation on achieving spectral accuracy for two-point BVPs in one dimension before tackling BVPs in irregular domains such as such as the L-shaped domain Ω_l; see the left plot in Figure 4. Due to specific structure of the basis matrix of Fourier extension method, we can exploit the FFT and the random matrix algorithm to speed up the FE method and introduce fast FE method [3], but we do not investigate it in this paper. Further details can be found in [4].

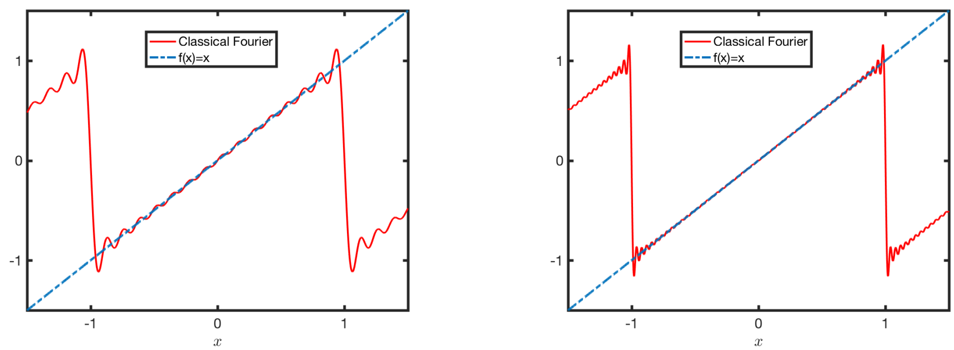

Figure 1: Illustration of the Gibbs phenomenon using the classical Fourier series of $f(x) = x$ with 15 terms (left) and 50 terms (right). The peak of the oscillation near the boundaries at $x = \pm 1$ does not decrease in magnitude.

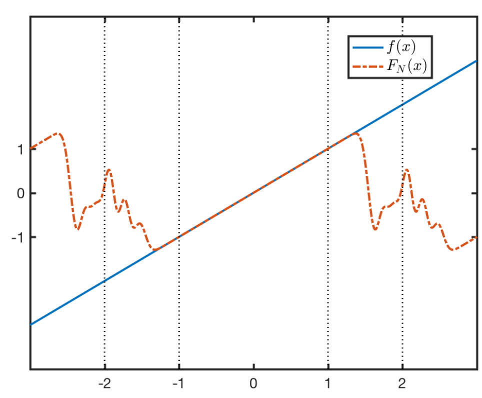

Figure 2: The Fourier extension $F_N(x)$ for $f(x) = x$ on $[-1, 1]$, overcomes the Gibbs phenomenon. The extended interval is $[-2, 2]$.

FE method for solving BVPs in 1D

Consider the linear two-point BVP

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y(x) = r(x), \quad x \in [-1, 1], \tag{1}$$

with the boundary conditions:

$$y(-1) = v_1, \quad \text{and} \quad y(1) = v_2, \tag{2}$$

In the direct FE approach, we suppose a solution of the form

$$y_F(x) = \sum_{j=-N}^{N} c_j e^{\frac{i\pi jx}{T}}, \quad T \text{ is the extension parameter.} \tag{3}$$

We find the first and second derivatives of y_F, then substituting these derivatives and equations (3) into the main problem (1) so we get:

$$\sum_{j=-N}^{N} c_j\, e^{\frac{i\pi jx}{T}} \left(-\frac{\pi^2}{T^2}j^2 + \frac{i\pi}{T}jp(x) + q(x)\right) = r(x). \tag{4}$$

Additionally, the solution needs to satisfy the BCs which for (3) read

$$\sum_{j=-N}^{N} c_j\, e^{\frac{-i\pi j}{T}} = v_1, \quad \text{and} \quad \sum_{j=-N}^{N} c_j\, e^{\frac{i\pi j}{T}} = v_2. \tag{5}$$

Our proposed solution (3) has $N_* = 2N + 1$ coefficients. If we divide the interval $[-1, 1]$ into $2\gamma N$ subintervals, then we have $M_* = 2\gamma N + 1$ collocation points. When approximating a function by FE we need more collocation points than coefficients; typically we take $\gamma = 2$ [1]. We impose equation (4) at the interior points leading to $(2\gamma N - 1)$ equations. These equations together with the two equation (5) form a system of M_* equations in N_* unknowns, written in the matrix form: $Ac = b$. If $\gamma > 1$, this system is overdetermined so, we use QR factorisation to solve the system and find the Fourier coefficients c.

Example 1. *We solve the BVP*

$$y''(x) + xy' = (2 + x^2)\cos x, \quad \textit{with} \quad y(\pm 1) = \sin(1). \tag{6}$$

The maximum error between the approximate solution and the exact solution $y(x) = x \sin x$, for several values of N is computed. Figure 3 shows the error when solving the BVP (6). We get spectral convergence, but we have to do oversampling $(\gamma = 2, 3)$ to get high accuracy to roundoff error.

FE for solving BVPs in 2D

In this section we generalise the direct FE method which is presented in the previous section to solve BVPs. We tackle the L-shaped domain Ω_l, and the Poisson equation is chosen as a test example.

Example 2. *On Ω_l, we solve the BVP*

$$\nabla^2 u(x, y) = u_{xx} + u_{yy} = \frac{-4xy(x^2 + y^2 + 3)}{(x^2 + y^2 + 1)^3}, \quad x, y \in [-1, 1], \tag{7}$$

where the boundaries are specified such that the exact solution is $u(x, y) = \frac{xy}{x^2+y^2+1}$.

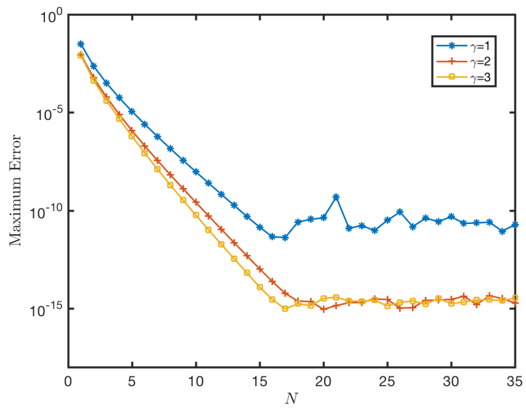

Figure 3: The maximum error of FE when solving the problem (6) with several values of N, $M_* = 2\gamma N + 1$ and $T = 2$.

We assume that the FE solution has the form

$$u_F(x,y) = \sum_{m=-N}^{N} \sum_{n=-N}^{N} \alpha_{m,n} e^{\frac{i\pi(mx+ny)}{T}}. \tag{8}$$

Then we compute the second derivatives with respect to x and y and substituting these derivatives into (7), so we get

$$\sum_{m=-N}^{N} \sum_{n=-N}^{N} \alpha_{m,n} \, (m^2+n^2) e^{\frac{i\pi(mx+ny)}{T}} = \frac{4T^2 xy(x^2+y^2+3)}{\pi^2(x^2+y^2+1)^3}. \tag{9}$$

The approximation formula (9) contains $N_*^2 = (2N+1)^2$ Fourier coefficients. We discretize the domain using the collocation points, then substitute these points into (9), so we get the linear system which need to be solved after imposing the boundary conditions. Figure 4 (right plot) represents the spectral accuracy when solving the Poisson equation using the direct FE method when $\gamma > 1$.

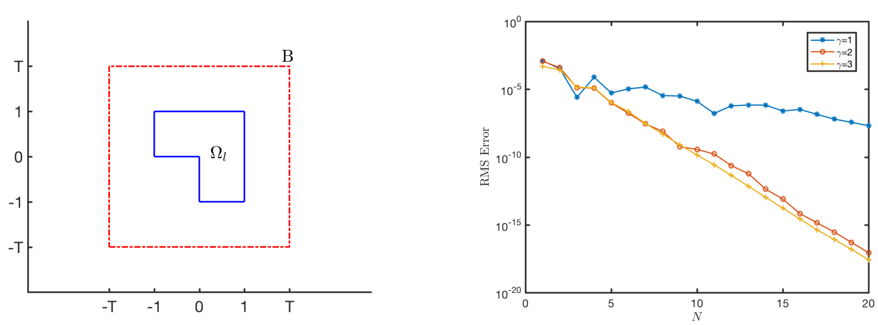

Figure 4: (Left) the L-shaped domain Ω_l inside the extended square domain $T = 2$. (Right) the RMS error when solving the BVP (7) on Ω_l using FE.

Conclusion

We constructed a method based on the equispaced collocation FE method to solve second-order BVPs in 1D and on arbitrary domain in 2D. The experiments demonstrate that spectral accuracy is obtained.

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One-sided fractional integrals in Morrey and vanishing Morrey spaces

Giorgi Imerlishvili

The boundedness of one-sided potentials defined, generally speaking, with respect to a Borel measure μ in Morrey spaces is established. Appropriate inequalities for power-type weights are derived in Morrey spaces and in complementary Morrey spaces. The similar conclusions are, also, given for power-type weights in vanishing Morrey spaces. The similar results for two-sided fractional integral operators were studied in [1]. This talk is natural continuation of the research carried out by us regarding two-sided fractional integrals in the classical Morrey spaces.

2010 Mathematics Subject Classifications: 26A33, 42B25, 47B34, 47B90

Keywords: Boundedness, one-sided potentials, Morrey spaces, complementary Morrey spaces, vanishing Morrey spaces

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Some Finite Cyclic Unit Graphs

Gülçin Karaca *1 *and Burcu Nişancı Türkmen* 2

In this study, Kirchhoff index, Hyper-Wiener index, Randic index, Szeged index, Pi index calculations of finite-cyclic unit graphs were made on the examples and classification of some finitely generated cyclic groups was reached with the help of graph theory.

2020 Mathematics Subject Classifications: 20A05, 05C90

Keywords: graph theory, unit connected graph

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Berezin radius inequalities with improvement of Young inequality

Hamdullah Başaran *[1] *and Mehmet Gürdal* [2]

The this work, we obtain the new Berezin radius inequalities by using improvement of Young inequality. Also, we present several refinements of Berezin radius inequalities in reproducing kernel Hilbert space operators. Our main results mainly extend and refine the inequalities in [8] and [11].

2010 Mathematics Subject Classifications: 47A30, 47A63

Keywords: Berezin symbol, Berezin number, Functional Hilbert space, Arithmetic-geometric mean inequality, Young inequality

Introduction

Let Υ be a subset of a topological space X such that boundary $\partial\Upsilon$ is nonempty. Let $\mathbb{H} = \mathbb{H}(\Upsilon)$ be an infinite-dimensional Hilbert space of functions defined on Υ. We say that $\mathbb{H}$ is a functional Hilbert space (FHS) or reproducing kernel Hilbert space if the following two conditions are satisfied:

(i) for any $\varepsilon \in \Upsilon$, the functionals $f \to f(\varepsilon)$ are continuous on $\mathbb{H}$;

(ii) for any $\varepsilon \in \Upsilon$, there exists $f_\varepsilon \in \mathbb{H}$ such that $f_\varepsilon(\varepsilon)$:

According to the classical Riesz representation theorem, the assumption (i) implies that for any $\varepsilon \in \Upsilon$ there exists $k_\varepsilon \in \mathbb{H}$ such that

$$f(\varepsilon) = \langle f, k_\varepsilon \rangle, \ f \in \mathbb{H}.$$

The function k_ε is called the reproducing kernel of $\mathbb{H}$ at point ε. Note that by (ii), we surely have $k_\varepsilon \neq 0$ and we denote $\widehat{k}_\varepsilon$ the normalized reproducing kernel, that is $\widehat{k}_\varepsilon = \frac{k_\varepsilon}{\|k_\varepsilon\|}$ (see, [1]).

The Berezin transform associates smooth functions with operators on Hilbert spaces of analytic functions.

Definition 1. *Let $\mathbb{H}$ be a FHS on a set Υ and let A be a bounded linear operator on $\mathbb{H}$.*

(i) For $\varepsilon \in \Upsilon$, the Berezin transform of A at ε (or Berezin symbol of A) is

$$\widetilde{A}(\varepsilon) := \left\langle A\widehat{k}_\varepsilon, \widehat{k}_\varepsilon \right\rangle_{\mathbb{H}}.$$

(ii) The Berezin range of A (or Berezin set of A) is

$$\operatorname{Ber}(A) := \operatorname{Range}(\widetilde{A}) = \left\{ \widetilde{A}(\varepsilon) : \varepsilon \in \Upsilon \right\}.$$

(iii) The Berezin radius of A (or Berezin number of A) is

$$\operatorname{ber}(A) := \sup_{\varepsilon \in \Upsilon} \left| \widetilde{A}(\varepsilon) \right|.$$

We also define the following so-called Berezin norm of operators $A \in \mathbb{B}(\mathbb{H})$:

$$\|A\|_{\text{Ber}} := \sup_{\varepsilon \in \Upsilon} \left\| A\widehat{k}_{\varepsilon} \right\|.$$

It is easy to see that actually $\|A\|_{\text{Ber}}$ determines a new operator norm in $\mathbb{B}(\mathbb{H}(\Upsilon))$ (since the set of reproducing kernels $\{k_{\varepsilon} : \varepsilon \in \Upsilon\}$ span the space $\mathbb{H}(\Upsilon)$). It is also trivial that $\text{ber}(A) \leq \|A\|_{\text{Ber}} \leq \|A\|$.

For each bounded operator A on $\mathbb{H}$, the Berezin transform $\widetilde{A}$ is a bounded real-analytic function on Υ. Properties of the operator A are often reflected in properties of the Berezin transform $\widetilde{A}$. The Berezin transform itself was introduced by F. Berezin in [3] and has proven to be a critical tool in operator theory, as many foundational properties of important operators are encoded in their Berezin transforms. The Berezin set and number, also denoted by $\text{Ber}(A)$ and $\text{ber}(A)$, respectively, were purportedly first formally introduced by Karaev in [16].

An important inequality for $\text{ber}(A)$ is the power inequality stating that

$$\text{ber}(A^n) \leq \text{ber}(A)^n \tag{1}$$

for $n = 1, 2, ...$; more generally, if A is not nilpotent, then

$$C_1 \text{ber}(A)^n \leq \text{ber}(A^n) \leq C_1 \text{ber}(A)^n$$

for some constants $C_1, C_2 > 0$.

In an FHS, the Berezin range of an operator A is a subset of the numerical range of A,

$$W(A) := \{\langle Ax, x\rangle : x \in \mathbb{H}(\Upsilon) \text{ and } \|x\| = 1\}.$$

Hence

$$\text{ber}(A) \leq w(A) := \sup\{|\langle Ax, x\rangle| : x \in \mathbb{H}(\Upsilon) \text{ and } \|x\| = 1\}$$

(the numerical radius of operator A). An operator's numerical range has a number of interesting characteristics. For instance, it is common knowledge that an operator's numerical range's closure contains the operator's spectrum. We refer to [2, 19, 20, 21, 22] for the fundamental properties of the numerical radius.

Berezin range and Berezin radius of operators are new numerical characteristics of operators on the FHS which are introduced by Karaev in [16]. For these new concepts fundamental characteristics and information, see [4, 5, 6, 7, 8, 9, 10, 12, 17].

Let $\mathbb{B}(\mathbb{H})$ define the C^*-algebra of all bounded linear operators on a Hilbert space. It is well-known that

$$\frac{\|A\|}{2} \leq w(A) \leq \|A\| \tag{2}$$

and

$$\text{ber}(A) \leq w(A) \leq \|A\| \tag{3}$$

for any $T \in \mathbb{B}(\mathbb{H}(\Upsilon))$.

In [13], Huban et al. obtained the following result.

$$\text{ber}(A) \leq \frac{1}{2}\left(\|A\|_{\text{ber}} + \left\|A^2\right\|_{\text{ber}}^{1/2}\right). \tag{4}$$

It has been shown in [11] that if $A \in \mathbb{B}(\mathbb{H}(\Upsilon))$, then

$$\frac{1}{4}\left\||A|^2 + |A^*|^2\right\|_{\text{ber}} \leq \text{ber}^2(A) \leq \frac{1}{2}\left\||A|^2 + |A^*|^2\right\|_{\text{ber}} \tag{5}$$

where $|A| = (A^*A)^{1/2}$ is the absolute value of A.

The this work, we obtain the new Berezin radius inequalities by using improvement of Young inequality. Also, we present several refinements of Berezin radius inequalities in reproducing kernel Hilbert space. Our main results mainly extend and refine the inequalities in [8] and [11].

Known lemmas

The following lemmas are needed for a better understanding of the work.

The first lemma is known generalized mixed Cauchy-Schwarz inequality in the literature and proved by Kittaneh (see, e.g., [15]).

Lemma 2. *Let $A \in \mathbb{L}(\mathbb{H})$ and $x, y \in \mathbb{H}$ be any vector.*

(i) If $0 \leq \alpha \leq 1$, then

$$|\langle Ax, y\rangle| \leq \sqrt{\left\langle |A|^{2\alpha} x, x\right\rangle \left\langle |A^*|^{2(1-\alpha)} y, y\right\rangle}, (0 \leq \alpha \leq 1) \tag{6}$$

(ii) If f, g are nonnegative continuous functions on $[0,\infty]$ satisfying $f(t).g(t) = t$, $(t \geq 0)$ then

$$|\langle Ax, y\rangle| \leq \|f(|A|)x\| \|g(|A^*|)y\|. \tag{7}$$

The second lemma follows from the spectral theorem for positive operators and Jensen's inequality and is known as Hölder-McCarthy inequality (see, e.g., [14]).

Lemma 3. *Let $A \in \mathbb{B}(\mathbb{H})$ be positive, and $x \in \mathbb{H}$ be any unit vector. Then*

$$\langle A^r x, x\rangle \leq \langle A^r x, x\rangle^r \text{ for } 0 < r \leq 1. \tag{8}$$

$$\langle Ax, x\rangle^r \leq \langle A^r x, x\rangle \text{ for } r \geq 1. \tag{9}$$

Lemma 4. *For $a, b \geq 0$, $0 \leq \alpha \leq 1$, $r \geq 1$ and $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$, we have*

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q} \leq \left(\frac{a^{pr}}{p} + \frac{b^{qr}}{q}\right)^{\frac{1}{r}}. \tag{10}$$

In [18, Theorem 2.1], Minclute expressed an improvement of the Young inequality as follows:

Lemma 5. *Let $a, b \geq 0$ and $\upsilon \in (0,1)$. Then*

$$a^\upsilon b^{1-\upsilon}\left(\frac{a+b}{2\sqrt{ab}}\right)^{2\mu} \leq \upsilon a + (1-\upsilon) b \tag{11}$$

where $\mu = \min\{\upsilon, 1-\upsilon\}$.

Notice that, if $0 < m \leq a, b \leq M$, then $\frac{m+M}{2\sqrt{mM}} \leq \frac{a+b}{\sqrt{ab}}$. Based on this fact, from the inequality (11) we get

$$a^\upsilon b^{1-\upsilon}\left(\frac{m+M}{2\sqrt{mM}}\right)^{2\mu} \leq \upsilon a + (1-\upsilon) b \tag{12}$$

where $\mu = \min\{\upsilon, 1-\upsilon\}$.

Main results

We have the following related inequality associated with inequality (12).

Theorem 6. *Let $\mathbb{H} = \mathbb{H}(\Upsilon)$ be a FHS. Let $T, R, S \in \mathbb{B}(\mathbb{H})$ such that T, R are positive, and*

$$m \leq T^r \leq M,\ m \leq R^r \leq M.$$

Then

$$\mathrm{ber}^r\left(T^\alpha S R^{1-\alpha}\right) \leq \left(\frac{m+M}{2\sqrt{mM}}\right)^{-2\mu} \|S\|^r \|\alpha T^r + (1-\alpha) R^r\|_{\mathrm{ber}} \tag{13}$$

for every $0 \leq \alpha \leq 1$, $r \geq 2$ and $\mu = \min\{\alpha, 1-\alpha\}$.

Proof. Let $\varepsilon \in \Upsilon$ be an arbitrary number. Then from the Cauchy-Schwarz inequality

$$\begin{aligned}
&\left|\left\langle T^{\alpha} S R^{1-\alpha} \widehat{k}_{\varepsilon}, \widehat{k}_{\varepsilon}\right\rangle\right|^{r} \\
&\leq \left|\left\langle S R^{1-\alpha} \widehat{k}_{\varepsilon}, T^{\alpha} \widehat{k}_{\varepsilon}\right\rangle\right|^{r} \leq \|S\|^{r}\left\|R^{1-\alpha} \widehat{k}_{\varepsilon}\right\|^{r}\left\|T^{\alpha} \widehat{k}_{\varepsilon}\right\|^{r} \\
&\leq \|S\|^{r}\left\langle R^{2(1-\alpha)} \widehat{k}_{\varepsilon}, \widehat{k}_{\varepsilon}\right\rangle^{r/2}\left\langle T^{2\alpha} \widehat{k}_{\varepsilon}, \widehat{k}_{\varepsilon}\right\rangle^{r/2} \\
&\leq \|S\|^{r}\left\langle R^{r} \widehat{k}_{\varepsilon}, \widehat{k}_{\varepsilon}\right\rangle^{(1-\alpha)}\left\langle T^{r} \widehat{k}_{\varepsilon}, \widehat{k}_{\varepsilon}\right\rangle^{\alpha} \text{ (by Lemma 3)} \qquad (14) \\
&\leq \left(\frac{m+M}{2\sqrt{mM}}\right)^{-2\mu}\|S\|^{r}\left((1-\alpha)\left\langle R^{r} \widehat{k}_{\varepsilon}, \widehat{k}_{\varepsilon}\right\rangle+\alpha\left\langle T^{r} \widehat{k}_{\varepsilon}, \widehat{k}_{\varepsilon}\right\rangle\right) \text{ (by (12))} \\
&\leq \left(\frac{m+M}{2\sqrt{mM}}\right)^{-2\mu}\|S\|^{r}\left\langle\left((1-\alpha) R^{r}+\alpha T^{r}\right) \widehat{k}_{\varepsilon}, \widehat{k}_{\varepsilon}\right\rangle
\end{aligned}$$

where $\mu = \min\{\upsilon, 1-\upsilon\}$. Taking the supremum over $\varepsilon \in \Upsilon$, we get

$$\sup_{\varepsilon\in\Upsilon}\left|\left\langle T^{\alpha} S R^{1-\alpha} \widehat{k}_{\varepsilon}, \widehat{k}_{\varepsilon}\right\rangle\right|^{r} \leq \left(\frac{m+M}{2\sqrt{mM}}\right)^{-2\mu}\|S\|^{r}\sup_{\varepsilon\in\Upsilon}\left(\left\langle\left((1-\alpha) R^{r}\alpha T^{r}\right) \widehat{k}_{\varepsilon}, \widehat{k}_{\varepsilon}\right\rangle\right)$$

where $\mu = \min\{\upsilon, 1-\upsilon\}$. Thus, we have

$$\operatorname{ber}^{r}\left(T^{\alpha} S R^{1-\alpha}\right) \leq \left(\frac{m+M}{2\sqrt{mM}}\right)^{-2\mu}\|S\|^{r}\left\|\alpha T^{r}+(1-\alpha) R^{r}\right\|_{\text{ber}}$$

where $\mu = \min\{\upsilon, 1-\upsilon\}$. □

As a result of above theorem, we get:

Corollary 7. *Suppose that the assumptions of Theorem 6 be satisfied. Then*

$$\operatorname{ber}^{r}\left(T^{\frac{1}{2}} S R^{\frac{1}{2}}\right) \leq \left(\frac{m+M}{2\sqrt{mM}}\right)^{-2\mu}\|S\|^{r}\left\|\frac{T^{\frac{1}{2}}+R^{\frac{1}{2}}}{2}\right\|_{\text{ber}}. \qquad (15)$$

The following theorem can be stated as well.

Theorem 8. *Let $\mathbb{H} = \mathbb{H}(\Upsilon)$ be a FHS. Let $T, R, S \in \mathbb{B}(\mathbb{H})$ such that T, R are positive, and*

$$m \leq T^{r} \leq M,\ m \leq R^{r} \leq M.$$

Then

$$\operatorname{ber}^{r}\left(\frac{T^{\alpha} S R^{1-\alpha}+R^{\alpha} S T^{1-\alpha}}{2}\right) \leq \left(\frac{m+M}{2\sqrt{mM}}\right)^{-2\mu}\|S\|^{r}\left\|\frac{T^{r}+R^{r}}{2}\right\|_{\text{ber}} \qquad (16)$$

for all $0 \leq \alpha \leq 1$, $r \geq 2$ and $\mu = \min\{\alpha, 1-\alpha\}$.

Proof. Let $\varepsilon \in \Upsilon$ be an arbitrary. From inequality (14), we get

$$\left|\left\langle T^{\alpha} S R^{1-\alpha} \widehat{k}_{\varepsilon}, \widehat{k}_{\varepsilon}\right\rangle\right|^{r} \leq \left(\frac{m+M}{2\sqrt{mM}}\right)^{-2\mu}\left\langle\left((1-\alpha) R^{r}+\alpha T^{r}\right) \widehat{k}_{\varepsilon}, \widehat{k}_{\varepsilon}\right\rangle$$

for every $0 \leq \alpha \leq 1$, $r \geq 2$ and $\mu = \min\{\alpha, 1-\alpha\}$. Therefore

$$\begin{aligned}
&\left|\left\langle\left(\frac{T^{\alpha}SR^{1-\alpha}+R^{\alpha}ST^{1-\alpha}}{2}\right)\widehat{k}_{\varepsilon},\widehat{k}_{\varepsilon}\right\rangle\right|^{r} \\
&\leq \left\langle\left(\frac{\left|T^{\alpha}SR^{1-\alpha}\right|+\left|R^{\alpha}ST^{1-\alpha}\right|}{2}\right)\widehat{k}_{\varepsilon},\widehat{k}_{\varepsilon}\right\rangle^{r} \\
&\leq \left\langle\left(\frac{\left|T^{\alpha}SR^{1-\alpha}\right|^{r}+\left|R^{\alpha}ST^{1-\alpha}\right|^{r}}{2}\right)\widehat{k}_{\varepsilon},\widehat{k}_{\varepsilon}\right\rangle \qquad (17) \\
&\leq \left(\frac{m+M}{2\sqrt{mM}}\right)^{-2\mu}\left\|\frac{S}{2}\right\|^{r}\left(\left\langle\left(\alpha R^{r}+(1-\alpha)T^{r}\right)\widehat{k}_{\varepsilon},\widehat{k}_{\varepsilon}\right\rangle+\left\langle\left((1-\alpha)R^{r}+\alpha T^{r}\right)\widehat{k}_{\varepsilon},\widehat{k}_{\varepsilon}\right\rangle\right) \\
&\leq \left(\frac{m+M}{2\sqrt{mM}}\right)^{-2\mu}\|S\|^{r}\left\langle\left(\frac{T^{r}+R^{r}}{2}\right)\widehat{k}_{\varepsilon},\widehat{k}_{\varepsilon}\right\rangle.
\end{aligned}$$

Taking the supremum over $\varepsilon \in \Upsilon$ in the above inequality, we get

$$\sup_{\varepsilon\in\Upsilon}\left|\left\langle\left(\frac{T^{\alpha}SR^{1-\alpha}+R^{\alpha}ST^{1-\alpha}}{2}\right)\widehat{k}_{\varepsilon},\widehat{k}_{\varepsilon}\right\rangle\right|^{r} \leq \left(\frac{m+M}{2\sqrt{mM}}\right)^{-2\mu}\|S\|^{r}\sup_{\varepsilon\in\Upsilon}\left\langle\left(\frac{T^{r}+R^{r}}{2}\right)\widehat{k}_{\varepsilon},\widehat{k}_{\varepsilon}\right\rangle.$$

Hence, we have

$$\operatorname{ber}^{r}\left(\frac{T^{\alpha}SR^{1-\alpha}+R^{\alpha}ST^{1-\alpha}}{2}\right) \leq \left(\frac{m+M}{2\sqrt{mM}}\right)^{-2\mu}\|S\|^{r}\left\|\frac{T^{r}+R^{r}}{2}\right\|_{\text{ber}}.$$

□

The following result concerning the sums of two operators can be stated as well:

Theorem 9. *Let $\mathbb{H} = \mathbb{H}(\Upsilon)$ be a FHS. Let $T, R \in \mathbb{B}(\mathbb{H})$. Then*

$$\operatorname{ber}^{r}(T+R) \leq \sqrt{\left\|\frac{1}{p}|T+R|^{2pr\alpha}+\frac{1}{q}\left|(T+R)^{*}\right|^{2qr(1-\alpha)}\right\|_{\text{ber}}} \qquad (18)$$

for $0 \leq \alpha \leq 1$, $r \geq 1$ with $\frac{1}{p}+\frac{1}{q}=1$.

Proof. Let $\widehat{k}_{\varepsilon}$ be a normalized reproducing kernel. Then

$$\begin{aligned}
&\left|\left\langle(T+R)\widehat{k}_{\varepsilon},\widehat{k}_{\varepsilon}\right\rangle\right|^{2r} \\
&\leq \left\langle|T+R|^{2\alpha}\widehat{k}_{\varepsilon},\widehat{k}_{\varepsilon}\right\rangle^{r}\left\langle\left|(T+R)^{*}\right|^{2(1-\alpha)}\widehat{k}_{\varepsilon},\widehat{k}_{\varepsilon}\right\rangle^{r} \text{ (by Lemma 2 } (i)\text{)} \\
&\leq \left\langle|T+R|^{2r\alpha}\widehat{k}_{\varepsilon},\widehat{k}_{\varepsilon}\right\rangle\left\langle\left|(T+R)^{*}\right|^{2r(1-\alpha)}\widehat{k}_{\varepsilon},\widehat{k}_{\varepsilon}\right\rangle \text{ (by Lemma 3 } (ii)\text{)} \\
&\leq \frac{1}{p}\left\langle|T+R|^{2r\alpha}\widehat{k}_{\varepsilon},\widehat{k}_{\varepsilon}\right\rangle^{p}+\frac{1}{q}\left\langle\left|(T+R)^{*}\right|^{2r(1-\alpha)}\widehat{k}_{\varepsilon},\widehat{k}_{\varepsilon}\right\rangle^{q} \text{ (by Lemma 4)} \\
&\leq \frac{1}{p}\left\langle|T+R|^{2pr\alpha}\widehat{k}_{\varepsilon},\widehat{k}_{\varepsilon}\right\rangle+\frac{1}{q}\left\langle\left|(T+R)^{*}\right|^{2qr(1-\alpha)}\widehat{k}_{\varepsilon},\widehat{k}_{\varepsilon}\right\rangle \text{ (by Lemma 3 } (ii)\text{)} \\
&\leq \left\langle\left(\frac{1}{p}|T+R|^{2pr\alpha}+\frac{1}{q}\left|(T+R)^{*}\right|^{2qr(1-\alpha)}\right)\widehat{k}_{\varepsilon},\widehat{k}_{\varepsilon}\right\rangle.
\end{aligned}$$

Taking the supremum over $\varepsilon \in \Upsilon$ in the above inequality, we have

$$\sup_{\varepsilon\in\Upsilon}\left|\left\langle(T+R)\widehat{k}_{\varepsilon},\widehat{k}_{\varepsilon}\right\rangle\right|^{2r} \leq \sup_{\varepsilon\in\Upsilon}\left\langle\left(\frac{1}{p}|T+R|^{2pr\alpha}+\frac{1}{q}\left|(T+R)^{*}\right|^{2qr(1-\alpha)}\right)\widehat{k}_{\varepsilon},\widehat{k}_{\varepsilon}\right\rangle.$$

Therefore, we get

$$\operatorname{ber}^r(T+R) \leq \sqrt{\left\| \frac{1}{p} |T+R|^{2pr\alpha} + \frac{1}{q} \left|(T+R)^*\right|^{2qr(1-\alpha)} \right\|_{\text{ber}}}.$$

This completes the proof. □

Remark 3. *If we take $R = T$, $p = q = 2$, $r = 1$, and $\alpha = \frac{1}{2}$ in the Theorem 9, we have*

$$\operatorname{ber}(T) \leq \frac{1}{2} \sqrt{\left\| 2\left(|T|^2 + |T^*|^2\right) \right\|_{\text{ber}}}$$

which is equivalent to

$$\operatorname{ber}(T) \leq \frac{1}{2} \left\| |T|^2 + |T^*|^2 \right\|_{\text{ber}}. \tag{19}$$

Also, inequality (19) have been found by Huban et al. in [11].

Corollary 10. *Let $T, R \in \mathbb{B}(\mathbb{H})$, such that*

$$\sqrt[r]{m} \leq |T+R|^{2\alpha} \leq \sqrt[r]{M},\ \sqrt[r]{m} \leq \left|(T+R)^*\right|^{2(1-\alpha)} \leq \sqrt[r]{M}.$$

Then by using the inequality (12), we deduce

$$\operatorname{ber}^r(T+R) \leq \left(\frac{m+M}{2\sqrt{mM}}\right)^{-1} \left\| \frac{|T+R|^{2r\alpha} + \left|(T+R)^*\right|^{2r(1-\alpha)}}{2} \right\|_{\text{ber}}.$$

In particular

$$\operatorname{ber}^r(T+R) \leq \left(\frac{m+M}{2\sqrt{mM}}\right)^{-1} \left\| \frac{|T+R|^{r} + \left|(T+R)^*\right|^{r}}{2} \right\|_{\text{ber}}.$$

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Remarks on certain families of arithmetic sequence based on Fibonacci and Lucas numbers

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The goal of this presentation is to give further remarks and observations on certain families of arithmetic sequences associated with the Fibonacci numbers and the Lucas numbers. We give relations between of these arithmetic sequences. We also give some properties for these sequences.

2010 Mathematics Subject Classifications: 05A15, 11B37, 11B39

Keywords: Arithmetic sequence, Fibonacci numbers, Lucas numbers, Golden ratio, Generating functions

Introduction

An arithmetic sequence has many applications all most all areas of sciences. The arithmetic sequence is a sequence of numbers such that the difference between the consecutive terms is constant, denoted by k. The n-th term of any arithmetic sequence, which have the common difference k and the first term a_1, is given by the following formula:

$$a_n = a_1 + (n-1)\,k$$

(*cf.* [1]-[11]).

In [2, 3], we constructed the following sequence, $(h_n(m,k))_{n=1}^{\infty}$:

$$h_n(m,k) = F_m + (n-1)\,k, \tag{1}$$

where $m, k \in \mathbb{N} \cup \{0\}$ and $n \in \mathbb{N}$ and F_m denotes the Fibonacci numbers, which are given by

$$F_{m+2} = F_{m+1} + F_m; \qquad m \in \mathbb{N} \cup \{0\}$$

with $F_0 = 0$ and $F_1 = 1$ (*cf.* [1]-[11]).

In [4, 5], we also constructed the following certain family of arithmetic sequence associated with the Lucas numbers, $(H_n(m,k))_{n=1}^{\infty}$:

$$H_n(m,d) = L_m + (n-1)\,k, \tag{2}$$

where $m, k \in \mathbb{N} \cup \{0\}$ and $n \in \mathbb{N}$ and L_m denotes the Lucas numbers, which are given by

$$L_{m+2} = L_{m+1} + L_m; \qquad m \in \mathbb{N} \cup \{0\}$$

where $L_0 = 2$ and $L_1 = 1$ (*cf.* [1]-[10]).

Formulas and relations for certain families of arithmetic sequences based on Fibonacci numbers and Lucas numbers

In this section, we give some formulas and relations for the arithmetic sequences $h_n(m,k)$ and $H_n(m,k)$.

Theorem 1. *Let $m,k \in \mathbb{N} \cup \{0\}$ and $n \in \mathbb{N}$. Then we have*

$$h_n(m,k) = \frac{H_n(m-1,k) + H_n(m+1,k) + 2(1-n)k}{5}.$$

Proof. By putting the following formula (*cf.* [7, 8]):

$$5F_m = L_{m-1} + L_{m+1} \tag{3}$$

in the equation (1), we obtain

$$h_n(m,k) = \frac{L_{m-1} + L_{m+1}}{5} + (n-1)k. \tag{4}$$

If we combine the following equation

$$H_n(m,k) + (1-n)k = L_m \tag{5}$$

with the equation (4), we get

$$h_n(m,k) = \frac{H_n(m-1,d) + H_n(m+1,k) + 2(1-n)k}{5},$$

where $m,k \in \mathbb{N} \cup \{0\}$ and $n \in \mathbb{N}$. Thus proof the theorem is completed. □

By substituting the following formula (*cf.* [7, 8]):

$$L_m = F_{m-1} + F_{m+1} \tag{6}$$

in the equation (5), we obtain the following result:

Corollary 2. *Let $m,k \in \mathbb{N} \cup \{0\}$ and $n \in \mathbb{N}$. Then we have*

$$H_n(m,k) = F_{m-1} + F_{m+1} + (1-n)k.$$

Theorem 3. *Let $m,k \in \mathbb{N} \cup \{0\}$ and $n \in \mathbb{N}$. Then we have*

$$\sum_{j=1}^{n} h_j(m,k) + H_j(m,k) = n(F_{m-1} + F_{m+2}) + kn^2.$$

Proof. By using partial sums of the sequences $h_n(m,k)$ and $H_n(m,k)$, we get

$$\sum_{j=1}^{n} h_j(m,k) + H_j(m,k) = n(L_m + F_m) + kn^2. \tag{7}$$

Substituting equation (6) into the above equation, after some arithmetical operations, we obtain

$$\sum_{j=1}^{n} h_j(m,k) + H_j(m,k) = n(F_{m-1} + F_{m+1} + F_m) + kn^2.$$

By using recurrence relation for the Fibonacci numbers, proof of theorem is completed. □

Theorem 4. *Let $m, k \in \mathbb{N} \cup \{0\}$ and $n \in \mathbb{N}$. Then we have*

$$\sum_{j=1}^{n} h_j(m,k) + H_j(m,k) = n\left(\frac{L_{m-1} + L_{m+2} + 4L_m}{5}\right) + kn^2.$$

Proof. Substituting equation (3) into the equation (7), after some arithmetical operations, we obtain

$$\sum_{j=1}^{n} h_j(m,k) + H_j(m,k) = n\left(\frac{L_{m-1} + L_{m+1} + 5L_m}{5}\right) + kn^2.$$

Combining a recurrence relation for the Lucas numbers with equation (7) into the above equation, after some arithmetical operations, we complete, proof of theorem. □

Conclusion

In our future investigates by the help of the sequences $h_n(m,k)$ and $H_n(m,k)$, we will investigate applications of these sequences in the real world problems. Furthermore, we will also investigate other sequences associated with the sequences $h_n(m,k)$ and $H_n(m,k)$.

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Modification of new generalization Kantorovich type Bernstein operators via (p,q)-calculus

Hayatem Hamal

The purpose of this paper is to present a new Kantorovich modification of Bernstein operators by means of (p,q)-calculus for $0<q<p\leq 1, n\in\mathbb{N}$ and $f\in C\left[0,1+l\right], l\in\mathbb{N}_0$, which is introduced as follows:

$$\Omega_{n,l,\alpha}^{p,q}\left(f,x\right)=\sum_{k=0}^{n+l} b_{n,k}^{l}(p,q,x)\int_0^1 f\left(\frac{p^{n+l-k}\left([k]_{p,q}+q^k t^{\alpha}\right)}{[n+l+1]_{p,q}}\right)\ d_{p,q}t,\quad \alpha>0,$$

where

$$b_{n,k}^{l,p,q}\left(x\right)\ =\ \frac{1}{p^{\frac{(n+l)(n+l+1)}{2}}}\left[\begin{array}{c} n+l \\ k \end{array}\right]_{p,q} p^{\frac{k(k-1)}{2}} r_{n,l}^{p,q}\left(x\right)\ \left(1-r_{n,l}^{p,q}\left(x\right)\right)_{p,q}^{n+l-k},$$

and

$$r_{n,l}^{p,q}\left(x\right)\ =\ \frac{[n]_{p,q}}{[n+l]_{p,q}}x,\ \text{for } x\in[0,1],\ 0<r_{n,l}^{p,q}\left(x\right)\leq 1.$$

We consider and investigate the special cases, $0<p<q\leq 1$, $1\leq p<q<\infty$ and $1\leq q<p<\infty$. We derive a recurrence formula for these newly defined operators to provide explicit formulas for the m-th order moments and central moments, which are essential in approximation theory.

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Keywords: (p,q)-calculus, Bernstein polynomials, Kantorovich type operators

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Bilinear multipliers of the spaces with fractional wavelet transform

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This work is motivated to consider the bilinear multipliers of the spaces with fractional wavelet transform in [4] and find examples of bilinear multipliers.

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Keywords: Bilinear multipliers, Fractional wavelet transform

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Analysis of a new class of q-combinatorial numbers and polynomials by its implementation in the Wolfram Language

Irem Kucukoglu

The main aim of this presentation is to introduce a new class of q-combinatorial numbers and q-combinatorial polynomials and to implement their computational formulas by writing a procedure in the Wolfram Language. As the outputs of this implementation and auxiliary commands, we also present some tables and plots which help to be able to analyze the values and characteristics of these numbers and polynomials.

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Keywords: Binomial coefficients, Factorials, Finite sums, Special numbers and polynomials, Combinatorial numbers, q-calculus, q-Stirling numbers of the second kind, Computational implementation

Introduction and Preliminaries

In this study, we are motivated to introduce a new class of numbers and polynomials, which contains some concepts of q-calculus and generalizes some combinatorial numbers and polynomials. The other motivation of this paper is to analyse this new class by an implementation in which we provide a procedure in the Wolfram Language and this procedure are able to return the numbers and polynomials introduced as members of the q-combinatorial class whose tables and graphs are presented at the end of this work.

Before giving the results of this study, we get start with recalling to mind some notations and definitions regarding the methods of q-calculus which are frequently used in obtaining these results.

Along this study, we let $\mathbb{C}$, $\mathbb{R}$ and $\mathbb{N}$ to denote respectively the set of complex numbers, the set of real numbers and the set of positive integers, as usual. Also, we let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

A q-analogue of any positive integer k (i.e. q-integer) is defined by

$$[k]_q := \frac{1-q^k}{1-q} = 1 + q + q^2 + \cdots + q^{k-1}; \qquad q \in \mathbb{C} \setminus \{1\}, \quad k \in \mathbb{N}_0 \tag{1}$$

such that

$$\lim_{q \to 1} [k]_q = k$$

(*cf.* [4]; and also see [2, 3]).

A q-analogue of binomial coefficients (i.e. q-binomial coefficients), is defined by

$$\left[\begin{array}{c} k \\ j \end{array}\right]_q := \frac{[k]_q!}{[j]_q!\,[k-j]_q!}; \qquad j=0,1,\ldots,k \tag{2}$$

in which $[k]_q!$ denotes a q-analogue of the well-known factorial concept (i.e. q-factorial), given as follows:

$$[k]_q! := \begin{cases} 1, & k=0, \\ [k]_q\,[k-1]_q \ldots [2]_q\,[1]_q\,, & k \in \mathbb{N} \end{cases} \tag{3}$$

(*cf.* [4]; and also see [2, 3]).

Note that the q-binomial coefficients, given in (2), are reduced to the ordinary binomial coefficients in the limit case when $q \to 1$ (*cf.* [2, 3, 4]).

A new class of q-combinatorial numbers and polynomials

In this section, we introduce a new class of q-combinatorial numbers and polynomials by the following definition:

Definition 1. *Let $n,k,r \in \mathbb{N}_0$, and $\lambda \in \mathbb{R}$ (or $\mathbb{C}$). Then, the q-combinatorial numbers $y_{6,q}(n,k;\lambda,r)$ and the q-combinatorial polynomials $y_{6,q}(x;n,k;\lambda,r)$ are respectively defined by the following finite sums:*

$$y_{6,q}(n,k;\lambda,r) = \frac{1}{[k]_q!}\sum_{j=0}^{k} q^{\binom{j}{2}} \left[\begin{array}{c} k \\ j \end{array}\right]_q^r [j]_q^n \lambda^j \tag{4}$$

and

$$y_{6,q}(x;n,k;\lambda,r) = \frac{1}{[k]_q!}\sum_{j=0}^{k} q^{\binom{j}{2}} \left[\begin{array}{c} k \\ j \end{array}\right]_q^r [x+j]_q^n \lambda^j. \tag{5}$$

Remark 4. *By setting $r=1$ in (4) and (5), we have*

$$y_{6,q}(n,k;\lambda,1) = y_{1,q}(n,k;\lambda) \tag{6}$$

and

$$y_{6,q}(x;n,k;\lambda,1) = y_{1,q}(x;n,k;\lambda) \tag{7}$$

where $y_{1,q}(n,k;\lambda)$ and $y_{1,q}(x;n,k;\lambda)$ denote members of another class of q-combinatorial numbers and polynomials recently introduced and investigated by Kucukoglu in [5] and [6].

Remark 5. *The case of (4) when q approaches to 1 implies*

$$\lim_{q\to 1} y_{6,q}(n,k;\lambda,r) = y_6(n,k;\lambda,r)$$

and

$$\lim_{q\to 1} y_{6,q}(x;n,k;\lambda,r) = P(x;n,k;\lambda,r)$$

where $y_6(n,k;\lambda,r)$ and $P(x;n,k;\lambda,r)$ denote members of a class of combinatorial numbers and polynomials recently introduced and investigated by Simsek in [7] and [9].

Remark 6. *The case of* (4) *when* q *approaches to* 1 *and* $r = 1$ *yields*

$$\lim_{q \to 1} y_{6,q}(n, k; \lambda, 1) = y_1(n, k; \lambda)$$

where $y_1(n, k; \lambda)$ *denote members of another class of combinatorial numbers recently introduced and investigated by Simsek in* [8].

Remark 7. *The case of* (4) *when* q *approaches to* 1 *and* $\lambda = 1$ *gives*

$$\lim_{q \to 1} y_{6,q}(n, k; 1, r) = \frac{1}{k!} \sum_{j=0}^{k} \binom{k}{j}^r j^n$$

which is associated with multifarious special finite sums. For details, see [9] *and the references cited therein.*

Remark 8. *In the special case when* $r = 1$ *and* $\lambda = -1$*, the equation* (4) *yields*

$$y_{6,q}(n, k; -1, 1) = q^{\binom{k}{2}} S_{2,q}(n, k) \tag{8}$$

where $S_{2,q}(n, k)$ *denotes the* q*-Stirling numbers of the second kind defined by Carlitz in* [1].

Computational Implementation of $y_{6,q}(x; n, k; \lambda, r)$ in the Wolfram Language

In this section, for symbolic computation of $y_{6,q}(x; n, k; \lambda, r)$, we provide a procedure in the Wolfram Language (see: Implementation 1). By executing this procedure in the Wolfram Cloud (*cf.* [10]) and using the commands `TableForm` and `Plot`, we present two tables for $y_{6,q}(x; n, k; \lambda, r)$ and plots for $y_{6,q}(x; n, k; \lambda, r)$ obtained just for a few special cases (among others).

Implementation 1: The following code, in the Wolfram Language, involves the procedure `y6` which returns the q-combinatorial polynomials $y_{6,q}(x; n, k; \lambda, r)$.

```
Unprotect[Power];
Power[0,0]=1;
Protect[Power];
qinteger[j_, q_]:=(1−q^j)/(1−q);
y6[q_, x_, n_, k_, \[Lambda]_, r_]:=(1/QFactorial[k, q])∗Sum[(q^Binomial[j, 2])
∗ ((QBinomial[k, j, q])^r)∗(qinteger[x + j, q]^n)∗ \[Lambda]^j, {j, 0, k}];
```

	k=1	k=2	k=3	k=4
n=1	λ	$\frac{1}{4}\lambda(9+2\lambda)$	$\frac{1}{96}\lambda(196+147\lambda+8\lambda^2)$	$\frac{\lambda(7200+8575\lambda+1575\lambda^2+32\lambda^3)}{5376}$
n=2	λ	$\frac{3}{4}\lambda(3+\lambda)$	$\frac{7}{192}\lambda(56+63\lambda+4\lambda^2)$	$\frac{5\lambda(1920+3430\lambda+735\lambda^2+16\lambda^3)}{7168}$
n=3	λ	$\frac{9}{8}\lambda(2+\lambda)$	$\frac{49}{384}\lambda(16+27\lambda+2\lambda^2)$	$\frac{75\lambda(512+1372\lambda+343\lambda^2+8\lambda^3)}{28672}$
n=4	λ	$\frac{9}{16}\lambda(4+3\lambda)$	$\frac{49}{768}\lambda(32+81\lambda+7\lambda^2)$	$\frac{75\lambda(2048+8232\lambda+2401\lambda^2+60\lambda^3)}{114688}$

Table 1: The values of $y_{6,q}(x, n, k; \lambda, r)$ for the special case when $q = \frac{1}{2}$, $x = 0$, $n \in \{1, 2, 3, 4\}$, $k \in \{1, 2, 3, 4\}$ and $r = 3$.

	k=1	k=2
n=1	$2^{-x}\left(2(-1+2^{x})+(-1+2^{1+x})\lambda\right)$	$\frac{1}{3}\times 2^{-2-x}\left(16(-1+2^{x})+27(-1+2^{1+x})\lambda+(-2+2^{3+x})\lambda^{2}\right)$
n=2	$4^{-x}\left(4(-1+2^{x})^{2}+(-1+2^{1+x})^{2}\lambda\right)$	$\frac{1}{3}\times 4^{-1-x}\left(32(-1+2^{x})^{2}+27(-1+2^{1+x})^{2}\lambda+(-1+2^{2+x})^{2}\lambda^{2}\right)$
n=3	$8^{-x}\left(8(-1+2^{x})^{3}+(-1+2^{1+x})^{3}\lambda\right)$	$\frac{1}{3}\times 8^{-1-x}\left(128(-1+2^{x})^{3}+54(-1+2^{1+x})^{3}\lambda+(-1+2^{2+x})^{3}\lambda^{2}\right)$
n=4	$16\left((-1+2^{-x})^{4}+(-1+2^{-1-x})^{4}\lambda\right)$	$\frac{2}{3}\left(16(-1+2^{-x})^{4}+54(-1+2^{-1-x})^{4}\lambda+8(-1+2^{-2-x})^{4}\lambda^{2}\right)$
n=5	$32\left((1-2^{-x})^{5}+(1-2^{-1-x})^{5}\lambda\right)$	$\frac{8}{3}\left(8(1-2^{-x})^{5}+27(1-2^{-1-x})^{5}\lambda+4(1-2^{-2-x})^{5}\lambda^{2}\right)$
n=6	$64\left((-1+2^{-x})^{6}+(-1+2^{-1-x})^{6}\lambda\right)$	$\frac{16}{3}\left(8(-1+2^{-x})^{6}+27(-1+2^{-1-x})^{6}\lambda+4(-1+2^{-2-x})^{6}\lambda^{2}\right)$

Table 2: The values of $y_{6,q}\,(x;n,k;\lambda,r)$ for the special case when $q=\frac{1}{2}$, $n\in\{1,2,3,4,5,6\}$, $k\in\{1,2\}$ and $r=3$.

Observe from Table 2 that the functions $y_{6,q}(x;n,k;\lambda,r)$ are polynomials of the variable λ, however, they are a linear combination of exponential functions when evaluated with respect to the variable x.

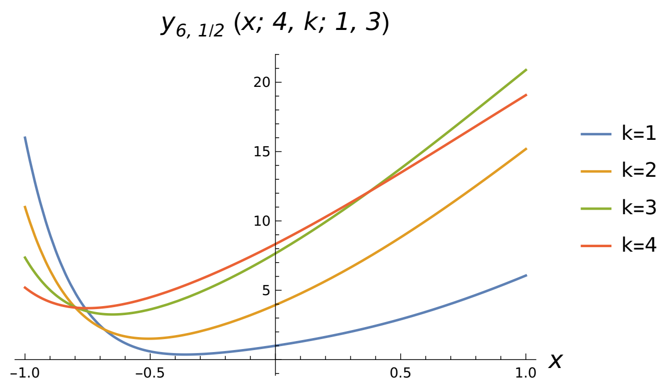

Figure 1: The plots of $y_{6,q}\,(x;n,k;\lambda,r)$ for the special case: $q=\frac{1}{2}$, $x\in[-1,1]$, $n=4$, $k\in\{1,2,3,4\}$, $\lambda=1$ and $r=3$.

Conclusion

In this study, various results and analysis on a new class of q-combinatorial numbers and polynomials has been presented that may find an application area in computational science and engineering.

Acknowledgments

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Notes on an initial value problem involving the generating functions of the Poisson-Charlier polynomials

Irem Kucukoglu

The two main aims of this presentation is to handle an initial value problem, which is mainly associated with the generating functions of the Poisson-Charlier polynomials, and to provide a research on which polynomial family the solution of this problem produces. Furthermore, in this study, we analyse some properties of this polynomial family mentioned above by calling them as Poisson-Charlier type polynomials. Finally, we conclude this presentation by presenting some remarks and observations on the findings of this study.

2010 Mathematics Subject Classifications: 05A10, 05A15, 11B83, 11S23, 33C47, 34A12

Keywords: Bernoulli numbers of the second kind, Cauchy numbers of the first kind, Existence and uniqueness, Generating function, Initial value problem, Integral representation, Orthogonal polynomials, Poisson-Charlier polynomials, Special numbers and polynomials, Stirling numbers of the first kind

Introduction and Preliminaries

In this study, we are concerned with the analysis of an initial value problem involving the generating functions of the Poisson-Charlier polynomials. As the result of this analysis, we introduce a new family of polynomials, which we call the Poisson-Charlier type polynomials.

Before deriving the aforementioned results of this study, we get start with recalling to mind some notations and definitions which are frequently used in obtaining these results. Along this study, we let

$$(w)_m = \prod_{j=1}^{m} (w - j + 1)$$

to denote the falling (descending) factorial in which m is a non-negative integer, w is a complex number such that $(w)_0 = 1$ (*cf.* [1], [8]).

The falling factorial $(w)_m$ can be given in term of the Stirling numbers of the first kind, $s(m,k)$, as in the following way (*cf.* [1], [8]):

$$(w)_m = \sum_{k=0}^{m} s(m,k)\, w^k. \tag{1}$$

The Poisson-Charlier polynomials, $C_n(x;a)$, are defined by the following generating function:

$$F_{pc}(t,x;a) = \left(\frac{t}{a}+1\right)^x \exp(-t) = \sum_{n=0}^{\infty} C_n(x;a)\frac{t^n}{n!}, \tag{2}$$

so that a is any nonzero number (*cf.* [3]-[10]).

To emphasize its importance, it may be noted here that the polynomials $C_n(x;a)$ forms a family of orthogonal polynomials satisfying the relation below:

$$\sum_{j=0}^{\infty} C_n(j;a)C_m(j;a)f(j|a) = a^{-n}n!\delta_{n,m} \tag{3}$$

where $f(j|a) = \frac{a^j \exp(-a)}{j!}$ and $\delta_{n,m}$ is the Kronecker delta (*cf.* [9], [10]; and see also cited references therein).

Because of the above feature, the Poisson-Charlier polynomials have been studied in different manners by many researchers hitherto, and as a result of these researches, it was seen that these polynomials are associated with many different concepts in various areas of probability theory, approximation theory and number theory. For instances, the Poisson-Charlier polynomials are members of the family of Sheffer-type sequences and are also known as orthogonal to the Poisson distribution. For other concepts associated with the Poisson-Charlier polynomials, refer to the works [3]-[10].

Note that the Poisson-Charlier polynomials can be computed by the computation formula below:

$$C_n(x;a) = \sum_{j=0}^{n} (-1)^{n-j}\binom{n}{j}\frac{(x)_j}{a^j}, \tag{4}$$

where $\binom{n}{j}$ is a binomial coefficient, and this formula yields the following first few values of $C_n(x;a)$:

$$C_0(x;a) = 1,$$
$$C_1(x;a) = -1+\frac{x}{a},$$
$$C_2(x;a) = \frac{a^2-2ax+x^2-x}{a^2},$$

and so on (*cf.* [10]).

One of the number families close to the Poisson-Charlier polynomials is the Cauchy numbers of the first kind, b_n, (also known as the Bernoulli numbers of the second kind), and these numbers are defined as

$$b_n = \int_0^1 (x)_n\,dx \tag{5}$$

and the relationship between these numbers and the Stirling numbers of the first kind is given by

$$b_n = \sum_{j=0}^{n}\frac{s(n,j)}{j+1}, \tag{6}$$

(*cf.* [1, p. 294], [4, p. 1908], [8, p. 114]).

With any of (5) and (6), the first few values of b_n are computed as follows:

$$b_0 = 1, \quad b_1 = \frac{1}{2}, \quad b_2 = -\frac{1}{6}, \quad b_3 = \frac{1}{4}, \quad b_4 = -\frac{19}{30}, \quad b_5 = \frac{9}{4},$$

and so on (*cf.* [2], [4], [6], [8, pp. 113–117]).

In the light of the preliminary information above, the main findings of this study has been presented in the next section.

Main Results

In this section, by using the equation (2), we start to define the function G by setting the following integral equation:

$$G(t,x;a) := y_0 + \int_{x_0}^{x} F_{pc}(t,u;a)\,du \tag{7}$$

where x_0 and y_0 are arbitrary real constants.

By the Fundamental Theorem of Calculus, we can conclude that the function $G(t,x;a)$ satisfy the following initial value problem:

$$\begin{cases} \frac{d}{dx}\{G(t,x;a)\} = F_{pc}(t,x;a) \\ G(t,x_0;a) = y_0, \end{cases} \tag{8}$$

since the function $F_{pc}(t,x;a)$ is continuous when $x \in (-\infty,\infty)$.

Observe that setting $x_0 = 0$ and $y_0 = 0$ in the equation (8) gives

$$\begin{cases} \frac{d}{dx}\{G(t,x;a)\} = F_{pc}(t,x;a) \\ G(t,0;a) = 0. \end{cases} \tag{9}$$

It is well-known that the Existence and Uniqueness Theorem guarantees that there will be a unique solution for the initial value problem in (9). Therefore, the unique solution of the initial value problem, given in (9), explicitly becames

$$G(t,x;a) = \int_0^x F_{pc}(t,u;a)\,du = \frac{\left(\left(\frac{t}{a}+1\right)^x - 1\right)\exp(-t)}{\log\left(\frac{t}{a}+1\right)}.$$

By the motivation of the above function, we now introduce a new polynomial family, denoted by $\mathcal{I}_{c,n}(x;a)$ and so-called Poisson-Charlier type polynomials, by the following definition:

Definition 1. *Let a be any nonzero number. Then the Poisson-Charlier type polynomials, $\mathcal{I}_{c,n}(x;a)$, are defined by the following generating functions:*

$$G(t,x;a) = \sum_{n=0}^{\infty} \mathcal{I}_{c,n}(x;a)\frac{t^n}{n!}. \tag{10}$$

By integrating the equation (2), with respect to u, from 0 to x, we have

$$\int_0^x F_{pc}(t,u;a)\,du = \sum_{n=0}^{\infty}\left(\int_0^x C_n(u;a)\,du\right)\frac{t^n}{n!}. \tag{11}$$

On the other hand, combination of (10) and the above equation yields

$$\sum_{n=0}^{\infty} \mathcal{I}_{c,n}(x;a)\frac{t^n}{n!} = \sum_{n=0}^{\infty}\left(\int_0^x C_n(u;a)\,du\right)\frac{t^n}{n!}. \tag{12}$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the equation just above implies the following theorem:

Theorem 2. *Let a be any nonzero number. Then Poisson-Charlier type polynomials, $\mathcal{I}_{c,n}(x;a)$, are of the following integral representation:*

$$\mathcal{I}_{c,n}(x;a) = \int_0^x C_n(u;a)\,du. \tag{13}$$

Remark 9. *From the work of Kucukoglu et al.* [3, Eq-(72)]*, we know that the polynomials $C_n(x;a)$ have a relationship with the numbers b_n as follows:*

$$\int_0^1 C_n(x;a)\,dx = \sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j}\frac{b_j}{a^j}.$$

Therefore, when $x = 1$, the combination of (13) *with the above equation yields the following first few values of $\mathcal{I}_{c,n}(1;a)$:*

$$\begin{aligned}
\mathcal{I}_{c,0}(1;a) &= 1\\
\mathcal{I}_{c,1}(1;a) &= -1 + \frac{1}{2a}\\
\mathcal{I}_{c,2}(1;a) &= 1 - \frac{1}{6a^2} - \frac{1}{a},\\
\mathcal{I}_{c,3}(1;a) &= -1 + \frac{1}{4a^3} + \frac{1}{2a^2} + \frac{3}{2a},
\end{aligned}$$

and so on.

Conclusion

By this study, it has been focused on finding which polynomial family the solutions of an initial value problem containing the generating function of the Poisson-Charlier polynomials produce and what some properties of this polynomial family are. Overall, the findings of this study may find an application area in a broad perspective for itself, particularly in computational science and engineering.

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Certain aspects of rough $\mathcal{I}_2$-statistical convergence in cone metric spaces

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The aim of this paper is to investigate the concept of rough $\mathcal{I}_2$-statistical convergence as an extension of rough convergence rough convergence in a cone metric space. In addition, we present the concept of rough $\mathcal{I}_2^*$-statistical convergence of sequences in a cone metric space and examine the relationship between rough $\mathcal{I}_2$-statistical and $\mathcal{I}_2^*$-statistical convergence of sequences.

2010 Mathematics Subject Classifications: 40A05, 40A99

Keywords: Cone metric spaces, $\mathcal{I}$-convergence, $\mathcal{I}^*$-convergence, Rough convergence

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Realizability Properties of Degree Sequences

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A set D of non-negative integers is called a degree sequence if its elements correspond to the vertex degrees of a graph. In such a case, D is said to be realizable. In this talk, we shall give some results on the number of realizations of a given degree sequence. Revent results on omega invariant will be used in combinatorial and topological calculations. Especially, this problem is solved combinatorically for molecular graphs.

2010 Mathematics Subject Classifications: 05C07, 05C10, 05C30

Keywords: degree sequence, realizability, omega invariant, molecular graph

Introduction

Let G be an undirected graph with vertex set V and edge set E. Let $n = |V|$ and $m = |E|$ be the order and size of G, respectively. The degree of a vertex $v \in V$ is denoted by dv and is defined as the number of edges meeting at v, that is the number of edges with one endpoint as v. Note that as we do not ask for the condition being simple, the graph G might have multiple edges and loops and a loop at a vertex v will contribute 2 to dv. A vertex of degree 0 is called an isolated vertex and a vertex of degree 1 is called a pendant vertex. An edge e having end vertices u and v will be denoted by $e = uv$. In such a case, e will be said of type (du, dv).

The vertex degrees of a graph G form a set D which is called the degree sequence of G. Each graph has a unique degree sequence consisting non-negative integers. The problem is that, for a given set D of non-negative integers, whether there exists a graph G having D as its degree sequence? If there is at least one graph with that condition, then D is called realizable. There are several algorithms to determine the realizability of a given set D. The most famous one is the Havel-Hakimi process, see [4, 5].

The main problem addressed here is the following problem:

Open problem 1. *What is the number of realizations of a given realizable set of non-negative integers?*

Like the number of graphs of a given order, there is no formula for the number of realizations of a given degree sequence. In this paper, we establish some partial results towards the solution of this problem.

Preliminaries

Here we recall the notion of a recently defined graph invariant called omega invariant and discover some of its properties that are needed to partially solve the open problem. From now on, when we say D, we shall understand a set $D = \{1^{(a_1)}, 2^{(a_2)}, 3^{(a_3)}, \cdots, \Delta^{(a_\Delta)}\}$.

Definition 2. *Let D be the degree sequence of a graph G. The $\Omega(G)$ is defined only in terms of the degree sequence as*

$$\begin{aligned}\Omega(G) = &\quad a_3 + 2a_4 + 3a_5 + \cdots + (\Delta - 2)a_\Delta - a_1 \\ = &\quad \sum_{i=1}^{\Delta}(i-2)a_i.\end{aligned}$$

Many properties related to omega have just been obtained in [1] and [2]. For a given degree sequence, it has been shown that if $\Omega \leq -4$, then the graph is certainly disconnected, if $\Omega = -2$ and the graph is connected, then the graph is certainly acyclic, and if $\Omega \geq 0$ and the graph is connected, then the graph is certainly cyclic. If Ω is odd, we can directly say that the given degree sequence is not realizable. In [1], it has been shown that omega invariant can be stated in terms of the numbers of vertices and edges as

$$\Omega(G) = 2(m - n). \tag{1}$$

It was shown that Ω of a graph G is additive over the set of the components of G. Let D be realizable as a (not necessarily connected) graph G with c components. The number r of faces of G is given, in [1], by

$$r = \frac{\Omega(G)}{2} + c. \tag{2}$$

In [1] and [2], the minimum and maximum number of components that a realization of D can respectively have was formulized as

$$c_{min} = max\left\{1, -\frac{\Omega(D)}{2}\right\} = max\left\{1, n - m\right\}, \tag{3}$$

$$c_{max} = \sum_{d_i\ even} a_i + \frac{1}{2}\sum_{d_i\ odd} a_i. \tag{4}$$

Let $D(c)$ be the set of all graph realizations of D having c components. Let $N(c) = |D(c)|$. To reach our aim, we shall apply the following algorithm:

Algorithm 3. *Given D, apply the following steps:*

- *Step 1. Calculate c_{max} and c_{min} by means of Eqn. 4 and Eqn. 3, respectively.*
- *Step 2. Draw all graphs with c_{max} components. That is, determine $D(c_{max})$.*
- *Step 3. Calculate $N(c_{max})$.s*
- *Step 4. Using the graphs in $D(c_{max})$, find all graphs in $D(c_{max}-1)$ with $c_{max}-1$ components by "joining" two components in a graph with c_{max} components.*
- *Step 5. Calculate $N(c_{max} - 1)$.*
- *Step 6. Repeat Step 4 and Step 5 by reducing the number of component by 1 at each round to obtain $N(c_{max} - 2)$, $N(c_{max} - 3)$, $\cdots$, $N(c_{min})$.*

- *Step 7. Find* $N(D) = N(c_{max}) + N(c_{max} - 1) + N(c_{max} - 2) + \cdots + N(c_{min})$ *by adding all obtained values.*

In this paper, our aim is to reduce the number of components one by one at each step according to Algorithm 3. We shall carry some components, mostly vertices and edges, onto others to reduce the number of components. To this aim, we give some results on which type of edges are carriable. Our first result is about the case we excluded in Theorem 6, that is, about carrying an edge of type $(1,1)$. Such an edge always form a single component and appears many times in the realizations with high number of components:

Theorem 4. *An edge of type* $(1,1)$ *is carriable onto another edge iff the graph is not simple.*

Proof. ($\Rightarrow$) Let the graph be simple. Let this edge be $e = uv$. Then $du = dv = 1$. To carry u (or v) onto another edge, u must have degree 2 as any vertex on an edge has degree at least 2. Therefore, such an edge cannot be carried onto another edge.
($\Leftarrow$) Let the graph have a loop at a vertex w and let the degree of w be dw. Then clearly, $dw \geq 2$. Let $e = uv$ be of type $(1,1)$. If $dw = 2$, then there is only one loop at w and we can carry e onto a path graph $P_3 = u, w, v$ reducing the number of components by 1. If $dw = 4$, then either there are two loops at w and we can carry e onto a graph consisting of a loop and two pendant vertices at w, or there is one loop and two pendant vertices at w and we can carry e onto this component to obtain a star graph S_5, again decreasing the number of components by 1. If $dw = 6$, then either there are two loops at w and we can carry e onto a graph consisting of a loop and two pendant vertices at w again reducing the number of components by 1. For larger even values of dw, similar arguments work. For odd values of $dw > 2$, the above cases are valid with one extra edge at w. □

Our second result is about the edges of type $(1, 2k)$. Such an edge cannot form a single component as the omega invariant is $2k - 3$ which is odd. A similar argument to the one in Theorem 4 shows that those edges are also not carriable onto another edges in the graph such that the number of components is reduced by 1:

Theorem 5. *An edge of type* $(1, 2k)$ *is carriable onto another edge iff the graph is not simple.*

Our last result is the general result about what types of edges can be carriable to another edge or edges:

Theorem 6. *Let* $e = uv$ *be an edge of a graph* G*. We can carry* e *onto another component(s) of* G *in the following three cases:*

- *If one end vertex, say* v*, is a pendant vertex, then* $e = uv$ *can be carried onto another edge* $f = wz$ *such that* u *becomes a vertex between* w *and* z *and it is connected to pendant vertex* v*, see Fig. 1.*
- *If one end vertex, say* v*, is of degree 3, and the other end vertex* u *is of odd degree* du*, then* $e = uv$ *can be carried onto another edge* $f = wz$ *such that* v *becomes a vertex between* w *and* z *and it is connected to another vertex* u *the edge either between* w *and* v *or between* v *and* z*, see Fig. 2.*
- *If both* du *and* dv *are greater than 2, then* u *and* v *can be carried onto any edges on another components and also they can be on the same edge.*

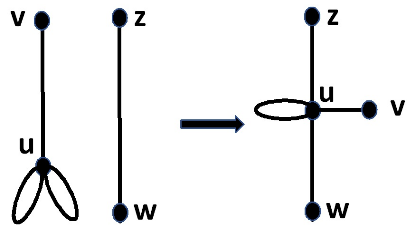

Figure 1: Carrying an edge of type $(1, 2k+1)$ for $k = 2$

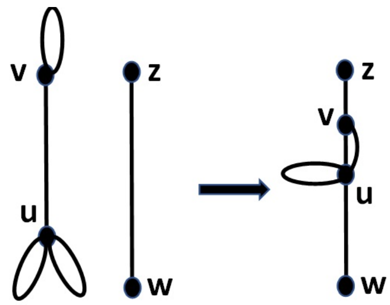

Figure 2: Carrying an edge of type $(3, 2k+1)$ for $k = 2$

Proof. In the second possibility, when we carry u and v onto same edge, all of the $du + dv - 2 = du$ edges adjacent to e must have the same vertex degrees which is impossible. The same reasoning appears in the third possibility. So we only have to consider the first case. In Theorems 4 and 5, we have shown the impossibility of carrying edges of type $(1, 1)$ and $(1, 2k)$. So it remains to carry the edges of type $(1, 2k+1)$. If u is the pendant vertex and v is the other vertex of odd degree $dv > 2$ of an edge uv, then it is possible to carry this edge onto an edge on another component by carrying v onto any edge and u as a pendant vertex adjacent to v. This reduces the number of components by one and uv is carriable. □

Conclusion

In this work, by omega invariant and its properties, we try to take first steps in answering an important open problem about the number of realizations of a given degree sequence.

Acknowledgments

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Stochastic Weather Modelling of Agricultural Drought in Arid Climates

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We propose a stochastic weather model in terms of precipitation, temperature, humidity and wind speed for modelling agricultural drought in arid climates. Stochastic differential equations are suggested to model temperature and humidity, in particular Ornstein-Uhlenbeck processes. For precipitation, a two-stage model is used: a Markov chain for precipitation occurrence and a probability distribution for precipitation amount. Markov chains are also used to model wind speed. By modelling these four weather factors, which are the key determinants of drought, we construct a drought index model that can be used to evaluate the intensity of drought and that can be used in pricing weather derivatives related to drought to help agricultural producers hedge against the financial impact of drought. Finally, we compare the drought index values between observation and simulation data. We use daily weather data from Qatar as a representative arid region.

2010 Mathematics Subject Classifications: 65C30, 00A69, 00A72

Keywords: Stochastic differential equations (SDEs), Reconnaissance drought index (RDI), Ornstein-Uhlenbeck (O-U) processes, Markov chains

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On Jordan algebra and its automorphism group $F_4(K)$ in fields K of characteristic two

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The purpose of this paper is to study certain properties of the Lie group $F_4(K) \leq E = E_6(K)$ for fields K of characteristic 2. We use the generalized quadrangle $(\mathcal{P}, \mathcal{L})$, where $\mathcal{P}$ is a set of points and $\mathcal{L}$ is a set of lines, to construct a symmetric trilinear form T on a 27-dimensional vector space $A = \langle e_x \mid x \in \mathcal{P} \rangle$ which leaves A invariant. We introduce an involution $g \to g^* := \left(g^t\right)^{-1}$ on E, algebra structure on A and a quadratic map $\hat{Q} : A \to A$. Then we prove the following results:

- The centralizer $C_{G_0}(S_L)$ is the group F isomorphic to $F_4(K)$, where $L \in \mathcal{L}$ and
$$S_L = \sum_{x \in L} e_x \text{ and } G_0 = \left\{ g \in GL(A) \mid \hat{Q}\left(a^g\right) = \hat{Q}(a)^{g^*} \right\}$$
- The map $B : (a, b) \to T(a, b, S_L)\,; a, b \in A$, is an alternating symmetric bilinear form on A with $\mathrm{Rad}(B) = \{a \in A \mid B(a, x) = 0, \forall x \in A\}$, where $\mathrm{Rad}(B)$ denotes the radical of B.
- F preserves $B, \hat{Q}(S_L) = S_L$ and $D(S_L) = 1$ where D is a cubic form on A.
- $F \leq S_P(26, K)$.
- $J = (A, +, \circ)$ is a commutative algebra, $F \leq \mathrm{Aut}(A, +, 0) = \mathrm{Aut}(J)$, where J can be considered as a Jordan algebra in characteristic 2.
- The group $\langle H, N_W(L) \rangle$ is isomorphic to the triality group of shape $\Omega_8^+(K) \cdot S_3$, where $W = Aut(\mathcal{P}, \mathcal{L})$ the Weyl group of type E_6, and H is the group generated by certain root subgroups.

2010 Mathematics Subject Classifications: 17A75, 17A45.

Keywords: Generalized quadrangle, Lie groups, triality, Jordan algebra, root base, root element, Weyl group.

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On the Multiple α-type Polynomials

Mehmet Ali Özarslan

In this paper, we focused on the investigation of two multiple polynomial families defined by the generating functions $(1-t_1)^{-\alpha_1}(1-t_2)^{-\alpha_2}F(x,t_1,t_2)$ and $(1-t_1-t_2)^{-\alpha}G(x,t_1,t_2)$. We obtain the difference recurrence relations and further generating functions for the multiple polynomials generated by the above mentioned functions.

2010 Mathematics Subject Classifications: 33E30, 33C45

Keywords: Multiple Laguerre polynomials, Lagrange expansion, Recurrence relations, Summation formula

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Umbral properties of generalized Bessel functions and hyperbolic like functions appearing in a magnetic transport problem

G. Dattoli [1], *Subuhi Khan* [2], *Mehnaz Haneef* [*3] *and Silvia Licciardi* [4]

The umbral methods originated from operational calculus. The umbral operational techniques offer a sturdy mechanism for simplifying the study of special functions appearing in mathematical physics. In this work, the umbral techniques are used to reformulate the theory of multivariable generalized Bessel functions and to derive addition and multiple argument expansion theorems. The umbral properties of families of exponential-like functions and their hyperbolic forms are also discussed. Further, the generalized forms of Mittag-Leffler functions are used to solve the technical problems concerning the transport of a charged beam in a solenoid magnet. The possibility of extending the umbral formalism to the study of other special functions of mathematical physics is also explored.

2020 Mathematics Subject Classifications: 33C10, 33C45, 33F10

Keywords: Umbral methods, Multi-variable Bessel functions, Pseudo-exponential functions, Pseudo-hyperbolic functions, Mittag-Leffler functions, Hypergeometric functions

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A note on mathematical model for conversion of sentences from active voice to passive voice

Melis Oyku Hazar [*1] *and Yilmaz Simsek* [2]

The purpose of this presentation is to further investigate the mathematical models that are used to convert any active sentences to passive sentences. After assessing some of the existing studies, we propose new open problems that will be helpful for future research involving comprehensive mathematical models for sentence transformations that include active and passive sentences.

Introduction

Language and mathematics are the most vital areas that are used as the oldest communication and calculation tool of human civilization. In the old ages, humans started to have their first communications by drawing pictures in the caves they lived in. Over time, small line drawings instead of numbers tried to meet their needs by knotting the ropes, but over time it became clear that this was not enough for them. The development of human civilization, which was fermented and blended with long efforts and developments, led to a new era in human history with both the invention of the alphabet and the construction of numbers. As a result, each civilization began to have both language and number sets unique to itself and its culture. Today, human civilization continues to contribute to incredible developments and discoveries, thanks to both the treatment of languages as a science and the inclusion of number theory as a major in mathematics.

In the light of these developments, although mathematics is now considered as a unique language used in the world, a language used by almost every country continues its development and use intensively. Based on these developments, many remarkable developments and theories have been built between language and the language of mathematics. With the use of mathematical modeling in many branches of science since the beginning of the 20th century, it was inevitable that linguistics would be affected. For this reason, many articles, books, and reports have been published on these topics (*cf.* [1]-[5]).

Motivation of this presentation, we study the some mathematical models relevant to transformation of the active and passive sentences.

Grammatical structure of the sentence in English linguistics

It is well-known that the grammatical structure of the sentence of different language is very different. Thus, in this study, we briefly study the sentence structures of the English language. The sentences in English literature can be described with the following eqaution:

$$Subject(S) + Verb(V) + Object(O).$$

All the tenses used in English grammar can be expressed with mathematical equations in similar manner because of the verb conjugations. Note here that some verbs may not have conjuctions and have the same conjugation when using different tenses in the English grammmer. This plays a key role when converting a sentence from active to passive structure. This is why it is extremely important ascept when constructing a mathematical model which transforms an active sentence to passive one.

The set of active sentences denote by $\mathcal{A}$:

$$S + V + O.$$

The set of passive sentences denote by $\mathcal{P}$:

$$O + be + V_3 + S.$$

Main question is given as follows:

How can one construct mathematical model for $\mathcal{A} \to \mathcal{P}$? That is, how can an active sentence be transformed into a passive sentence in any linguistics grammar?

One can find variety of answers for the above question. Recently, to answer this question, many mathematical techniques involving equations with the aid of functions or homomorphism. Applying mathematical equations and functions, Pandey and Dhami [5] gave a transformation for $\mathcal{A} \to \mathcal{P}$.

It is still of interest to the researchers to find new solutions to the above question by searching new mathematical methods and modeling techniques.

Open questions for the mathematical model for $\mathcal{A} \to \mathcal{P}$ and future investigations

In a future study, our plan is to extend the transformations by converting direct speech to indirect speech or vice versa. Our goal is then to investigate the feasibilty of applying these models to a real world problem which involves a reported speech tool. We are specifically interested in applying these models to the reported speech tool aiding ESL students. These students have diffuculty in grasping complex speech conversions which require change in sentence structure in addition to utilization of different tense forms. We will also look into whether speech conversions can be combined with our previous conversion equation.

With this our open problems are as follows:

One of the main motivations for our future work and research is to search a new solutions to $\mathcal{A} \to \mathcal{P}$.

Then, we will look for a mathematical methods and modeling techniques for the function $\mathcal{P} \to \mathcal{A}$.

Lastly, how can we interpret the function $\mathcal{A} \to \mathcal{P}$ and $\mathcal{P} \to \mathcal{A}$?

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Bivariate Hybrid Chlodowsky-Jain-Appell operators

Merve Çil [*1] *and Mehmet Ali Özarslan* [2]

In this paper we introduce and investigate hybrid bivariate operators of Bernstein-Chlodowsky operators and the Jain-Appell operators. We examine the degree of approximation of the operators with te help of complete modulus of continuity and the partial moduli of continuities. We investigate a quantitative error estimate of the operators by using bivariate extension of Holhos' weighted modulus of continuity and obtain the local approximation properties in terms of Lipschitz type maximal functions. We also introduce and investigate the GBS version of the bivariate hybrid Chlodowsky-Jain-Appell operators

2010 Mathematics Subject Classifications: 41A10, 41A25

Keywords: Jakimovski-Leviathan operators, Chlodowsky Szasz-Mirakyan Operators, Jain-Appell operators, modulus of continuity, weighted Korovkin theorem, GBS

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Hermite–Hadamard–Fejer type inequalities and weighted quadrature formulae

Mihaela Ribičić Penava

The Hermite–Hadamard–Fejér inequalities state

$$h\left(\frac{a+b}{2}\right)\int_a^b u(x)\,dx \le \int\limits_a^b u(x)h(x)\,dx \le \left[\frac{1}{2}h(a)+\frac{1}{2}h(b)\right]\int_a^b u(x)\,dx,$$

where $h:[a,b]\to\mathbf{R}$ is a convex function and $u:[a,b]\to\mathbf{R}$ is nonnegative, integrable and symmetric about $\frac{a+b}{2}$. If $u\equiv 1$, then we are talking about the Hermite–Hadamard inequalities.

Hermite–Hadamard and Hermite–Hadamard–Fejér type inequalities have many applications in mathematical analysis, numerical analysis, probability and related fields. Their generalizations, refinements and improvements have been an important topic of research in the past years.

The main purpose of this talk is to present some new Hermite–Hadamard–Fejér type inequalities for higher-order convex functions. As applications of the main results, certain Hermite–Hadamard–Fejér type estimates for various classical quadrature formulae will be obtained.

2010 Mathematics Subject Classifications: 25D15, 65D30, 65D32

Keywords: Hermite–Hadamard–Fejér inequalities, weighted quadrature formulae, higher-order convex functions, w-harmonic sequences of functions

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A brief distributional approach to plurisuperharmonic functions

Mika Koskenoja

We present an approach to plurisuperharmonic functions based on the Levi form in the sense of distributions in $\mathbb{C}^n$. We observe that the Sobolev space $W^{1,2}_{\mathrm{loc}}$ is a favourable function space to consider plurisuperharmonic functions. Using the distributional approach we derive several properties of plurisuperharmonic functions. For example, we show that plurisuperharmonic functions satisfy so called "ess lim inf" property.

2020 MATHEMATICS SUBJECT CLASSIFICATIONS: 32U05, 31C10

KEYWORDS: Levi form, positive semidefinite, weak (super)solution, pluriharmonic, plurisuperharmonic, essential lower bound

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A polynomial approach to optimal control switched systems and applications

Mohamed Ali Hajji

Switched systems are a class of hybrid systems that model phenomena whose dynamics is controlled by a continuous control signal(s). An optimal control problem for a switched system is the problem of finding the optimal trajectory of a constrained switched system. In this talk we present a numerical approach based on polynomial approximation and the theory of moments for solving optimal control problems for nonlinear switched systems. The main idea is to transform the nonlinear switched system into an equivalent non-switched system using polynomial representation. Then the method of moments is used to convexify the new control variable to obtain semidefinite programs (SDP) which can be solved by SDP solvers. An important application of optimal control switched system in the modelling of epidemics will be discussed.

2010 Mathematics Subject Classifications: 49J15, 49M25, 49M25, 78M05, 78M50

Keywords: Switched systems, Method of moments, Semidefinite programs

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Transversal hypersurfaces in Loentz Minkowski space

Mohammed Abdelmalek

Given M^{n+1} an $(n+1)$-dimensional pseudo-Riemannian manifold of index $q \geq 0$ and let M^n be a non-degenerate oriented hypersurface of M^{n+1} with regular boundary $\partial M \subset P^n$, where P^n is a totally geodesic hypersurface of M^{n+1}.

In this work we proove that the hypersurfaces M^n and P^n are transverse along the boundary ∂M if for some $1 \leq r \leq n$, the Newton transformation T_r is positif defined.

We give in the end some examples and applications.

Keywords: Pseudo Riemannian manifold, Newton transformations, Higher order mean curvature

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Some Diophantine equations involving the product of two k-fibonacci numbers and Fermat or Mersenne numbers

Mohand Ouamar Hernane [*1], *Salah Eddine Rihane* [2], *Safia Seffah* [3] *and Alain Togbé* [4]

For a positive integer $k \geq 2$, we consider a generalization of Fibonacci sequence, which is called the k-generalized Fibonacci sequence or, for simplicity, the k-Fibonacci sequence. The k-Fibonacci sequence $(F_n^{(k)})_{n\geq 2-k}$ is given by the recurrence

$$F_n^{(k)} = F_{n-1}^{(k)} + F_{n-2}^{(k)} + \cdots + F_{n-k}^{(k)} \quad \text{for all } n \geq 2,$$

with the initial conditions $F_{-(k-2)}^{(k)} = F_{-(k-3)}^{(k)} = \cdots = F_0^{(k)} = 0$ and $F_1^{(k)} = 1$. In this talk, we use Baker's method to show that there are no Fermat or Mersenne numbers expressible as product of two k-Fibonacci numbers greater than 1 (see [3]).

2010 Mathematics Subject Classifications: 11B39, 11J86

Keywords: k-Fibonacci numbers, Fermat numbers, Mersenne numbers, Linear form in logarithms, reduction method

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A Note on Color Compositions and the Patterns

Mustafa Alkan *[1] *and Busra Al* [2]

In this paper, we focus on decomposing both the composition set and the palindrome composition set of an integer. These decompositions allow us to investigate the generating functions for the numbers of the elements of these sets. Moreover, generalizing the concept of $n-$color compositions, we find out the general form for the numbers of some color compositions of an integer.

2010 Mathematics Subject Classifications: 05A15, 05A17, 05A18, 11B39,11B99

Keywords: Compositions of the integers, the n-color combination of the integers, Palindromes compositions, Fibonacci numbers, Generating Function

Introduction

Partitions of an integer and Compositions of an integer have attracted the attention of many researchers from past to present. They have made incredible contributions to almost all branches of science. These discoveries led to an increase in the importance of mathematical analysis and number theory.

Recently, many researchers ([1], [2], [13]) have been interested $n-$color compositions of an integer m , defined as composition of m for which a part of size n can take on n colors. There are eight $n-$color compositions of 3:

$$(3_1),(3_2),(3_3),(1_1,2_1,),(1_1,2_2),(2_1,1_1),(2_2,1_1),(1_1,1_1,1_1).$$

When we assign the color for each parts like the following Figure 1;

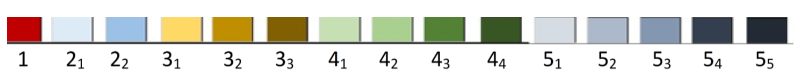

Figure 1: The $n-$color compositions of 3

we can represent the $n-$color compositions of 3 in rectangles with dimension 3×1 as in Figure 2;

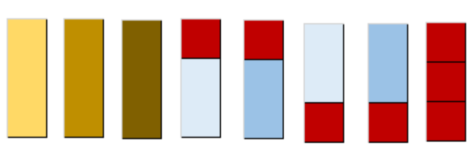

Figure 2: The $n-$color compositions of 3

In [8], Guo interested in n-color compositions with no parts of size 1. In these papers, they paid attention to the size of parts in $n-$color compositions.

In this study, we focus on decompositions of the composition sets and both the color and the position of parts in compositions of an integer. Our main aim is to find out the generating functions for the numbers of elements of some color compositions of an integer.

Decompositions of the composition sets

In this section, we focus on decomposing the composition sets and relations among the pattern of an integer n. Moreover, we generalize the concept of n–color composition and we define the palindromes color for an integer.

We denote the composition set of an integer n as follows

$$P_n = \left\{(a_1, a_2, ..., a_t) : \ a_1 + a_2 + ... + a_t = n, \quad a_i, t \in \mathbb{Z}^+\right\}.$$

Then from [3], we recall the following operations for the element $a = (a_1, a_2, ..., a_t) \in P_n$ and an integer j;

$$j \odot a = (j, a_1, a_2, ..., a_t) \text{ and } j \oplus a = (a_1 + j, a_2, ..., a_t).$$

Then we use the notations $j \oplus P_n$ and $j \odot P_n$ for the following sets

$$j \oplus P_n = \{l \oplus a : a \in P_n\} \text{ and } j \odot P_n = \{l \odot a : a \in P_n\}.$$

Theorem 1. *Let n, r be positive integers ($r \leq n$). Then the set P_n is the disjoint union of the sets $(r \oplus P_{n-r})$ and $(i \odot P_{n-i})$ for all $i \in \{1, \ldots, r\}$, i.e.*

$$P_n = (\cup_{i=1}^{r}(i \odot P_{n-i}) \cup (r \oplus P_{n-r}).$$

Proof. It is sufficient to prove the inclusion $P_n \subseteq (\cup_{i=1}^{r}(i \odot P_{n-i}) \cup (r \oplus P_{n-r})$.

Let $x = (a_1, \ldots, a_m) \in P_n$. If $a_1 \leq r$ then $x \in \cup_{i=1}^{r}(i \odot P_{n-i})$.

Now assume that $r < a_1$. Then $b = a_1 - r$ and so define the element $y = (b, a_2, a_3, \ldots, a_m) \in P_{n-r}$. Then it is clear that $x = r \oplus y \in (r \oplus P_{n-r})$.

It is also clear that $(r \oplus P_{n-r}) \cap (i \odot P_{n-i}) = \varnothing$ for all $i \in \{1, \ldots, r\}$. □

Then we also get the following result in [3, Theorem 5].

Corollary 2. *For a positive integer n, we have*

$$P_{n+1} = (1 \oplus P_n) \cup (1 \odot P_n).$$

Definition 3. *Let k be a positive integer and α, β be sequence of colors (the color sequence α means that the part with size k can take on colors α_k).*

We define (α-β)-color compositions of an integer m as the composition of an integer m such that its first part with size k can take on colors α_k and the other parts with size k can take on colors β_k.

The pattern obtained by combining of (α-β)-color compositions of an integer m is called the (α-β)-pattern of m.

This definition is a generalization of n–color composition in [1]. Now we wonder about the number of (α-β)-color compositions of an integer and about the (α-β)-pattern of an integer.

By using Corollary 2, we investigate the generating function for the numbers of (α-β)-color compositions of an integer;

Theorem 4. *[5] Let α, β be sets of colors. If $T(\beta, t)$ is a generating function of the number of β-color composition of an integer and $S(\alpha, t)$ is a generating function for the elements of α then the generating function of the number of (α, β)-color composition is*

$$T(\alpha, \beta, t) = S(\alpha, t)(1 + T(\beta, t)).$$

Proof. Let $T(\beta, n)$ is the number of β-color composition of an integer n. Assume that $T(\beta, t)$ is a generating function of the number of β-color composition of an integer,

$$T(\beta, t) = \sum_{n=1}^{\infty} T(\beta, n)t^n.$$

Then by Corollary 2, it follows that

$$T(\alpha, \beta, n+1) = \alpha_1 T(\beta, n) + \sum_{y=1}^{n} \alpha_{1+y} T(\beta, n-y) = \sum_{y=1}^{n+1} \alpha_y T(\beta, n+1-y)$$

and so

$$T(\alpha, \beta, t) = \alpha_1 t + \sum_{n=1}^{\infty} \sum_{y=1}^{n+1} \alpha_y T(\beta, n+1-y)t^{n+1} = S(\alpha, t)(1 + T(\beta, t)).$$

□

Theorem 4 make us reacquire some well known identities and obtain the generating function for the numbers of composition with any coloring rules.n-color compositions

i) Let $\alpha = \beta$ then $T(\alpha, \beta, t) = T(\beta, t)$ and so it follows that

$$T(\beta, t) = \frac{S(\alpha, t)}{1 - S(\alpha, t)}.$$

ii) Let $\alpha = \beta$ be color sequence such that every part with size c can take on one color. Then it follows that

$$S(\alpha, t) = \frac{t}{(1-t)}$$

and so

$$T(\beta, t) = \frac{t}{1-2t}$$

is the generating function for the number of the compositions of an integer. Therefore, we have regained the well known result that the number of compositions of an integer m is 2^{m-1} ([5]).

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Some Identities Between Pell numbers and Pell Lucas numbers

Mustafa Alkan

In this work, we obtain the matrix which represent Pell numbers and Pell Lucas numbers. Then by using the matrix, we investigate some new identities containing both Pell numbers and Pell Lucas numbers.

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Keywords: Horadam Numbers, Pell Numbers, Fibonaaci numbers, Pell-Lucas Polynomials, Binet Formula

Introduction

In the present day, since there are many applications of Fibonacci numbers from Physis to Art, there are many sequences of integers derived the Fibonacci numbers such as Fibonacci polynomials, Lucas numbers and polynomials, Pell numbers and polynomials, Pell-Lucas numbers and polynomials, Jacobstal numbers and polynomials, Jacobstal-Lucas numbers and polynomials, Chebyshev numbers and polynomials and so on. Many researcher have studied these generalizations to get some combinatorial identities and generating functions, (For example; ([2], [3], [4], [5], [6], [10], [14], [20], [21], [23], [24], [26], [28], [29], [30]).

In the literature, Horadam numbers and polynomials also were defined as a generalization of Fibonacci numbers and polynomials.

Let p, q, a and b be elements of Rational numbers. Then for a positive integer n, Horadam numbers was defined by the recurrence relation

$$w_{n+1} = pw_n + qw_{n-1} \tag{dw}$$

with the initial conditions $w_0 = a$, $w_1 = b$, denoted $w_n(a, b, p, q)$ or briefly $\{w_n\}$.

Horadam polynomials $h_n(x, a, b, p, q)$ (briefly h_n) was also defined in the following

$$h_n = ph_{n-1} - qh_{n-2} \ (n \geq 3)$$

with the initial conditions $h_1 = a$, and $h_2 = bx$.

For integer n greater than 1, we define a generalization of Lucas sequences with the initial condition $w_0 = 2, w_1 = p$;

$$h_n = w_{n+1} + qw_{n-1}. \tag{1}$$

For different values p, q, and, Horadam sequence is a generalization of some well-known sequences such as Fibonacci, Lucas, k-Fibonacci, k-Lucas, Pell, Pell-Lucas, Jacobsthal and Jacobsthal-Lucas

We may get the value of h_n;

n	0	1	2	3	4
h_n	2	p	p^2+2q	p^3+3pq	$p^4+4p^2q+2q^2$

.

When $p = 2$ and $q = 1$, w_n is Pell numbers and h_n is Pell-Lucas Numbers.
Then Pell numbers is that $1, 2, 5, 12, 29$.......
The Pell–Lucas numbers is that $2, 6, 14, 34, 82, ...$

In this work we focus on Pell numbers and Pell Lucas numbers to investigate the relations between them. Then we obtain the matrix which represent the Pell numbers and Pell Lucas numbers. Then by using the matrix, we get the Binet Formula for Pell numbers and Pell Lucas numbers . And also we investigate some new identities containing both Pell numbers and Pell Lucas numbers by using the matrix.

The matrix of Pell Numbers and Pell Lucas numbers

In this section, we set the matrix to obtain some new identities and gain an alternative simple proof for some well known ones.

Lemma 1. *For an integer $n \geq 2$, we have that*

$$h_n = ph_{n-1} + qh_{n-2} \tag{2}$$

where $h_0 = 2$ and $h_1 = p$.

Proof. By Definitions, for an integer $n \geq 2$ we get that

$$\begin{aligned} h_n &= w_{n+1} + qw_{n-1} \\ &= (pw_n + qw_{n-1}) + q\,(pw_{n-2} + qw_{n-3}) \\ &= ph_{n-1} + qh_{n-2}. \end{aligned}$$

□

Using Definitions, we also get the matrix $T = \begin{bmatrix} p & q \\ 1 & 0 \end{bmatrix}$.

Theorem 2. *[1]For an integer n, we have*

$$T^n = \begin{bmatrix} w_{n+1} & qw_n \\ w_n & qw_{n-1} \end{bmatrix}.$$

Let α, β be a real numbers such that

$$x^2 - px - q = (x - \alpha)(x - \beta)$$

and $\Delta = \sqrt{p^2 + 4q}$. Then we set the matrices

$$P_1 = \frac{-1}{q} \begin{bmatrix} 1 & 1 \\ \beta_2 & \alpha \end{bmatrix}, P = \begin{bmatrix} 1 & 1 \\ \frac{1}{\sqrt{\Delta}} & \frac{-1}{\sqrt{\Delta}} \end{bmatrix}.$$

Then to construct the matrix S, we compute

$$\begin{aligned} S &= \left(PP_1^{-1}\right) T \left(P_1 P^{-1}\right) \\ &= \frac{1}{2} \begin{bmatrix} p & \Delta \\ 1 & p \end{bmatrix}. \end{aligned}$$

After this point, we fix the notations: $\alpha, \beta, \Delta = p^2 + 4q$ and S.

When $p = 2$ and $q = 1$, w_n is Pell numbers and h_n is Pell-Lucas Numbers. Then $\Delta = 8$ and $x^2 - 2x - 1$.It follows that $\alpha = (1 + \sqrt{2}), \beta = (1 - \sqrt{2})1$.Then in the following result, we give a characterization the power of S;

Theorem 3. *For any integer n, we get that*

$$S^n = \frac{1}{2}\begin{bmatrix} h_n & \triangle w_n \\ w_n & h_n \end{bmatrix}.$$

Proof. It is clear that $S^n = PD^nP^{-1}$ for any positive integer n where $D = \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix}$ and so it follows that

$$\begin{aligned} S^n &= P\left(P_1^{-1}T^nP_1\right)P^{-1} \\ &= \frac{1}{2}\begin{bmatrix} w_{n+1}+qw_{n-1} & \Delta w_n \\ \frac{4qw_n-p^2w_n+2p.pw_n}{\Delta} & w_{n+1}+qw_{n-1} \end{bmatrix} \\ &= \frac{1}{2}\begin{bmatrix} h_n & \Delta w_n \\ w_n & h_n \end{bmatrix}. \end{aligned}$$

□

By using the matrix S, we obtain the Binet formulas for the sequence w_n and h_n easily;

Theorem 4. *(Binet Formulas) For any integer n, we have*

$$w_n = \frac{\alpha^n - \beta^n}{\sqrt{\Delta}} \text{ and } h_n = \alpha^n + \beta^n.$$

Proof. With the above notations, we have that

$$\begin{aligned} S^n &= PD^nP^{-1} \\ &= \begin{bmatrix} 1 & 1 \\ \frac{1}{\sqrt{\Delta}} & \frac{-1}{\sqrt{\Delta}} \end{bmatrix}\begin{bmatrix} \alpha^n & 0 \\ 0 & \beta^n \end{bmatrix}\frac{1}{2}\begin{bmatrix} 1 & \sqrt{\Delta} \\ 1 & -\sqrt{\Delta} \end{bmatrix} \\ &= \frac{1}{2}\begin{bmatrix} \alpha^n+\beta^n & \sqrt{\Delta}\left(\alpha^n-\beta^n\right) \\ \frac{\alpha^n-\beta^n}{\sqrt{\Delta}} & \alpha^n+\beta^n \end{bmatrix} \end{aligned}$$

for any integer n and using the matrix equality, we get the formula for the sequences

$$w_n = \frac{\alpha^n - \beta^n}{\sqrt{\Delta}} \text{ and } h_n = \alpha^n + \beta^n.$$

□

Theorem 5. *For any integer n, The Binet formula for Pell numbers P_n and Pell-Lucas Numbers p_n,are*

$$P_n = \frac{(1+\sqrt{2})^n - (1+\sqrt{2})^n}{\sqrt{8}} \text{ and } p_n = (1+\sqrt{2})^n + (1-\sqrt{2})^n.$$

Theorem 6. *For any integers n and m, we get that*

1. $4(-2)^n = p_n^2 - 8P_n^2,$
2. $2p_{n+m} = p_np_m + 8P_nP_m,$
3. $2P_{n+m} = P_np_m + p_nP_m.$

Proof. For Pell numbers P_n and Pell-Lucas Numbers p_n, the matrix

$$S^n = \frac{1}{2}\begin{bmatrix} p_n & 8P_n \\ P_n & p_n \end{bmatrix}$$

. For any integers n, we compute the determinant of S^n and obtain

$$\det(S^n) = \frac{1}{4}(p_n - 8P_n^2) = (-2)^n.$$

Moreover, by Theorem 3, it follows that

$$\begin{aligned} S^{n+m} &= \frac{1}{2}\begin{bmatrix} p_n & 8P_n \\ P_n & p_n \end{bmatrix}\frac{1}{2}\begin{bmatrix} p_m & 8P_m \\ P_m & p_m \end{bmatrix} \\ &= \frac{1}{4}\begin{bmatrix} p_np_m + 8P_nP_m & 8(P_np_m + p_nP_m) \\ P_nh_m + p_nP_m & h_nh_m + (P_nP_m \end{bmatrix} \end{aligned}$$

Then the matrices equality has completed the proof. □

Moreover, Theorem 6 makes us get the following identities;

Corollary 7. *For any integer m, we get that*

1. $2p_{2m} = p_m^2 + 8P_m^2$,
2. $P_{2m} = P_m p_m$,
3. $2p_{2m+1} = p_{m+1}p_m + 8P_{m+1}P_m$,
4. $2P_{2m+1} = P_{m+1}p_m + p_{m+1}P$.

Theorem 8. *For any integers n and m, we get that*

$$\begin{aligned} 2(-2)^m p_{n-m} &= p_np_m - 8P_nP_m, \\ 2(-2)^m P_{n-m} &= P_np_m - h_nP_m, \\ p_np_m &= (-8)^m p_{n-m} + p_{n+m}, \\ P_np_m &= (-8)^m P_{n-m} + P_{n+m}. \end{aligned}$$

Proof. The inverse of the matrix S^m is $S^{-m} = \frac{1}{(-2)^m}\frac{1}{2}\begin{bmatrix} p_m & -8P_m \\ -P_m & p_m \end{bmatrix}$ and so we get that

$$\begin{aligned} S^{n-m} &= S^nS^{-m} \\ &= \frac{1}{4(-2)^m}\begin{bmatrix} p_np_m - 8P_nP_m & 8(P_np_m - p_nP_m) \\ P_np_m - p_nP_m & p_np_m - 8P_nP_m \end{bmatrix}. \end{aligned}$$

Then the matrices equalities have completed the proof for the identities

$$\begin{aligned} 2(-2)^m p_{n-m} &= p_np_m - 8P_nP_m \\ 2(-2)^m P_{n-m} &= P_np_m - p_nP_m. \end{aligned}$$

For the the other identities, we compute

$$\begin{aligned} S^{n+m} + (-q)^m S^{n-m} &= S^n\left[S^m + (-q)^m (S^m)^{-1}\right] \\ &= \frac{1}{2}\begin{bmatrix} p_n & 8P_n \\ P_n & p_n \end{bmatrix}\begin{bmatrix} p_m & 0 \\ 0 & p_m \end{bmatrix} \\ &= \frac{1}{2}\begin{bmatrix} p_np_m & 8P_np_m \\ P_np_m & p_nP_m \end{bmatrix} \end{aligned}$$

□

Then the matrix equalities have completed the proof.

Corollary 9. *For positive integer m, we have that*

1. $4\left(-2\right)^m = p_m - 8P_m^2,$
2. $2p\left(-q\right)^m = h_{m+1}p_m - 8w_{m+1}P_m,$
3. $2\left(-2\right)^m = P_{m+1}p_m - p_{m+1}P_m.$
4. $p_m = 2\left(-q\right)^m + p_{2m},$
5. $P_m p_m = P_{2m},$
6. $P_{m+1}p_m = 2\left(-2\right)^m + h_{2m+1},$
7. $P_{m+1}p_m = \left(-2\right)^m + w_{2m+1}.$

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Statistical Assessment of Geochemical Properties of Alanya Barites (Antalya/Türkiye)

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Barite deposits, which can be observed in the color range of white-gray-black in nature, are used as raw material consisting of barium sulfate ($BaSO_4$). Detailed geochemical investigations of this raw material, which is used for several commercial purposes in the industry such as glass manufacturing, paint industry, oil industry, and drilling fluid, are important. The main subject of the study is to interpret the geochemical contents of barite deposits in the Alanya district of Antalya province using "descriptive statistics", which is one of the multivariate statistical methods. A total of 30 samples were collected primarily from barite quarries and their surroundings in Demirtas and Konakl in Alanya (Antalya). In this context, the geochemical contents of the samples collected from the study area were revealed using the X-Rays Fluorescence (XRF) Analysis. Descriptive statistics of the geochemical contents were analyzed. The mean values standard error of the mean of the major and trace elements constituting Alanya barites are listed in descending order as follows: Ba (44.09 2.10) > SO_3 (27.80 1.40) > TiO_2 (8.89 1.21) > SiO_2 (7.52 1.83) > Zr (2.650.10) > CaO (1.47 0.82) > Sr (1.38 0.12) > MgO (1.31 0.67) > V (1.120.12) > Nd (1.030.07) > Na_2O (0.800.49) > Al_2O_3 (0.60 0.11) > Fe_2O_3 (0.530.16) > K_2O (0.250.24) > MnO (0.11 0.05) > Sc (0.100.03) > Sn (0.090.04) > Zn (0.060.06) > I (0.040.01) > Cl (0.030.01) > Sb (0.020.01) > Cu (0.010.00) = Te (0.010.00) = Ta (0.010.00) = In (0.010.00) = Re (0.010.00) > Os (0.0020.00) > Hf (0.0010.00) = Au (0.0010.00) = As (0.0010.00) = Pb (0.0010.00). The results of the interpretation of the geochemical contents of Alanya barites using descriptive statistics revealed that the barium and sulfate had higher concentrations compared to other major and trace elements. Since the barite deposit was enriched with rock-forming components in the environment, it was interpreted that the barite deposit had a syngenetic source.

2010 Mathematics Subject Classifications: 62H10, 62H05

Keywords: Barite, Multivariate Statistical Analyses, Descriptive Statistics, Alanya, Antalya

Introduction

Barite ($BaSO_4$) is the most common barium-containing mineral in nature. It forms under a wide range of physical and chemical conditions. The barite has an extremely low resolution. Due to this physical property, fluids enriched with Ba and SO4 allow barite to precipitate in various geological environments [1]; [2]; [3]. Barite is observed to be a gangue mineral in magmatic rocks such as carbonatites, dacites, and rhyolites [3]. Barite mineral porphyry might also be observed in metallic mineral deposits such as Porphyry Cu, Cu-Mo, and W deposits [1]; [3]. In the literature, four barite precipitation environments are listed as follows: marine/pelagic barite, hydrothermal barite, cold seeps barite, and diagenetic barite [4]; [5]; [3]. Marine/pelagic barite is formed by the deterioration of oversaturated organic material compared to barite in microenvironments [2]; [3]. Hydrothermal barite, on the other hand, precipitates in rich hydrothermal liquids. They are usually formed in fluid environments with seawater interference associated with volcanic activity. Cold seeps barite deposits refer to barites that precipitate between sediment and fluid. Diagenetic barite is formed by the second precipitation of the previously formed barite [2]; [3]. The study on barite deposits in North East Jordan reported the mineralogical, geochemical, and stable isotope characteristics of barite samples. Barites, which are among the Eocene-aged marine carbonates, have a content of 56% BaO and 0.40% Sr. The elements of Cr and V have been enriched with BaO. According to the results of geochemical studies, the formation of barites in the region was found to be compatible with the seawater effect [6]. The study on the mineralogical and geochemical characteristics of barites, which were observed together with hematite in Iran, revealed that barite formation was associated with tectonic activity in the region. Hematite and barite enrichment has formed seams with tectonism in the Eocene-aged volcanic wall rocks [7]. Geological and geochemical studies were carried out on the Dadagl barite deposits in Kahramanmara province in Trkiye. Barites, which have been formed in limestones, have been enriched with seam-type mineralization in the cavities and pores of limestones [8]. Multivariate statistical methods were used for interpreting the geochemical data in the study conducted for Gazipaa barite deposits in the east of Antalya province in Trkiye. The mean $BaSO_4$ content was found to be between 86-99%. The mineralization in the region was mostly observed in dolomitic limestones [9]. Alanya district, which is located in the eastern part of Antalya, hosts many industrial and metallic ores. The distance between the study area, which is 180 km away from the Antalya city center, and the coastline is 50 km. In the literature, the geochemical properties of the barite deposits in Alanya have not been interpreted by using "Descriptive Statistics", which is one of the multivariate statistical methods. This study aims to interpret the geochemical properties of Alanya barites using descriptive statistical methods. The descriptive statistics such as arithmetic mean, median, minimum, maximum, total, range, and standard deviation of the elemental contents involved in the formation of barites and the skewness and kurtosis of their distributions were revealed using this method.

Main results

Descriptive statistics of 30 barite samples collected from Demirtas and Konakl in Alanya (Antalya) are given in (Table 1).

Table 1: Descriptive statistics of barite samples collected from Demirtas and Konakl in Alanya (Antalya)

	Mean	Std. Error of Mean	Std. Deviation	Skewness	Kurtosis
Ba		44.092.10	11.50	-1.81	5.24
SO_3		27.801.40	7.64	-1.45	6.69
TiO_2		8.89 1.21	6.65	1.62	3.59
SiO_2		7.521.83	10.01	3.23	12.78
Zr		2.650.10	0.55	-2.02	5.97
CaO		1.47 0.82	4.49	3.94	15.54
Sr		1.38 0.12	0.67	0.59	-0.99
MgO		1.310.67	3.69	3.20	9.83
V		1.120.12	0.66	0.31	0.71
Nd		1.030.07	0.40	1.04	5.63
Na_2O		0.800.49	2.70	5.02	26.47
Al_2O_3		0.60 0 .11	0.62	5.04	26.61
Fe_2O_3		0.530.16	0.89	2.60	5.86
K_2O		0.250.24	1.33	5.47	29.97
MnO		0.11 0 .05	0.25	2.56	6.07
Sc		0.100.03	0.14	4.30	21.04
Sn		0.090.04	0.24	5.47	29.95
Zn		0.060.06	0.31	5.47	29.98
I		0.040.01	0.03	-0.38	-1.92
Cl		0.030.01	0.04	4.85	25.44
Sb		0.020.01	0.06	5.43	29.60
Cu		0.01 0.00	0.03	3.51	13.90
Te		0.010.00	0.01	-1.38	0.58
Ta		0.010.00	0.01	0.42	-1.16
In		0.010.00	0.00	-1.36	1.04
Re		0.010.00	0.01	0.62	-0.18
Os		0.0020.00	0.00	2.01	4.86
Hf		0.0010.00	0.00	1.96	2.49
Au		0.0010.00	0.00	1.25	0.29
As		0.0010.00	0.00	2.05	4.49
Pb		0.0010.00	0.00	1.53	1.49

As one of the multivariate statistics, the kurtosis value, which provides information about the shape of the distribution, refers to the steepness or flatness of the distribution curve. When the shape of the curve is steep, the kurtosis value becomes positive; on the other hand, the kurtosis value becomes negative when the curve is flattened [10]; [11]; [12]. The skewness value ranges between $-\infty$ $+\infty$. If the skewness value is between -3 and 3, the distribution is considered normal [13]; [14]; [15].

Conclusion

Considering the average concentrations of the geochemical contents of the samples collected from the Alanya barite deposits, the barium in the environment was observed to be together with the rock-forming components. Barium, which is an industrial raw material, has formed together with wall rocks in the region. Therefore, it can be interpreted that the barite in the region has a syngenetic source. The kurtosis values of the barite samples collected from Alanya are listed in descending order as follows: Zn (29.98) > K_2O (29.97) > Sn (29.95) > Sb (29.60) > Al_2O_3 (26.61) > Na_2O (26.47) > Cl (25.44) > Sc (21.04) > CaO (15.54) > Cu (13.90) > SiO_2 (12.78) > MgO (9.83) > SO_3 (6.69) > MnO (6.07) > Zr (5.97) > Fe_2O_3 (5.86) > Nd (5.63) > Ba (5.24) > Os (4.86) > As (4.49) > TiO_2 (3.59) > Hf (2.49) > Pb (1.49) > In (1.04) > V (0.71) > Te (0.58) > Au (0.29) > Re (-0.18) > Sr (-0.99) > Ta (-1.16) > I (-1.92). This is directly proportional to the contents of the barite samples in descending order. It can be stated that the curves of the elements with high concentrations are steep while the curves of the elements with low concentrations are flattened. The skewness values of the barite samples collected from Alanya are listed in descending order as follows: K_2O (5.47) = Zn (5.47) = Sn (5.47) > Sb (5.43) > Al_2O_3 (5.04) > Na_2O (5.02) > Cl (4.85) > Sc (4.30) > CaO (3.94) > Cu (3.51) > SiO_2 (3.23) > MgO (3.20) > Fe_2O_3 (2.60) > MnO (2.56) > As (2.05) > Os (2.01) > Hf (1.96) > TiO2 (1.62) > Pb(1.53) > Au (1.25) > Nd (1.04) > Re (0.62) > Sr (0.59) > Ta (0.42) > V (0.31) > I (-0.38) > In (-1.36) > Te (-1.38) > SO_3 (-1.45) > Ba (-1.81) > Zr (2.02). In this context, the skewness values of the curves of K_2O, Zn, Sn, Sb, Al_2O_3, Na_2O, Cl, Sc, CaO, Cu, SiO_2, and MgO exceed the range between -3 and 3 and show a left-skewed distribution while the Fe_2O_3, MnO, As, Os, Hf, TiO_2; Pb, Au, Nd, Re, Sr, Ta, V, I, In, Te, SO_3, Ba, Zr concentrations have a normal distribution.

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Correlation of Geochemical Contents of Alanya (Antalya) Barites

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Barite deposits, the main components of which are Ba^{2+} and SO_4^{2-}, are formed in different geological environments. Pearson correlation analysis, one of the multivariate statistical analyses, was applied to the chemical data to reveal the geological formation conditions of the barite deposits in Alanya (Antalya). In this context, X-Rays Fluorescence Spectrometer (XRF) analysis was applied to the samples collected from the study area. The results obtained constitute the major and trace elemental contents of the Alanya barite deposit. Pearson correlation analysis of chemical data revealed positive and negative correlations between the major and trace elements of Alanya barites. According to the interpretations made on the results of the correlation analysis, the barite enrichment in the region was found to increase with the rock-forming components.

2010 Mathematics Subject Classifications: 62H10, 62H05

Keywords: Multivariate Statistics, Correlation, Barite, Antalya, Alanya

Introduction

The trace elements S, O, Ba, Sr, Ca, Ra, Pb, and Nd play a major role in the formation of barite [1]. While the main components of barite are Ba^{2+} and S, the main component of its oxidized form is SO_4^{2-}. Barite ($BaSO_4$), which has a high density, is used as industrial raw material [2]. Barite, which forms in marine environments, is found in the paragenesis of metallic minerals Pb, Zn, Cu, and Au under high sulfidation. Mostly, the sedimentary processes are effective in the formation of barite deposits. Magmatic, hydrothermal, and metamorphic fluids also play a role in the formation of barite deposits [1]; [2]; [3]; [4]; [5]; [6]; [7]. Suitable environmental conditions for the formation of barite deposits are listed as follows: biochemical interaction/process, fluid-rock alteration, and the mixture of two or more fluids [8]; [2]. The barite deposit with the highest reserve in Trkiye was formed as a result of Alpine and Hercynian orogenies [9]; [2]. Tectonic belts provide suitable formation environments for barite deposits. Information about the formation and source of the deposit was obtained thanks to various mineralogical and geochemical studies conducted on the Chenarvardeh barite deposit in Iran. Barite enrichments occurred as layers and lenses in pyroclastic rocks in the region. In the formation of this deposit, barite precipitation occurred when hydrothermal fluids encountered seawater containing sulfates, and the formation of the Chenarvardeh deposit was attributed to this reaction [10]. The geology, geochemistry, and isotope chemistry of the Farsesh barite deposit in Iran were examined. Barites showed hydrothermal mineralization with epigenetic character in dolomitized limestones [11]. In the study conducted on barite galena deposits in the Baghin region in Iran, the results of the geochemical analysis were interpreted using multivariate statistical analysis. In this context, elemental relationships were evaluated using correlation analysis. In the barite formation environment, the iron-oxy-hydroxides were found to show a high positive correlation with the elements added to the clay structure [12]. Alanya district, which is located in the southeastern part of the Antalya province, hosts many raw materials used in industry and mining. In the literature, the interactions of the major and trace elements that played a role in the formation of the barite deposits in the Alanya district have not been assessed using correlation analysis, which is one of the multivariate statistical methods. In this context, the relationship between the chemical analysis results of 30 barite samples collected from the study area was evaluated by correlation analysis [13]; [14]; [15]; [16]; [17]; [18]; [19].

Main results

The "Pearson Correlation Analysis" of the chemical analysis results of the samples collected from the Alanya barite deposit is presented in (Table 1).

	Cl	V	Cu	Zn	As	Sr	In	Sn	Sb	Te	I	Ba	Nd	Hf	Ta	Au	Pb	MgO	Al_2O_3	SiO_2	SO_3	K_2O	CaO	TiO_2	MnO	Fe_2O_3	Na_2O	Sc	Zr	Re	Os
Cl	1																														
V	-0.352	1																													
Cu	-0.129	-0.222	1																												
Zn	.963**	-0.317	-0.1	1																											
As	-0.163	-0.099	0.267	-0.116	1																										
Sr	-0.212	0.078	-0.237	-0.211	-0.148	1																									
In	-0.03	-0.103	0.147	-0.019	-0.015	0.208	1																								
Sn	.963**	-0.32	-0.099	.999**	-0.119	-0.204	-0.002	1																							
Sb	.959**	-0.329	-0.11	.997**	-0.113	-0.208	-0.022	.997**	1																						
Te	-.378*	0.027	0.2	-.384*	0.257	-0.165	0.011	-.385*	-.419*	1																					
I	-0.166	-0.045	0.305	-0.226	0.202	0.113	0.114	-0.212	-0.206	-0.105	1																				
Ba	-.664**	0.062	0.137	-.698**	0.174	0.194	0.281	-.677**	-.684**	.378*	.512**	1																			
Nd	-.473**	.510**	-0.017	-.489**	0.154	0.165	0.176	-.486**	-.511**	0.215	0.018	.385*	1																		
Hf	.447*	-0.349	0.32	.495**	-0.076	-0.261	0.201	.502**	.485**	-0.003	0.204	-0.161	-0.3	1																	
Ta	-0.118	-0.115	.385*	-0.127	-0.02	-0.17	-0.108	-0.115	-0.135	0.334	.434*	.439*	-0.101	.410*	1																
Au	0.289	-0.127	-0.019	0.327	0.11	-.483**	-0.069	0.335	0.356	-0.156	0.089	0.038	-0.269	0.309	0.247	1															
Pb	-0.098	-0.226	.567**	-0.116	0.283	-0.278	-0.127	-0.119	-0.099	0.028	0.075	0.01	0.144	0.025	0.063	0.082	1														
MgO	.516**	-.538**	-0.152	.426*	-0.186	-0.256	-0.087	.413*	.427*	-0.072	-.391*	-.545**	-.404*	0.111	-0.321	-0.008	0.213	1													
Al_2O_3	.936**	-.390*	-0.046	.974**	-0.055	-0.233	-0.084	.969**	.970**	-.381*	-0.256	-.762**	-.563**	.450*	-0.157	0.309	-0.082	.474**	1												
SiO_2	.804**	-.443*	0.204	.831**	-0.019	-0.332	-0.134	.820**	.823**	-0.32	-0.229	-.795**	-.546**	.507**	-0.08	0.232	0.189	.514**	.885**	1											
SO_3	-.744**	.409*	-0.063	-.686**	0.118	.376*	0.033	-.680**	-.682**	0.231	0.212	.574**	0.318	-.430*	0.097	-0.274	-0.226	-.735**	-.672**	-.758**	1										
K_2O	.964**	-0.324	-0.097	1.000**	-0.113	-0.213	-0.017	.999**	.997**	-.385*	-0.228	-.698**	-.496**	.494**	-0.125	0.331	-0.115	.428*	.976**	.834**	-.688**	1									
CaO	0.155	-.453*	-0.123	0.045	-0.142	-0.198	-0.149	0.028	0.044	-0.022	-0.327	-0.354	-0.286	-0.089	-0.315	-0.145	0.107	.812**	0.145	0.249	-.506**	0.051	1								
TiO_2	-0.309	.901**	-0.184	-0.245	-0.112	0.098	-0.182	-0.254	-0.259	-0.089	-0.181	-0.142	.499**	-0.322	-0.232	-0.19	-0.14	-.424*	-0.296	-0.284	0.267	-0.252	-0.359	1							
MnO	0.039	-0.297	-0.142	-0.048	-0.038	-0.144	-.391*	-0.067	-0.05	-0.015	-0.262	-0.286	-0.089	-0.19	-0.225	-0.144	.384*	.732**	0.03	0.177	-.431*	-0.049	.758**	-0.188	1						
Fe_2O_3	.576**	-.611**	-0.038	.493**	-0.069	-0.326	-0.072	.481**	.501**	-0.134	-0.308	-.578**	-.488**	0.2	-0.286	0.093	0.272	.960**	.556**	.633**	-.788**	.498**	.801**	-.492**	.631**	1					
Na_2O	.943**	-0.318	-0.087	.972**	-0.092	-0.274	-0.017	.973**	.967**	-0.317	-0.173	-.630**	-.509**	.525**	-0.033	.362*	-0.144	.385*	.932**	.780**	-.644**	.971**	0.011	-0.277	-0.104	.459*	1				
Sc	-0.253	0.352	-0.108	-0.142	-0.097	0.17	-0.118	-0.154	-0.147	-0.091	-0.215	-0.243	-0.026	-0.2	-0.305	-0.202	-0.148	-0.234	-0.059	-0.149	.569**	-0.145	-0.205	.390*	-0.178	-0.277	-0.168	1			
Zr	-.677**	0.36	0.01	-.704**	0.115	0.359	0.316	-.687**	-.705**	0.273	0.332	.848**	.779**	-0.316	0.192	-0.173	0.01	-.589**	-.798**	-.839**	.560**	-.708**	-.422*	0.232	-0.246	-.679**	-.684**	-0.154	1		
Re	-0.22	-0.126	0.037	-0.22	0.269	0.038	0.251	-0.201	-0.21	0.272	0.291	.600**	-0.005	-0.045	.451*	.407*	-0.06	-0.323	-0.254	-0.321	0.225	-0.211	-0.296	-0.261	-.369*	-0.249	-0.152	-0.281	.367*	1	
Os	-0.013	0.029	0.194	-0.033	-0.07	-0.082	-0.223	-0.031	-0.048	0.127	0.155	0.143	-0.079	-0.001	.624**	0.132	0.134	-0.174	-0.026	0.014	0.02	-0.032	-0.172	-0.048	0.038	-0.176	0.025	-0.14	0.051	0.295	1

Table 1: Pearson correlation analysis of the major and trace elements of Alanya barite deposit

Conclusion

According to the results obtained from the correlation analysis of the samples collected from the barite deposit in Alanya (Antalya), there are very strong positive correlations between the following elements: Cl and Zn (0.963**), Cl and Sn (0.963**), Cl and Sb (0.959**), Cl and Al_2O_3 (0.936**), Cl and K_2O (0.964**), Cl and Na_2O (0.943**); V and TiO_2 (0.901**); Zn and Sn (0.999**), Zn and Sb (0.997**), Zn and Al_2O_3 (0.974**), Zn and K_2O (1.000**), Zn and Na_2O (0.972**); Sn and Sb (0.997**), Sn and Al_2O_3 (0.969**), Sn and K_2O (0.999**), Sn and Na_2O (0.973**); Sb and Al_2O_3 (0.970**), Sb and K_2O (0.997**), Sb and Na_2O (0.967**); MgO and Fe_2O_3 (0.960**); Al_2O_3 and K_2O (0.976**), Al_2O_3 and Na_2O (0.932**); K_2O and Na_2O (0.971**). There are strong positive correlations between the following elements: Zn and SiO_2 (0.831**); Sn and SiO_2 (0.820**); Sb and SiO_2 (0.823**); Ba and Zr (0.848**); MgO and CaO (0.812**); Al_2O_3 and SiO_2 (0.885**); SiO_2 and K_2O (0.834**); CaO and Fe_2O_3 (0.801**). There are moderate positive correlations between the following elements: Cl and SiO_2 (0.804**); Nd and Zr (0.779**); MgO and MnO (0.732**); SiO_2 and Na_2O (0.780**); CaO and MnO (0.758**). There is a strong negative correlation between SiO_2 and Zr (-0.839**). There are moderate negative correlations between the following elements: Cl and SO_3 (-0.744**); Sb and Zr (-0.705**); Ba and Al_2O_3 (-0.762**), Ba and SiO_2 (-0.795**); MgO and SO_3 (-0.735**); Al_2O_3 and Zr (0.798**); SiO_2 and SO_3 (-0.758**); SO_3 and Fe_2O_3 (-0.788**); K_2O and Zr (-0.708**). The very strong positive correlations between vanadium, which is one of the transition metals, and K_2O, the oxidized state of potassium, which is one of the alkali metals, and between Sn, an element in the carbon group and titanium family, and Al_2O_3, an earth metal from the scandium family indicate that some of the transition metals in the barite formation environment act together with the main rock-forming oxide components.

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Approximation properties of the generalized Kantorovich type Szász-Mirakjan operators based on q-integers

Mustafa Kara

In this paper, a new generalized Kantorovich variant based on q-integers of Szász-Mirakyan operators is defined as follows:

$$K_{n,q}^{l}(f;x)=\sum_{k=0}^{\infty}s_{n,k}\left(q,x\right)\int_{0}^{1}...\int_{0}^{1}f\left(\frac{q^{(1-k)}\left[k\right]_{q}+t_{1}+...+t_{l}}{\left[n\right]_{q}}\right)dt_{1}...dt_{l},$$

Using Krovokin's type theorem, we show that operator converges uniformly. Next, the local approximation properties in terms of modulus of continuity are studied for these operators. Moreover, quantitative Vorononskaja's theorem is provided. Finally, last section is devoted to graphical representation and numerical results for these operators.

2010 Mathematics Subject Classifications: 41A25, 41A36, 47A58

Keywords: q-Szász-Mirakjan operators, Lipschitz class, Voronovskaya type theorem

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Asymptotic normality for regression function estimate under right censored data and association

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The goal of this work is to establish asymptotic normality of the kernel estimator of regression function, in the random right censored model for associated data. An application to prediction and confidence bands are also given. Simulations are drawn to lend further support to our theoretical results for finite samples sizes.

KEYWORDS: Associated data, Asymptotic normality, Censored data, Kaplan-Meier estimator, Kernel regression estimator

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Extending Quasi-alternating links

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Knot theory is the branch of low dimensional topology concerned with the study of knots and links. The ultimate goal of this theory is the classification of these objects up to natural deformations. Quasi-alternating links represent an important class of links in the three-sphere introduced by Ozsváth and Szabò [1] as a generalization of alternating links. While alternating links are known to have a simple diagrammatic definition, this new class of links are defined in a recursive way. In general, using the recursive definition, it is very hard to determine whether a given link is quasi-alternating. Over the past fifteen years, several obstruction criteria for quasi alternateness of links have been introduced in terms of link homology and polynomial invariants.

In this talk, we show that a link obtained by extending a quasi-alternating crossing in a quasi-alternating link diagram to an alternating tangle of same type is quasi alternating. This extends the work of Champanerkar-Kofman [2] and permits to introduce new examples of quasi-alternating links.

2010 Mathematics Subject Classifications: 57M25, 57M27

Keywords: Quasi-alternating links, Jones polynomial, alternating tangles

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Analytical solution of the fractional linear multi-delayed systems and their Ulam-Hyers stability

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We introduce the delayed Mittag-Leffler type matrix functions, delayed fractional cosine, delayed fractional sine and use the Laplace transform to obtain an analytical solution to the IVP for a Hilfer type fractional linear time-delay system $D_{0,t}^{\mu,\nu} z(t) + Az(t) + \Omega z(t-h) = f(t)$ of order $1 < \mu < 2$ and type $0 \leq \nu \leq 1$, with nonpermutable matrices A and Ω. Moreover, we study Ulam-Hyers stability of the Hilfer type fractional linear multi-delayed system.

2010 Mathematics Subject Classifications: 26A33, 34A12, 34K40, 47H08

Keywords: Hilfer fractional derivative, multi delay systems, stability

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Relations for special polynomials obtained from trigonometric and hyperbolic functions

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The aim of this presentation is to study trigonometric and hyperbolic functions with the special polynomials and numbers. By using methods of the generating functions with special functions and derivative operator, some identities and recurrence relations associated with the Bernoulli numbers, the Euler numbers and special polynomials are given.

2010 Mathematics Subject Classifications: 05A15, 11B68, 26D05

Keywords: Bernoulli numbers, Euler numbers, Trigonometric and hyperbolic functions, Special polynomials, Generating functions

Introduction

The trigonometric functions, the hyperbolic functions and the generating functions are commonly used in mathematics, physics, engineering and all other sciences. Besides, it is well-known that these functions are also closely related to each other. Namely, the Bernoulli numbers and the Euler numbers not only appear in Taylor series expansions of some trigonometric and hyperbolic functions, but also some recurrence relations are computed for the coefficients of these functions. Therefore, the main motivation for this presentation is to investigate special polynomials involving trigonometric and hyperbolic functions and give some interesting results. In order to give these results, we remind of some definitions and notations as follows.

The Bernoulli numbers B_n and the Euler numbers E_n are defined by means of the following generating functions:

$$\frac{w}{e^w - 1} = \sum_{n=0}^{\infty} B_n \frac{w^n}{n!} \tag{1}$$

and

$$\frac{2}{e^w + 1} = \sum_{n=0}^{\infty} E_n \frac{w^n}{n!}, \tag{2}$$

respectively (*cf.* [1]-[7]).

The relations among the trigonometric functions, the hyperbolic functions, the Bernoulli numbers and the Euler numbers are given as follows:

$$\frac{w}{2} \cot\left(\frac{w}{2}\right) = \sum_{n=0}^{\infty} (-1)^n B_{2n} \frac{w^{2n}}{(2n)!}, \tag{3}$$

$$\frac{w}{2}\coth\left(\frac{w}{2}\right)=\sum_{n=0}^{\infty}B_{2n}\frac{w^{2n}}{(2n)!}, \tag{4}$$

$$w\tan(w)=\sum_{n=0}^{\infty}(-1)^n 2^{2n}\left(1-2^{2n}\right)B_{2n}\frac{w^{2n}}{(2n)!} \tag{5}$$

and

$$-\tanh\left(\frac{w}{2}\right)=\sum_{n=0}^{\infty}E_{2n+1}\frac{w^{2n+1}}{(2n+1)!} \tag{6}$$

(*cf.* [2, 4, 5]).

In [7], Simsek defined new classes of generating functions for special numbers and polynomials, which are related to the hyperbolic functions as follows:

$$\frac{aw}{4\sinh\left(\frac{w(k+2)}{2}\right)\cosh\left(\frac{wk}{2}\right)}=\sum_{n=0}^{\infty}\mathcal{Y}_n(k,a)\frac{w^n}{n!} \tag{7}$$

and

$$\frac{awe^{wx}}{4\sinh\left(\frac{w(k+2)}{2}\right)\cosh\left(\frac{wk}{2}\right)}=\sum_{n=0}^{\infty}Q_n(x,k,a)\frac{w^n}{n!}, \tag{8}$$

where a be a real (or complex) numbers and k be a integers with $k\neq -2$.

The polynomials $Q_n^{(C)}(x,y,k,a)$ and the polynomials $Q_n^{(S)}(x,y,k,a)$ are defined by means of the following generating functions:

$$\frac{aw\cos(yw)e^{wx}}{4\sinh\left(\frac{w(k+2)}{2}\right)\cosh\left(\frac{wk}{2}\right)}=\sum_{n=0}^{\infty}Q_n^{(C)}(x,y,k,a)\frac{w^n}{n!} \tag{9}$$

and

$$\frac{aw\sin(yw)e^{wx}}{4\sinh\left(\frac{w(k+2)}{2}\right)\cosh\left(\frac{wk}{2}\right)}=\sum_{n=0}^{\infty}Q_n^{(S)}(x,y,k,a)\frac{w^n}{n!}, \tag{10}$$

respectively (*cf.* [1]).

Setting $y=0$ in (9), we have

$$Q_n^{(C)}(x,0,k,a)=Q_n(x,k,a).$$

Main results

In this section, we give main results of this presentation.

Theorem 1. *Let n be a nonnegative integer with $n\geq 2$. Then we have*

$$\begin{aligned}&(n-1)\,Q_n^{(S)}(x,y,k,a)-xnQ_{n-1}^{(S)}(x,y,k,a)\\ =&\sum_{r=0}^{\left[\frac{n-2}{2}\right]}\binom{n}{2r+2}k^{2r+2}(r+1)\,Q_{n-2-2r}^{(S)}(x,y,k,a)\,E_{2r+1}\\ &-\sum_{r=0}^{\left[\frac{n}{2}\right]}\binom{n}{2r}B_{2r}Q_{n-2r}^{(S)}(x,y,k,a)\left((k+2)^{2r}+(-1)^{r+1}(2y)^{2r}\right).\end{aligned}$$

Proof. By applying the derivative operator $\frac{\partial}{\partial w}$ to the equation (10), we get

$$\frac{aw\sin(yw)\,e^{wx}\left(-\frac{wk}{2}\tanh\left(\frac{wk}{2}\right)-\frac{w(k+2)}{2}\coth\left(\frac{w(k+2)}{2}\right)\right)}{4\sinh\left(\frac{w(k+2)}{2}\right)\cosh\left(\frac{wk}{2}\right)}$$

$$+\frac{aw\sin(yw)\,e^{wx}\left(wx+1+yw\cot(yw)\right)}{4\sinh\left(\frac{w(k+2)}{2}\right)\cosh\left(\frac{wk}{2}\right)}=\sum_{n=0}^{\infty}nQ_n^{(S)}(x,y,k,a)\frac{w^n}{n!}.$$

Combining the above equation with (3), (4), (6) and (10), we find that

$$\begin{aligned}
&\frac{w^2}{2}\sum_{n=0}^{\infty}\sum_{r=0}^{\left[\frac{n}{2}\right]}\binom{n}{2r}\frac{k^{2r+2}}{2r+1}Q_{n-2r}^{(S)}(x,y,k,a)\,E_{2r+1}\frac{w^n}{n!}\\
&-\sum_{n=0}^{\infty}\sum_{r=0}^{\left[\frac{n}{2}\right]}\binom{n}{2r}Q_{n-2r}^{(S)}(x,y,k,a)\,(k+2)^{2r}B_{2r}\frac{w^n}{n!}\\
&+xw\sum_{n=0}^{\infty}Q_n^{(S)}(x,y,k,a)\frac{w^n}{n!}+\sum_{n=0}^{\infty}Q_n^{(S)}(x,y,k,a)\frac{w^n}{n!}\\
&+\sum_{n=0}^{\infty}\sum_{r=0}^{\left[\frac{n}{2}\right]}\binom{n}{2r}Q_{n-2r}^{(S)}(x,y,k,a)\,(-1)^r(2y)^{2r}B_{2r}\frac{w^n}{n!}\\
=\ &\sum_{n=0}^{\infty}nQ_n^{(S)}(x,y,k,a)\frac{w^n}{n!}.
\end{aligned}$$

After some calculations, comparing the coefficients of $\frac{w^n}{n!}$ on both sides of the above equation, we arrive at the Theorem 1. □

Theorem 2. *Let n be a nonnegative integer with $n\geq 2$. Then we have*

$$\begin{aligned}
&(n-1)\,Q_n^{(C)}(x,y,k,a)-xnQ_{n-1}^{(C)}(x,y,k,a)\\
=&\sum_{r=0}^{\left[\frac{n-2}{2}\right]}\binom{n}{2r+2}k^{2r+2}(r+1)\,Q_{n-2-2r}^{(C)}(x,y,k,a)\,E_{2r+1}\\
&-\sum_{r=0}^{\left[\frac{n}{2}\right]}\binom{n}{2r}B_{2r}Q_{n-2r}^{(C)}(x,y,k,a)\left((k+2)^{2r}+(-1)^r(2y)^{2r}\left(1-2^{2r}\right)\right).
\end{aligned}$$

Proof. By applying the derivative operator $\frac{\partial}{\partial w}$ to the equation (9), then combining the final equation with (4), (5), (6) and (9), we have

$$\begin{aligned}
&\frac{w^2}{2}\sum_{n=0}^{\infty}\sum_{r=0}^{\left[\frac{n}{2}\right]}\binom{n}{2r}\frac{k^{2r+2}}{2r+1}Q_{n-2r}^{(C)}(x,y,k,a)\,E_{2r+1}\frac{w^n}{n!}\\
&-\sum_{n=0}^{\infty}\sum_{r=0}^{\left[\frac{n}{2}\right]}\binom{n}{2r}Q_{n-2r}^{(C)}(x,y,k,a)\,(k+2)^{2r}B_{2r}\frac{w^n}{n!}+\sum_{n=0}^{\infty}Q_n^{(C)}(x,y,k,a)\frac{w^n}{n!}\\
&-\sum_{n=0}^{\infty}\sum_{r=0}^{\left[\frac{n}{2}\right]}\binom{n}{2r}Q_{n-2r}^{(C)}(x,y,k,a)\,(-1)^r(2y)^{2r}\left(1-2^{2r}\right)B_{2r}\frac{w^n}{n!}\\
&=\sum_{n=0}^{\infty}nQ_n^{(C)}(x,y,k,a)\frac{w^n}{n!}-xw\sum_{n=0}^{\infty}Q_n^{(C)}(x,y,k,a)\frac{w^n}{n!}.
\end{aligned}$$

After simple calculations, comparing the coefficients of $\frac{w^n}{n!}$ on both sides of the above equation, we arrive at the Theorem 2. □

Setting $y = 0$ into the Theorem 2, we get the following corollary:

Corollary 3. *Let n be a nonnegative integer with $n \geq 2$. Then we have*

$$\begin{aligned} &(n-1) Q_n(x,k,a) - xnQ_{n-1}(x,k,a) \\ = &\sum_{r=0}^{\left[\frac{n-2}{2}\right]} \binom{n}{2r+2} k^{2r+2}(r+1) Q_{n-2-2r}(x,k,a) E_{2r+1} \\ &-\sum_{r=0}^{\left[\frac{n}{2}\right]} \binom{n}{2r} (k+2)^{2r} Q_{n-2r}(x,k,a) B_{2r}. \end{aligned} \tag{11}$$

On the other hand, Simsek gave a recurrence relation for the polynomials $Q_n(x,k,a)$ as follows:

$$\begin{aligned} &(n-1) Q_n(x,k,a) - xnQ_{n-1}(x,k,a) \\ = &-\frac{2}{a}\sum_{r=0}^{\left[\frac{n}{2}\right]} \binom{n}{2r} Q^{(2)}_{n-2r}(x,k,a)\left((k+1)^{2r+1}+1\right), \end{aligned} \tag{12}$$

where

$$Q^{(2)}_m(x,k,a) = \sum_{j=0}^{m} \binom{m}{j} \mathcal{Y}_j(k,a) Q_{m-j}(x,k,a)$$

(*cf.* [7, Theorem 10]).

Combining (11) with (12), we obtain the following result:

Corollary 4. *Let n be a nonnegative integer with $n \geq 2$. Then we have*

$$\begin{aligned} &\sum_{r=0}^{\left[\frac{n}{2}\right]} \binom{n}{2r} \left(Q^{(2)}_{n-2r}(x,k,a)\left((k+1)^{2r+1}+1\right) - \frac{a}{2}(k+2)^{2r} Q_{n-2r}(x,k,a) B_{2r}\right) \\ &\qquad = -\frac{a}{2}\sum_{r=0}^{\left[\frac{n-2}{2}\right]} \binom{n}{2r+2} k^{2r+2}(r+1) Q_{n-2-2r}(x,k,a) E_{2r+1}. \end{aligned}$$

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Evaluation of the Relationships Between Mechanical and Physical Properties of Cementitious Materials

Niyazi Ugur Kockal *[1] *and Nazli Ipek Kul Gul* [2]

As a result of rapidly increasing urbanization, environmental pollution has increased, and natural resources have been depleted. For this reason, the use of wastes in construction materials (like cementitious ones) as raw materials and energy sources is took part in both industrial and scientific studies.

In this study, the use of plastic wastes with different chemical compositions in cementitious mixtures as an aggregate was investigated. Physical and mechanical properties of these mixtures were tested. These properties were compressive strength, flexural strength, water absorption, unit weights and fresh unit volume weight. Some mathematical equations derived to show the relationships between the physical and mechanical properties of these mixtures. It could be concluded that the ratio and types of the wastes were affected these relationships differently and there some correlations between the physical and the mechanical properties of these mixtures.

KEYWORDS: Cementitious mixtures, Construction materials, Mathematical derivations, Mechanical Properties, Physical Properties, Plastic waste

Introduction

The concept of sustainability has gained importance in both academic and industrial studies, owing to the developing technology and high consumption. Therefore, studies in many areas are evaluated within this concept. Due to increasing environmental pollution, depleted energy and raw material resources, one of the important points of sustainability has been the protection of the environment. The evaluation of many industrial wastes and by-products such as alternative raw materials or fuels is also become even more important for the academic studies [1, 2, 3]. In this context, various wastes in the construction industry, especially for construction materials, are used to meet different needs.

If it is not possible to prevent the generation of waste, it is vital to ensure reuse according to the waste hierarchy [4, 5]. In this study, plastic wastes were added to the cementitious mixtures instead of aggregates in order to reuse the plastic wastes. Moreover, the physical and mechanical properties of these mixtures were analyzed and the relationships between the results were detected.

Main results

Materials and Methods

Certain materials were used to prepare the cementitious mixtures. The materials were ordinary Portland cement, a superplasticizer type of chemical additive, fine crushed limestone aggregate (0-4 mm), plastic wastes of polystyrene and polycarbonate and water. The plastic wastes were used as aggregate as distinct from the traditional cement mortar. The wastes of polystyrene and polycarbonate were coded as S and K, respectively.

The mixtures were prepared like as traditional mortars and they were cured by standard water curing during 28 days. The cured and hardened samples' flexural and compressive strengths were evaluated according to the equations 1 and 2 respectively. The water absorption and surface saturated dried (SSD) unit weight of the samples were calculated by the equations 3 and 4 respectively.

$$\sigma_f = \frac{3 \times F \times l}{2 \times b \times d^2} \tag{1}$$

where

σ_f : Flexural strength (MPa)

F : maximum load (N)

l : Distance between two supporting pins (mm)

b : Width of specimen (mm)

d : Thickness of specimen (mm)

$$\sigma_C = \frac{F}{A} \tag{2}$$

where

σ_C : Compressive strength (MPa)

F : Maximum load (N)

A : The cross section of the area of the material resisting the load (mm^2)

$$\text{water absorption } (\%) = \frac{m_{SSD-A} - m_{OD}}{m_{OD}} \times 100 \tag{3}$$

$$UW_{SSD}\left(\frac{g}{cm^3}\right) = \frac{m_{SSD-A}}{m_{SSD-A} - m_{SSD-W}} \tag{4}$$

where

m_{OD} : oven dried mass of the samples (g)

m_{SSD-A} : SSD mass of the samples at air (g)

m_{SSD-W} : SSD mass of the samples at water (g)

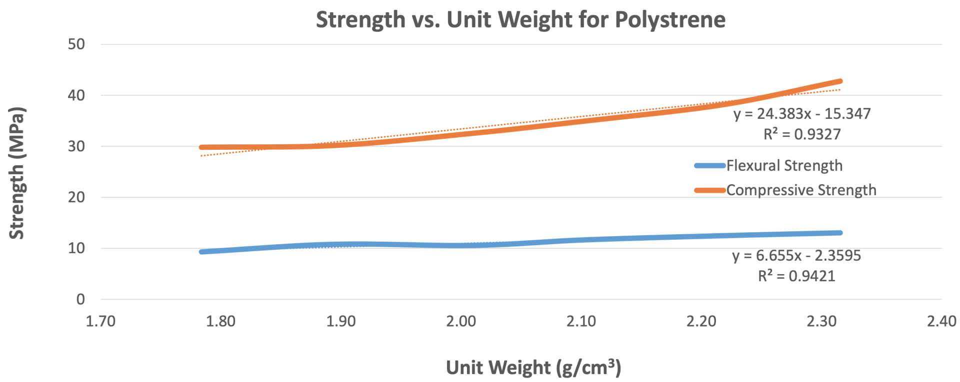

Figure 1: The relation between strength and unit weight for polystyrene included mixtures

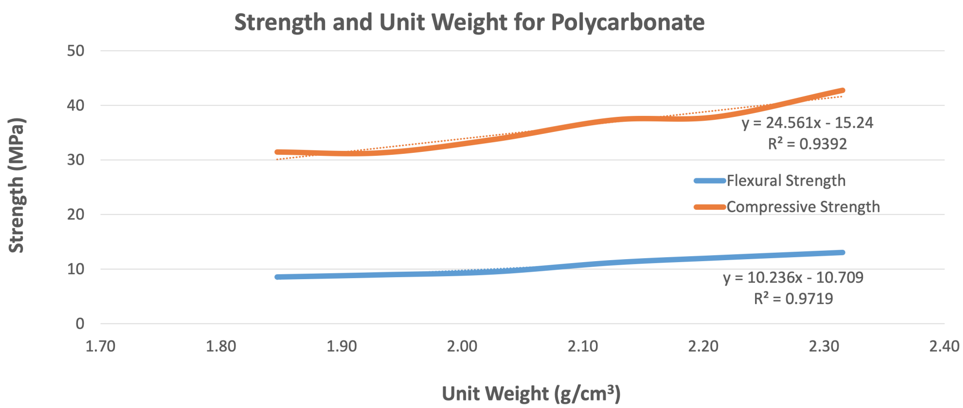

Figure 2: The relation between strength and unit weight for polycarbonate included mixtures

Results

The results of the physical and mechanical properties of the samples including polystyrene and polycarbonate wastes were given in Figures 1-5. The relationships between the properties of the mixtures obtained as a result of experimental studies were examined and also given in Figure 1-5.

It was determined that the strengths were directly proportional to the unit weight and the trends for both types of plastic wastes are quite similar (Figure 1 and 2). It was observed that the increase in the strengths started especially with the unit weight exceeding 2 g/cm^3, and this increase was higher in the compressive strength.

The equations 5 and 6 belonged to the polystyrene included mixtures and the equations 7 and 8 belonged to the polycarbonate included mixtures.

$$\sigma_C = 24.383 \times UW_{SSD} - 15.347 \qquad \mathrm{R}^2 = 0.9327 \qquad (5)$$
$$\sigma_f = 6.655 \times UW_{SSD} - 2.3595 \qquad \mathrm{R}^2 = 0.9421 \qquad (6)$$
$$\sigma_C = 24.561 \times UW_{SSD} - 15.24 \qquad \mathrm{R}^2 = 0.9392 \qquad (7)$$
$$\sigma_f = 10.236 \times UW_{SSD} - 10.709 \qquad \mathrm{R}^2 = 0.9719 \qquad (8)$$

The high amount of waste is very vital in the production of sustainable construction materials. For this reason, the variation of the strength properties of cementitious mixtures with the increasing waste ratio was investigated. In Figures 3 and 4, the relationship between strength and waste ratio was examined.

As could be seen from these graphs, the strengths decreased as the amount of waste increased. But the trend and reasonableness of this downward is actually more important. In addition, it has been observed that the flexural strength is less affected by the amount of waste than the compressive strength. The equations 10 and 11 belonged to the polystyrene included mixtures and the equations 12 and 13 belonged to the polycarbonate included mixtures.

$$\sigma_C = 2.6034 \times \text{waste ratio} + 25.628 \qquad \mathrm{R}^2 = 0.9411 \qquad (9)$$
$$\sigma_f = 0.7084 \times \text{waste ratio} + 8.8318 \qquad \mathrm{R}^2 = 0.9447 \qquad (10)$$
$$\sigma_C = 2.2874 \times \text{waste ratio} + 27.743 \qquad \mathrm{R}^2 = 0.9266 \qquad (11)$$
$$\sigma_f = 0.9572 \times \text{waste ratio} + 7.1913 \qquad \mathrm{R}^2 = 0.9668 \qquad (12)$$

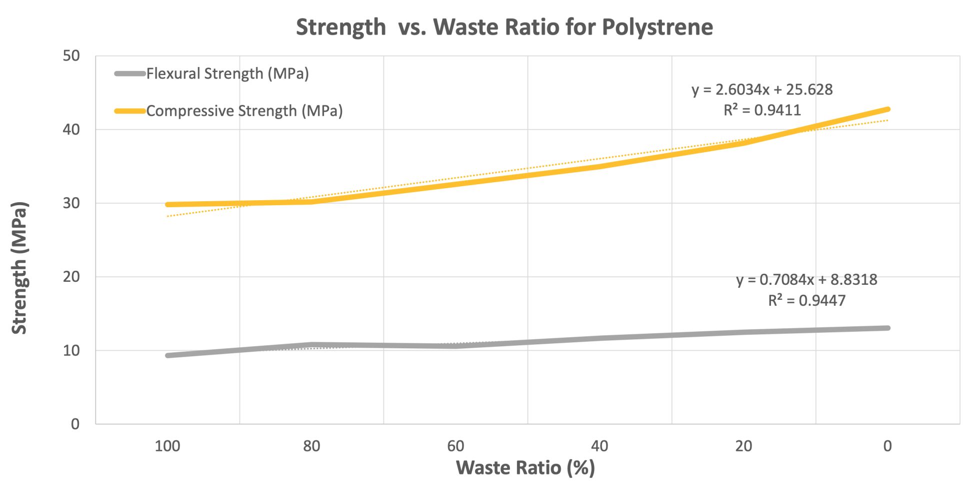

Figure 3: The relation between strength and waste ratio for polystyrene included mixtures

In many usage areas of cementitious mixtures, low water absorption is expected. In this respect, it is desirable that the amount of water absorption does not increase while the unit weights of these mixtures decrease. In Figure 5, the relationship between unit weight and water absorption was examined. The relations of these properties were given in Equation 13 and 14 for polycarbonate and polystyrene included mixtures, respectively.

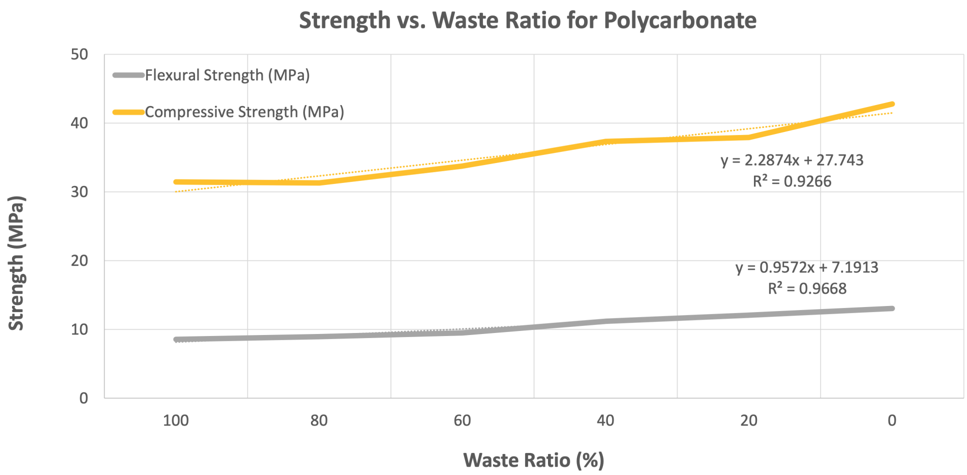

Figure 4: The relation between strength and waste ratio for polycarbonate included mixtures

$$WA = -3.6272 \times UW_{SSD}^2 + 13.868 \times UW_{SSD} - 9.2382 \qquad \mathrm{R}^2 = 0.9127 \qquad (13)$$
$$WA = -1.85 \times UW_{SSD} + 7.5826 \qquad \mathrm{R}^2 = 0.9725 \qquad (14)$$

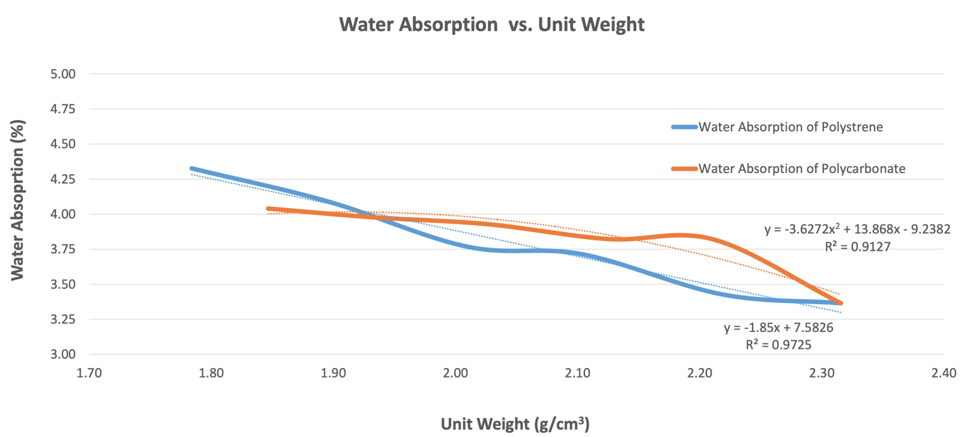

Figure 5: The relation between water absorption and unit weight for all mixtures

Conclusions

The relationship between the material properties is important for the products that behave differently. On that sense, knowing a property of the material could enable the prediction of another property. In this way, both time savings and cost reductions will be achieved, and these relationships will create the basis for other models.

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A spectrum of Zariski topology over multiplication Krasner hypermodules

Burcu Nişancı Türkmen [1] *and Öznur Kulak* [*2]

In this study, we define the concept of pseudo-prime subhypermodules as a generalization of the prime hyperideal of commutative hyperrings in [4]. Firstly we examine a Zariski topology on the spectrum of pseudo-prime subhpermodules of certain hypermodules, then we give some relevant properties of the hypermodule in this topological hyperspace.

2010 Mathematics Subject Classifications: 16W99, 20N99, 54B20

Keywords: Pseudo-prime spectrum, Zariski topology, Spectral hyperspace

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Some Approximation Properties of q-Stancu-Durrmeyer Operators

Pembe Sabancıgil

For $f \in C[0,1]$, $\alpha \geq 0$ and $0 < q < 1$, new q-Stancu-Durrmeyer operators are defined as

$$S_n^{q,\alpha}(f,x) = p_{n,0}^{\alpha}(q,x)f(0) + p_{n,n}^{\alpha}(q,x)f(1) + [n-1]\sum_{k=1}^{n-1} q^{1-k} p_{n,k}^{\alpha}(q,x) \int_0^1 p_{n-2,k-1}(q,qt)f(t)d_qt,$$

where for $n = 1$ the sum is empty, i.e., equal to 0. The advantage of these operators is that they preserve linear functions. In this paper we investigate some approximation properties of q-Stancu-Durrmeyer operators.

2010 Mathematics Subject Classifications: 41A25, 41A36, 47A58

Keywords: q-calculus, q-Bernstein Polynomials, q-Durrmeyer Operators, q-Stancu Operators, Moments

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On Linear Operators Associated with the Umbral Algebra

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Methods of the umbral calculus give us some convenient properties of some special polynomials such as Sheffer polynomials. In this work, we study some operators by using the methods of the umbral algebra. We give the actions of these operators on the special polynomials. Then, we get varied properties of these polynomials.

2010 Mathematics Subject Classifications: 05A40, 47C05, 11B83

Keywords: Umbral algebra, linear operators, special polynomials

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Some new results on Hardy spaces of monogenic functions in the octonionic setting

Rolf Sören Kraußhar

Functions in the kernel of the octonionic Cauchy-Riemann operator are often called octonionic monogenic functions. However, in contrast to Clifford analysis, the lack of associativity implies that octonionic monogenic functions do not form an octonionic module. Consequently, requiring that an octonionic operator $\mathcal{T}$ defined in terms of an O-valued inner product viz $\mathcal{T}(f) := (f, g)$ is O-linear is too strong, since such a setting does not offer a Cauchy-Schwarz inequality. One way to overcome this problem is to weaken the property of being an O-linear functional in the sence just requiring that it is O-para-linear in the sense of [3]were we only require that $\Re(\mathcal{T}(f\alpha)) = \Re(f\alpha, g) = \Re((f, g)\alpha)$. Within this context we are able to develop a generalized theory of Hardy spaces with reproducing kernel functions in the octonionic monogenic setting where the reproduction property is based on the reproduction of the real parts.

2010 Mathematics Subject Classifications: 30G35

Keywords: Octonionic monogenic Hilbert spaces, para-linearity

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Contribution to the theoretical and experimental study of organic fresh vegetables drying products in southern Algeria

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Nowadays, the existence of living organisms is in real danger due to climate change, this climate change is caused by poor ways of using fossil fuels, and we know very well that it is impossible to eliminate the use of fossil fuels, but we can reduce their use by using renewable energies such as wind, solar and hydraulic energy.

Renewable energies also contribute to economic and human development, reduce poverty, and a solution for liberation from dependence on non-fuel-producing countries. The present work is a contribution to the theoretical and experimental study of a solar energy-dryer chain operating in forced convection for drying applications of organic fresh vegetables produced in the region of Adrar. Through this study, we have determined the instantaneous thermal performance of the designed system. It turns out that the effectiveness of this element is very sensitive to variations in climatic parameters.

The experimental results obtained compared to those resulting from the existing models prove conclusive. We have also experimentally determined according to several operating parameters, aerothermal air and the amount of product to be dried, the evolution of drying kinetics in order to establish the best conditions. The study of the kinetics of the drying of such a product through the parameters considered in this study suggests that the possibility of solar drying is feasible.

KEYWORDS: Solar, Dryer, Performance, Vegetable, Quality, Efficiency, Economy

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A Special Number Sequence Derived From Suborbital Graphs for the Modular Group Γ

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In this paper, we show the existence of a special number sequence, the Fibonacci sequence, derived from some special suborbital graphs for the Modular group Γ.

2010 Mathematics Subject Classifications: 05CXX, 11B39, 11F06, 20H05, 22E40

Keywords: Modular group, Suborbital graph, Fibonacci sequence

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Mathematical models describing the correlation between electrical conductivity and ash content of adulterated Turkish molasses

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Molasses *(pekmez)*, which is a traditional Turkish food and generally produced using the concentration of fruit juices, is subjected to fraudulent activities like several other foodstuffs. This study presents mathematical models obtained usingelectrical conductivity and ash values to control the quality of molasses and detect adulteration, particularly by using fructose sugar and glucose sugar. Therefore, adulterated mixtures were prepared by adding glucose and fructose syrups in certain proportions to the molasses samples produced as a liquid product, immediately after their production. It was found that the adulteration could be determined according to the results of the regression analysis, using the electrical conductivity and ash values. Considering these results, it was found that there was a linear relationship between the ash values and the electrical conductivity values. According to these results, it will be quite advantageous to use the equations obtained particularly using the ash values and electrical conductivity values in order to detect the adulteration in molasses and to avoid waste of time.

2010 Mathematics Subject Classifications: 62H20, 62Q05, 93A30

Keywords: Mathematical modeling, correlation, electrical conductivity, Ash content, Adulterated Turkish molasses

Introduction

Molasses is a product obtained by thickening fruit syrup by boiling it under normal atmospheric pressure or vacuum after deacidification. Although molasses can be produced from various fruits such as mulberries, plums, carob, figs, apples, and pears, grapes is the most common raw material used [1].

Good molasses does not experience sugar crystallization. According to the Turkish Food Codex Regulation, it is prohibited to produce grape molasses with names such as fruit sugar syrup, grape dessert, and grape molasses syrup by diluting and/or increasing it with commercial sugars such as glucose, fructose, etc. [2].

Adulteration in food refers to the production of foods by adding lower quality products to the high-quality product and offering the consumer at the same price. The adulteration leads to unfair competition by reducing the product quality, as well as, seriously endangering human health. Determining the quality of foods produced or consumed and preventing adulteration are important for the world economy [3].

Food traceability is an integral part of quality control. With the increase in consumer awareness, consumers want to obtain more accurate information about the origin and processing method of the food they consume. Adulteration is an important evaluation for the food traceability in the food chain [4].

Usually, glucose and fructose syrups are added to molasses for the purpose of adulteration, as they are relatively cheap. The addition of glucose and fructose syrups in molasses causes changes in the physical and chemical properties of the product. Several methods are used to detect the adulteration in molasses. In this study, changes in electrical conductivity and ash valuesof various molasses varieties were observed by adding glucose and fructose syrup in increasing proportions. Particularly, electrical conductivity and ash amount are important criteria for the quality of molasses. Therefore, mathematical equations were developed to calculate the amount of glucose and fructose syrups used in adulteration based on the changes in these properties that occurred after adding syrup.

Material

Sample mixtures were prepared by using liquid grape, mulberry, fig, carob, and date molasses, as well as, glucose and fructose syrups as research material (Table 1). The prepared mixtures were stored at room.

	Addition of glucosesyrup (GS) (%)					Addition of fructosesyrup (FS) (%)				
Grape molasses (GM)	0	10	30	50	70	0	10	30	50	70
Mulberry molasses (MM)	0	10	30	50	70	0	10	30	50	70
Carob molasses (CM)	0	10	30	50	70	0	10	30	50	70
Fig molasses (FM)	0	10	30	50	70	0	10	30	50	70

Table 1: Sugar syrup concentrations applied to the adulteration samples

Methods

Electrical conductivity was measured at 20C using a bench conductivity/TDS meter (Jenway, 4510, UK). The readings were recorded as direct results in μs/cm [5].

Total ash amounts of the samples were measured in accordance with the AOAC 1984 [6].

Statistics

A total of 3 measurements were conducted for 2 samples prepared in parallel for each molasses variety. Experimental results were analyzed using the MS Excel Analysis Tool Package. The regression equalities were generated in order to determine the adulteration ratio of pekmez samples. The performance of the models was evaluated using coefficient of determination (R^2).

It can be interpreted that the relationship between them is weak if the regression coefficients are low while the relationship between them is high if these coefficients are high [7].

Conclusion and Discussion

The composition of molasses varies depending on many factors such as the type and composition of the fruit used in production, the production technique of molasses, and the storage conditions. The values calculated in pure molasses samples and adulterated molasses samples are given in Table 2, Table 3, Table 4, and Table 5.

	EC(µs/cm)	Ash(%)
Grape molasses (GM)	2.26	2.185
GM+GS (90%+10%)	1.82	1.595
GM+GS (70%+30%)	1.54	1.123
GM+GS (50%+50%)	0.80	0.846
GM+GS (30%+70%)	0.36	0.219
GM+FS (90%+10%)	2.08	1.924
GM+FS (70%+30%)	1.99	1.809
GM+FS (50%+50%)	1.84	1.599
GM+FS (30%+70%)	1.34	1.321

Table 2: Properties of grape molasses (GM) samples adulterated with glucose syrup (GS) and fructose syrup (FS)

	EC(µs/cm)	Ash(%)
Carob molasses (CM)	8.84	3.55
CM+GS (90%+10%)	7.47	3.01
CM+GS (70%+30%)	5.02	2.78
CM+GS (50%+50%)	2.70	2.11
CM+GS (30%+70%)	1.20	0.9
CM+FS (90%+10%)	7.81	3.22
CM+FS (70%+30%)	6.75	2.96
CM+FS (50%+50%)	4.14	2.35
CM+FS (30%+70%)	2.6	2.09

Table 3: Properties of carob molasses (CM) samples adulterated with glucose syrup (GS) and fructose syrup (FS)

	EC(µs/cm)	Ash(%)
Fig molasses (FM)	9.73	3.798
FM+GS (90%+10%)	8.20	3.276
FM+GS (70%+30%)	4.43	2.363
FM+GS (50%+50%)	2.03	1.872
FM+GS (30%+70%)	0.93	0.562
FM+FS (90%+10%)	9.11	3.452
FM+FS (70%+30%)	6.06	2.997
FM+FS (50%+50%)	4.54	1.872
FM+FS (30%+70%)	2.26	1.252

Table 4: Properties of fig molasses (FM) samples adulterated with glucose syrup (GS) and fructose syrup (FS)

	EC(µs/cm)	Ash(%)
Mulberry molasses (MM)	4.15	2.340
MM+GS (90%+10%)	3.56	1.926
MM+GS (70%+30%)	2.18	1.221
MM+GS (50%+50%)	1.13	0.568
MM+GS (30%+70%)	0.24	0.211
MM+FS (90%+10%)	3.95	2.293
MM+FS (70%+30%)	2.56	2.178
MM+FS (50%+50%)	1.32	1.942
MM+FS (30%+70%)	0.85	1.790

Table 5: Properties of mulberry molasses (MM) samples adulterated with glucose syrup (GS) and fructose syrup (FS)

Regression equations and regression coefficients were calculated, and regression charts were preparedfor electrical conductivity and ash values of samples of molasses varieties, mixed with fructose and glucose syrups in the specified concentrations.

Electrical Conductivity

Electrical conductivity provides more information about the amounts of mineral salts, organic acids, and protein. Electrical conductivity is high in products containing high amounts of mineral salts, organic acids, and protein [8].

	Regression equation	**Coefficient of determination, R^2 (%)**
GM+GS	y_3= -0.0266x + 2.208	0.982
GM+FS	y_3= -0.0118x + 2.283	0.907
CM+GS	y_3= -0.1174x + 9.002	0.978
CM+FS	y_3= -0.0951x + 9,226	0.982
FM+GS	y_3= -0.1307x + 9.226	0.961
FM+FS	y_3= -0.1106x + 9.918	0.989
MM+GS	y_3= -0.0567x + 4.066	0.992
MM+FS	y_3= -0.0517x + 4.212	0.973

Table 6: Correlation between electrical conductivity values and different adulteration ratio of molases samples

Y1: Electrical conductivity value (μs/cm); x: Adulteration proportion (%)

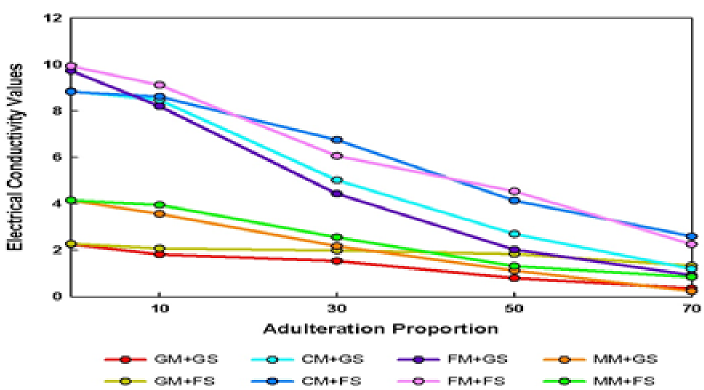

Figure 1: Correlation between electrical conductivity values and different adulteration ratio of molases samples

Conductivity values of the grape molasses, carob molasses, fig molasses, and mulberry molasses samples were found to be 2.26, 8.82, 9.93, and 2.15(μs/cm) respectively. In the case of 70% glucose syrup addition to the grape molasses, carob molasses, fig molasses, and mulberry molasses samples, their conductivity values were found to be 0.36, 1.20, 0.93, and 0.24(μs/cm) respectively.

On the other hand, the addition of 70% fructose syrup resulted in the conductivity values of 1.34, 2.06, 2.26, 0.85(μs/cm) respectively. The conductivity values were observed to decrease with the addition of both sugar syrups. The decrease observed with the addition of glucose syrup was found to be higher than the decrease observed with the addition of fructose syrup in all molasses varieties. The decreases in the conductivity observed with the addition of both sugar syrups were found to be linear.

Therefore, it is seen that the addition of glucose syrup, which does not contain mineral salts, organic acids, and protein, to natural molasses samples reduces the conductivity value because it reduces the concentrations of these ingredients in the samples.

Trkben et al. (2016) studied the electrical conductivity of the grape molasses samples made from fourteen different grapes, and they found the electrical conductivity ranging between 1.96 and 4.51 mS cm^{-1} [9]. In her study, Erbil (2020) found the average value of the electrical conductivity of carob molasses samples as 2.83 [10]. Tosun and Kele (2012) studied mulberry molasses samples and found their average conductivity value as 4.29 [11].

It is stated that molasses variety, sugar syrup type, and sugar syrup proportion have an effect on the conductivity. A study reported that the sucrose-added molasses samples had the lowest conductivity (3.03 mS cm^{-1}) while the glucose-added molasses samples had the highest conductivity (3.15 mS cm^{-1}) [12].

Ash

Mineral substances in foods or inorganic substances remaining as a result of burning are called ash. The composition of the fruit, from which the molasses is produced, varies depending on the type of fruit, its ripeness, and the characteristics of the region where it is grown.

	Regression equation	Coefficient of determination, R^2 (%)
GM+GS	y4 = -0.0255x + 2.010	0.962
GM+FS	y4 = -0.0112x + 2.127	0.968
CM+GS	y4 = -0.0197x + 2.816	0.973
CM+FS	y4 = -0.0087x + 2.765	0.979
FM+GS	y4 = -0.0476x + 3.754	0.991
FM+FS	y4 = -0.0373x + 3.866	0.981
MM+GS	y4 = -0.0309x + 2.243	0.983
MM+FS	y4 = -0.0081x + 2.369	0.982

Table 7: Correlation between ash values and different adulteration ratio of molasses samples

y2: Ash value(%); x: Adulteration proportion (%)

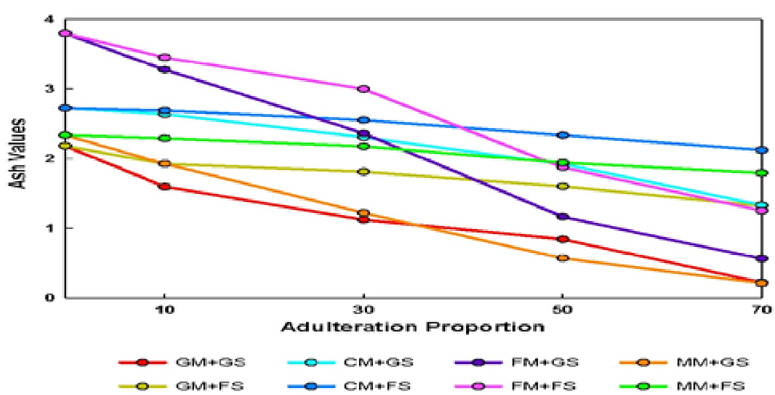

Figure 2: Correlation between ash values and different adulteration ratio of molases samples

The ash values of the grape molasses, carob molasses, fig molasses, and mulberry molasses samples were found to be 2.28, 3.55, 3.73, and 2.48, respectively. In the case of 50% glucose syrup addition to the grape molasses, carob molasses, fig molasses, and mulberry molasses samples, their ash values were found to be 0.78, 0.90, 1.93, and 0.44, respectively.The addition of 50% fructose syrup, on the other hand, resulted in the ash values of 1.24, 2.09, 2.63, and 0.74, respectively. The ash values were observed to decrease with the addition of both sugar syrups. The decrease observed with the addition of glucose syrup was found to be higher than the decrease observed with the addition of fructose syrup. The decreases in the ash value observed with the addition of both sugar syrups were found to be linear (r^2 =0.96).

A study on the mulberry molasses reported the ash value as 2.02% [11]. In a study on the chemical compositions of different varieties of carob molasses, the researchers found the average ash content of different carob molasses varieties as 2.8% [13]. In their study, Eki and Artk (1984) found the total ash content of the carob molasses as 1.57% [14]. In their study, Aksu and Nas (1996) found the total ash value of the mulberry molasses samples as 2.05% [15]. It has been reported that the ash value can be considered a method for detecting food adulteration made with syrup addition [16].

Conclusions

In Turkey, adulteration of molasses is a serious problem due to its nutritional value. Varieties of molasses are produced and consumed widely in many regions of Turkey. In this study, four most consumed molasses samples and two most used sugar syrups were used in adulteration trials. A significant correlation was obtained between the electrical conductivity and ash content measurements. This study revealed the potential of the electrical conductivity and ash parameters for detecting possible adulterations with glucose syrup and fructose syrup additions.This method can also be used for quantitative estimation of the adulterants in authentic Turkish foods.

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FastCombPerm: A Fast Package For Creating Combinations and Permutations With And Without Repetition

Seyyed Ali Mohammadiyeh *[1] *and Ali Reza Ashrafi*

This paper introduces the package FastCombPerm that can help researchers who are working in combinatorics. All possible combinations and permutations of sets or multisets can be easily created by this software. Sometimes we need to find the list of all possible without spending time designing recursive or non-recursive algorithms to find a list of permutations or combinations. This software is publicly available an open-source software under the GPL-3 license.

2010 Mathematics Subject Classifications: 05A05, 05A18, 05A19, 68W01, 68N30

Keywords: Combinations, Permutations, Algorithm

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On combinatorics of some higher order generalised geometric polynomials

Sithembele Nkonkobe

Introducing bars in-between blocks of ordered set partitions, one gets a barred preferential arrangement. In this study we interpret the generalised stirling numbers $S(n, k, \alpha, \beta, \gamma)$ introduced by Hsu and Shiue (1998) in terms of preferential arrangements. We also give an overview as to how barred preferential arrangements can be used to interpret several classes of higher order generalised geometric polynomials. We also discuss some asymptotic results.

2010 Mathematics Subject Classifications: 05A15, 05A16, 05A18, 05A19, 11B73, 11B83

Keywords: Barred preferential Arrangements, Geometric polynomials

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Asymptotic behavior of solutions to a time-space fractional diffusive Volterra equation

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In this talk we are interested to study the time asymptotic behavior of the bounded solution of the time-space fractional diffusive Volterra equation.

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Development of an electronic encyclopedia of number pyramids based on the mathematical apparatus of compositae

Dmitry Kruchinin *[1] *and Vladimir Kruchinin* [2]

In this paper, we consider the problem of automating the processes of working with number pyramids. To do this, it is proposed to use the developed electronic encyclopedia of number pyramids. In addition, we show the main attributes of the data representation model used in this database and present types of commands to search for information in this database.

2010 Mathematics Subject Classifications: 05A15, 68V35

Keywords: Bivariate generating function, Coefficient, Composita, Number pyramid, Electronic encyclopedia

Introduction

The mathematical apparatus of compositae is a good basis for automating the process of calculating the coefficients of the powers of generating functions. For example, compositae of generating functions can be used to obtain explicit expressions for polynomials [2] and this approach has been automated in the form of a library for Wolfram Mathematica [3]. The next step in the development of this direction is the creation of an electronic encyclopedia of number pyramids. The database of number pyramids will allow convenient storage and visualization of such structures. Thus, we will get the automation of the process of calculating the coefficients of generating functions, as well as obtaining knowledge about such coefficients.

Existing databases of numbers are aimed at obtaining new knowledge for numbers or number sequences. For example, there are such large projects as "World! Of Numbers" [1] and "OEIS" [4]. The main function of the OEIS is to search and edit integer sequences of two types: number sequences and number triangles. The proposed database of number pyramids focuses on bivariate generating functions and their powers.

Main results

Let consider a bivariate generating function

$$U(x,y)=\sum_{n\geq 0}\sum_{m\geq 0} u(n,m)\, x^n y^m,$$

whose coefficients $u(n,m)$ form a number triangle.

Then, the number pyramid for the generating function $U(x, y)$ is the three-dimensional table formed by the expression

$$T(n, m, k) = [x^n y^m]U(x, y)^k.$$

The electronic encyclopedia of number pyramids is developed in the form of a website, similar to OEIS. Consider a data representation model that consists of the following attributes:

- Number pyramid identifier num;
- Generating function expression $U_{num}(x, y)$;
- Explicit formula of the coefficients $T_{num}(n, m, k)$;
- Program for calculating the coefficients $T_{num}(n, m, k)$ using the corresponding explicit formula;
- Program for calculating the coefficients $T_{num}(n, m, k)$ using the series expansion for the corresponding generating function;
- List of related pyramids;
- List of links to related sequences in the OEIS;
- List of pyramid properties;
- Representation of the generating function $U_{num}(x, y)$ as:
  - LaTeX-text;
  - MathML-text;
  - Maxima-program;
  - Mathematica-program;
- Representation of the explicit formula of the coefficients $T_{num}(n, m, k)$ as:
  - LaTeX-text;
  - MathML-text;
  - Maxima-program;
  - Mathematica-program.

In addition, this electronic encyclopedia of number pyramids allows to automate the search process. To do this, use one of the following commands:

- F:<expression of the generating function $U_{num}(x, y)$>;
- T:<expression of an explicit formula of the coefficients $T_{num}(n, m, k)$>;
- D:<list of strings with some values of the coefficients $T_{num}(n, m, k)$>;
- <number pyramid identifier num>.

Conclusion

The developed electronic encyclopedia of number pyramids allows to perform an automated process of searching for stored records. Now, this database contains records about 1500 generating functions and their coefficients.

Acknowledgments

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Remarks on earthquakes involving polynomials-type Rocking Bearing

Yasemin Leventeli *[1] *and Ilyas Yilmazer* [3]

One of the aims of this study is to understand the seismic waves propagate through the soil and/or rocks. Because there is misconception among people like it is not the earthquake, the building that kills although experience shows otherwise. The seismic waves propagate through the rock with great speed, high frequency, low wavelength, and appreciably higher attenuation rate. And in the soil, it is just the opposite. The shorter the wavelengths and higher the frequency corresponds with greater energy consumption whilst propagation. So, the longer the wavelengths and lower the frequency results in lower energy intake.

The other one aims of this presentation is to investigate and study various properties of earthquake facts using polynomials-type Rocking Bearing and geological structures. The second aim is to investigate the mathematical properties of such polynomials. Future research on how to construct mathematical models of these polynomials to better understand earthquake realities.

KEYWORDS: Earthquake, Polynomials, Rocking Bearing, Geotechnics, Geology

Introduction

Recently, the polynomial Racing Bearing related to earthquake waves and their application has been investigated by many researchers. This polynomial has been appeared in seismic isolation system on irregular bridges, in seismic performance of rocking base-isolated structures, and also analyzed of earthquake loads. The polynomial Racing Bearing is given by the following form:

$$G(X) = c_1X^6 + c_2X^4 + c_3X^2,$$

where c_1, c_2, c_3 are arbitrary constant [1].

Since 1970s every occasion of catastrophic earthquakes, particularly in Anatolia, has been observed and investigated. All of earthquake catastrophes have been experienced only soil plains which are national wealth. The recent earthquakes in Anatolia with their epicenters are shown in Figure 1 and Table 1. When the significant earthquakes of the last period with different depths and magnitudes are examined, it can be seen that the earthquake effects are zero on the rock and cause problems only on the soil [2, 3, 4, 5].

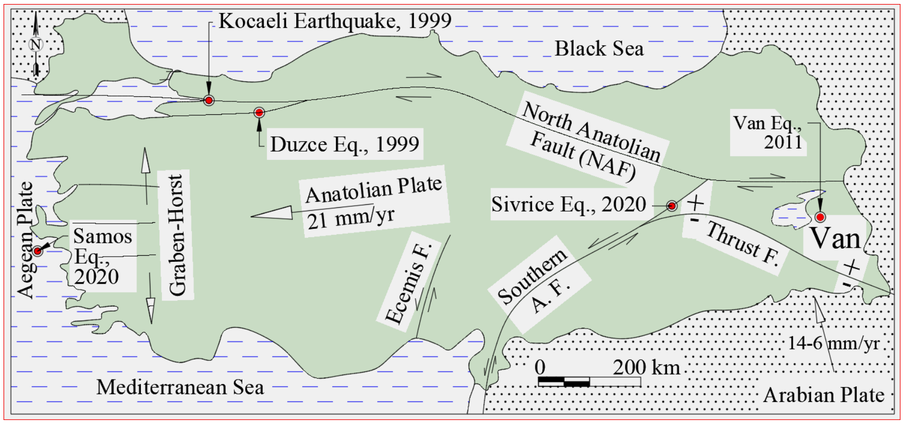

Figure 1: Tectonic outline of the Anatolia (Turkey) and epicenter for four of catastrophic recent earthquakes

All fault zones created invaluable farm fields. For instance, Menderes graben of Graben-Horst system produces over 50% of exported dried fig and Gediz graben yield over 36% of the world dried seedless grape (Sultani). Moreover, the best quality tobacco (its nicotine< 2%) in the world is grown in the Aegean graben-horst system. Earthquake catastrophe occurs in graben, not in horst comprising rock. Consequently, one may easily conclude that settlement has to be promoted outside of the graben boundary to solve earthquake problem and protect fertile soil plains and save them for farming.

The soil plain/rock boundary has been drawn in 1960s and then digitized. It is precisely known and defined by the points where abrupt change in topographic slope. Unfortunately, linear structures such as highways, railways, pipelines and new residential areas are located on 1^{st} class farm fields and invite catastrophe throughout the country. Almost all developing countries and underdeveloped countries besides some developed countries decision makers ignore this subject in land use planning.

Location	Date	Magnitudes (Mw)	Depth to Hypocenter (km)
Marmara (Kocaeli)	17.08.1999	7.6	17
Van-Tabanlı	23.10.2011	7.2	5
Erdemkent	09.11.2011	5.7	5
Sivrice	24.01.2020	6.8	8
Samos	30.10.2020	6.9	11.8

Table 1: The recent earthquakes in Anatolia

By analogy seismic wave energy attenuates at a high rate in rock whereas in soil it attenuates at a lower rate. A simple test configuration is depicted in Figure 2. Their viscosity is 10-3, 10-1 and 10 N.s, respectively. When a ball of the same mass is released from the same height into three pools filled with water, olive oil, and honey; (i) Water: the wave propagates to the edge of the pool. (ii) Olive oil: Wave is limited to a few cm. (iii) Honey: the wave is with very high frequency and it could not be seen via naked eye. one may conclude that as viscosity increases wavelength decreases (frequency increases) and wave energy attenuates rapidly.

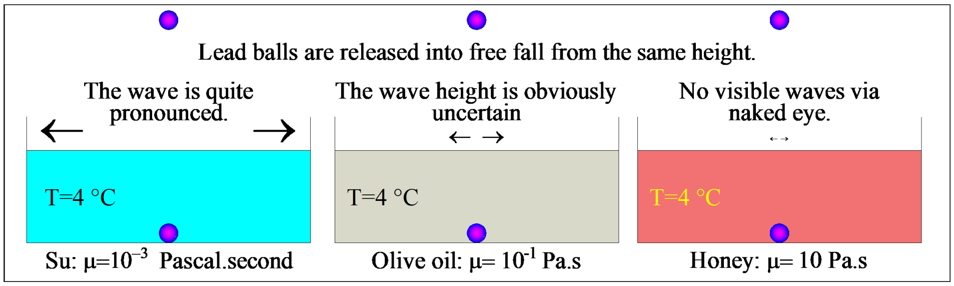

Figure 2: Viscosity is a quantity expressing the magnitude of internal friction in a fluid, as measured by the force per unit area resisting uniform flow and it is directly proportional with wave propagation energy attenuation [Rock has higher strength properties than soil does. Hence, seismic wave energy attenuates quickly in rocky medium which in turn does not harm structures]

Main results

Almost all of the relevant engineers focus on; (a) constructing buildings resistive to earthquake forces and (b) predicting when and where will an earthquake hit. Naturally such researches have to be continued in academic medium. But it shouldnt prevent selecting proper rocky sites for construction and preserve soil land for agricultural purposes. The authors strongly recommend that engineers should study the damage to structures and casualties after every destructive earthquake by accounting whether the structures are founded in/on soil or rock. Definitely they will reach a common conclusion that rocky ground is superior to soil ground in order to minimize/avoid earthquake catastrophe. Then, the reality would be recognized globally, and the proposed solution would be practiced in all countries subjected to earthquake catastrophe.

For the solution of the following questions about Polynomials-type Rocking Bearing and geological structures, the subject of mathematical structures will be investigated:

On which domain will the Polynomials-type Rocking Bearing polynomials be defined?
How will the value sets of these polynomials be determined?
How can the mathematical structure be constructed between these clusters and the earthquake reality?

Solutions to such problems are among the research topics in the near future.

The main point should be kept in mind to protect the innocent peoples of the world from the disasters of earthquakes, floods, and landslides, is a simple solution to a well-defined difficult problem is more reliable than an advance solution to an unidentified simple problem..

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Mass movements investigation: Suluada (Antalya)

Sercan Sarikaya [1] *and Yasemin Leventeli* [*2]

Antalya is known as the capital of tourism and there are many areas that we can define as "natural wonders". Suluada, which is located on the Teke Peninsula and close to Adrasan, is one of these places, known as the Maldives of the Mediterranean. However, many and different types of mass movements develop on the island due to geological and atmospheric conditions. It is also confirmed by the locals that the island is constantly losing land. In the long term, it is thought that the island is in danger of extinction due to mass movements. For this reason, the determination of existing mass movements and improvement methods has been the aim of this study. In addition, potential mass movements are also required for planning prevention efforts. This situation can be seen in most places, but Suluada is distinguished from others due to its natural wonder with tourism and economic return. This study showed that, it is very important and possible to leave such a beauty to future generations, to protect nature and also to contribute to the economy for many years by scientific approach.

KEYWORDS: Mass movement, Stability, Geotechnics, Suluada, Antalya

Introduction

The mass movement or landslide is defined as the displacement of rock or soil under the influence of gravity as well as factors such as lithological features, water condition, topographic slope. The movements sometimes create natural beauties, sometimes cause loss of life and property.

Suluada is one of the important tourism centers of Antalya (Figure 1) and is a piece of paradise with its white sand and turquoise water. The mass movements have occurred in different ways in the long term and continue to come. These movements change its shape and also it shrinks in area and/or volume. The island, which is one of the important attractions of the region, needs to be protected. The aim of this study is to reveal the mechanisms of current and potential mass movements that cause soil loss on the island in the long term and to determine the prevention studies.

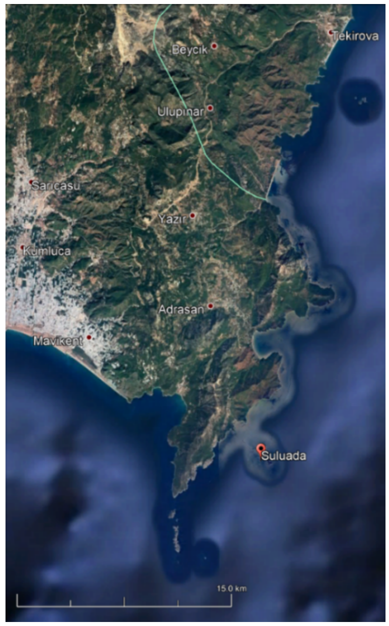

Figure 1: The location of Suluada (Antalya-Turkey)

Methods

In this study; first of all, the causes of potential mass movements such as ground conditions, presence of ground and surface water, geomorphological situation and human effects have been investigated. After the field studies, samples have been taken and physical and mechanical analyzes were made in accordance with the standards. Than, the engineering and geotechnical properties of the units were determined. With the help of the obtained parameters, the formation mechanisms of the existing mass movements were revealed and appropriate methods will be tried to be determined to prevent potential mass movements.

The geological properties of the study area have been studied different researchers. Kalafatolu (1973) studied the geology of the western part of Antalya Bay, revealed the stratigraphy and tectonic structure of the region; zgl (1976) distinguished the units reflecting different basin conditions in the Western Taurus; Demirtal (1977) examined the ophiolitic mlange and nappes in the western part of the Taurus Belt; Robertson and Woodcock (1980) divided the Antalya Complex into five different zones: Beydalar, Kumluca, Gdene, Kemer, Tekirova zones; Yaln and nal (2018) investigated the natural radioactivity levels, anomalies and their effects on human health of ophiolites in the western part of Antalya.

Conclusions and recommendations

The various types of landslides can be differentiated by the kinds of material involved and the mode of movement. A classification system based on these parameters is shown in Table 1 [6].

TYPE OF MOVEMENT		TYPE OF MATERIAL		
		BEDROCK	ENGINEERING SOILS	
			Predominantly coarse	Predominantly fine
FALLS		Rock fall	Debris fall	Earth fall
TOPPLES		Rock topple	Debris topple	Earth topple
SLIDES	ROTATIONAL	Rock slide	Debris slide	Earth slide
	TRANSLATIONAL			
LATERAL SPREADS		Rock spread	Debris spread	Earth spread
FLOWS		Rock flow	Debris flow	Earth flow
		(deep creep)	(soil creep)	
COMPLEX	Combination of two or more principal types of movement			

Table 1: Mass movements classification [7]

Limestone, ophiolitic melange and breccia were observed predominantly in the study area. Deep weathering traces were observed in the whitish gray, yellowish red colored limestones. In addition to the rock, debris can be seen in dark gray-greenish serpentinites. Weatherings and debris flows were detected in reddish brown breccias consisting of white limestone blocks. There are also many discontinuities. Some of them are open, some are clay filled and some are calcite filled. Besides these lithological units there are also Quaternary aged talus and beach sands.

According the table and results of the investigation; the main types of the mass movements in Suluada are rock falls and debris flow. While the aforementioned rock-falls are mainly observed in limestones, debris flow is present in all types of units including talus. The atmospheric conditions such as wind, sun and rain play an important role in mass movements, besides the discontinuities.

Falls are abrupt movements of masses of geologic materials, such as rocks and boulders, that become detached from steep slopes or cliffs. Separation occurs along discontinuities such as fractures, joints, and bedding planes, and movement occurs by free-fall, bouncing, and rolling. Falls are strongly influenced by gravity, mechanical weathering, and the presence of interstitial water. A debris flow is a form of rapid mass movement in which a combination of loose soil, rock, organic matter, air, and water mobilize as a slurry that flows downslope. Debris flows are commonly caused by intense surface-water flow, due to heavy precipitation or rapid snowmelt, that erodes and mobilizes loose soil or rock on steep slopes. Debris flows also commonly mobilize from other types of landslides that occur on steep slopes, are nearly saturated, and consist of a large proportion of silt- and sand-sized material. Debris-flow source areas are often associated with steep gullies, and debris-flow deposits are usually indicated by the presence of debris fans at the mouths of gullies [6].

Mass failure mechanisms have been determined, and the next step is improvement or prevention studies. Mass failure mechanisms have been determined, and the next step is improvement or prevention studies. Our claim is “Precaution is a much more engineering approach than improvement.”. Therefore; preventing or minimizing these mass movements in the form of rockfall and debris flow, which are common on the island, will ensure that the nature is preserved for future generations and will continue to contribute to the country’s economy in the future as it does today.

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Development of combinatorial generation algorithms for some lattice paths using the method based on AND/OR trees

Yuriy Shablya *[1] *and Alina Tokareva* [2]

In this paper, we consider the problem of developing combinatorial generation algorithms for lattice paths. To do this, it is proposed to use the method that is based on AND/OR trees. This method allows to develop algorithms for ranking, unranking and listing combinatorial sets, that are represented as AND/OR trees. As an example, we have applied this method to some well-known lattice paths and have obtained combinatorial generation algorithms for them.

2010 Mathematics Subject Classifications: 68R05, 05C38

Keywords: Combinatorial generation, Lattice path, AND/OR tree, Recurrence, Algorithm

Introduction

Methods of combinatorial generation make it possible to obtain different ways of generating elements of the sets of discrete structures [1]. There are several basic general methods for developing combinatorial generation algorithms. To solve the task of developing combinatorial generation algorithms, we propose to use the method that is based on representing a combinatorial set in the form of an AND/OR tree structure [2]. This method allows to develop new combinatorial generation algorithms (algorithms for ranking, unranking and listing combinatorial sets) and requires to know the cardinality function that belongs to the algebra $\{\mathbb{N}, +, \times, R\}$.

In this article, we consider the case of applying this method to obtain combinatorial generation algorithms for some lattice paths. Lattice paths are often used to enumerate other discrete structures due to the appropriate bijection. Therefore, algorithms for generating lattice paths can also be used to generate objects similar to them.

Main results

North-East lattice paths

A North-East lattice path is a lattice path in the plane which begins at $(0, 0)$, ends at (n, k), and consists of steps $(0, 1)$ and $(1, 0)$.

The total number of North-East lattice paths is defined by the following binomial coefficient (sequence $A007318$ in OEIS [3]):

$$L_n^k = \binom{n+k}{k}.$$

The value of L_n^k can also be calculated using the following recurrence that belongs to the required algebra $\{\mathbb{N}, +, \times, R\}$:

$$L_n^k = L_n^{k-1} + L_{n-1}^k, \quad L_n^0 = L_0^k = 1. \tag{1}$$

Applying (1), we can construct the AND/OR tree structure and develop algorithms for ranking, unranking and listing North-East lattice paths.

Dyck paths

A Dyck path is a lattice path in the plane which begins at $(0,0)$, ends at (n,n), consists of steps $(0,1)$ and $(1,0)$, and never rise above the diagonal $y = x$.

The total number of Dyck paths is defined by the Catalan numbers (sequence $A000108$ in OEIS [3]):

$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

The value of C_n can also be calculated using the following recurrence that belongs to the required algebra $\{\mathbb{N}, +, \times, R\}$:

$$C_n = \sum_{i=0}^{n-1} C_i C_{n-1-i}, \quad C_0 = 1. \tag{2}$$

Applying (2), we can construct the AND/OR tree structure and develop algorithms for ranking, unranking and listing Dyck paths.

Schroder paths

A Schroder path is a lattice path in the plane which begins at $(0,0)$, ends at (n,n), consists of steps $(0,1)$, $(1,0)$ and $(1,1)$, and never rise above the diagonal $y = x$.

The total number of Schroder paths is defined by the large Schroder numbers (sequence $A006318$ in OEIS [3]):

$$S_n = \sum_{i=0}^{n} \frac{1}{i+1}\binom{n+i}{n}\binom{n}{i}.$$

The value of S_n can also be calculated using the following recurrence that belongs to the required algebra $\{\mathbb{N}, +, \times, R\}$:

$$S_n = S_{n-1} + \sum_{i=0}^{n-1} S_i S_{n-1-i}, \quad S_0 = 1. \tag{3}$$

Applying (3), we can construct the AND/OR tree structure and develop algorithms for ranking, unranking and listing Schroder paths.

Delannoy paths

A Delannoy path is a lattice path in the plane which begins at $(0,0)$, ends at (n,k), and consists of steps $(0,1)$, $(1,0)$ and $(1,1)$.

The total number of Delannoy paths is defined by the Delannoy numbers (sequence $A008288$ in OEIS [3]):

$$D_n^k = \sum_{i=0}^{\min(n,k)} 2^i \binom{n}{i}\binom{k}{i}.$$

The value of D_n^k can also be calculated using the following recurrence that belongs to the required algebra $\{\mathbb{N}, +, \times, R\}$:

$$D_n^k = D_n^{k-1} + D_{n-1}^k + D_{n-1}^{k-1}, \quad D_n^0 = D_0^k = 1. \tag{4}$$

Applying (4), we can construct the AND/OR tree structure and develop algorithms for ranking, unranking and listing Delannoy paths.

Motzkin paths

A Motzkin path is a lattice path in the plane which begins at $(0,0)$, ends at (n,n), and consists of steps $(0,2)$, $(2,0)$ and $(1,1)$, and never rise above the diagonal $y = x$.

The total number of Motzkin paths is defined by the Motzkin numbers (sequence $A001006$ in OEIS [3]):

$$M_n = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \frac{1}{i+1} \binom{n}{2i} \binom{2i}{i}.$$

The value of M_n can also be calculated using the following recurrence that belongs to the required algebra $\{\mathbb{N}, +, \times, R\}$:

$$M_n = M_{n-1} + \sum_{i=0}^{n-2} M_i M_{n-2-i}, \quad M_0 = M_1 = 1. \tag{5}$$

Applying (5), we can construct the AND/OR tree structure and develop algorithms for ranking, unranking and listing Motzkin paths.

Conclusion

The method based on AND/OR trees is an effective way to develop new combinatorial generation algorithms for various combinatorial sets. For example, using this method, new algorithms for ranking, unranking and listing some lattice paths have been obtained. The main requirement for applying this method is the need to know the cardinality function that belongs to the algebra $\{\mathbb{N}, +, \times, R\}$.

Acknowledgments

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Formulas derived from certain family operator and p-adic bosonic integral

Yilmaz Simsek

The main aim of this presentation is to give some properties for the special operator, which was given in [12] (*Construction method for generating functions of special numbers and polynomials arising from analysis of new operators*, *Math Meth Appl Sci.* 2018): By the aid of this opeartor and p-adic bosonic integral on the ring of p-adic intergers, many novel formula involving special polynomials, the Bernoulli polynomials, the Stirling numbers, and the combinatorial numbers are given.

2010 Mathematics Subject Classifications: 12D10, 11B68, 11S40, 11S80, 26C05, 26C10, 30B40, 30C15

Keywords: Generating function, Bernoulli polynomials, Stirling numbers, Daehee numbers, Operators, p-adic bosonic integral

Definitions, relations and notations

Even with a brief glance at the history of mathematics, we can easily observe the extent to which numbers played a role in the development of mathematics. Therefore, it is well known that numbers and number sets play a major role in the invention of dazzling developments and innovations in every phase of science. Fibonacci was among the mathematicians who discovered the first special class of numbers. In the spirit of discovery of Fibonacci's special number family, new classes of special numbers and their generating functions were invented. As a result of this class of functions, special polynomials began to be invented. Especially in the last two hundred years, we have witnessed groundbreaking discoveries in these fields. Today, this special family of functions is not only used specifically for numbers and polynomials, but also in quantum physics, differential equations, engineering, mathematical modeling, approximation theory, etc. have taken their place among the indispensable subjects of such fields.

Even with this brief historical overview, it is important to stress that the families of special functions, including generating functions, have been the main source of inspiration for the academic work of many mathematicians, physicists, engineers, and social scientists for centuries. This type of special functions even today. Because perhaps a new function has been or will be defined in the class of generating functions today. Because with the help of these functions, many different properties of special numbers and special polynomials can be studied and also new formulas and relations can be obtained for them.

Similar to the obvious effect of the theory of special functions, operator theory also shows similar effects and uses in all scientific fields of the last century.

For this reason, we focused the motivation of this presentation on the issues we briefly mentioned above. Very recently a new operator class has been defined by me and some properties have been given to it, see for detail [12] and [14].

Throughout this paper, we use the following notations and definitions: Let $\mathbb{N} = \{1, 2, 3, \ldots\}$, $\mathbb{N}_0 = \{0, 1, 2, 3, \ldots\}$. Let $\mathbb{Z}$ denote the set of integers, $\mathbb{R}$ denote the set of real numbers, and $\mathbb{C}$ denote the set of complex numbers.

$$\binom{\lambda}{v} = \frac{\lambda(\lambda-1)\cdots(\lambda-v+1)}{v!} = \frac{(\lambda)_v}{v!}$$

and

$$\binom{\lambda}{0} = 1$$

$(v \in \mathbb{N},\ \lambda \in \mathbb{C})$(*cf.* [1]-[16]).

The Bernoulli polynomials $B_n(x)$ are defined by

$$\frac{te^{tx}}{e^t-1} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!}, \tag{1}$$

where $|t| < 2\pi$ (*cf.* [1]-[16]).

The Stirling numbers of the first kind $S_1(n,k)$ are defined by means of the following generating function:

$$\frac{(\log(1+t))^k}{k!} = \sum_{n=0}^{\infty} S_1(n,k)\frac{t^n}{n!}, \tag{2}$$

and

$$\binom{x}{n} = \frac{1}{n!}\sum_{k=0}^{n} S_1(n,k)x^k \tag{3}$$

(*cf.* [1]-[16]).

The operators $\mathcal{O}_\lambda[f;a,b]$ and $T_\lambda[f;a,b]$

The operator $\mathcal{O}_\lambda[f;a,b]$ is defined by

$$\mathcal{O}_\lambda[f;a,b](x) = \lambda E^a[f](x) + E^b[f](x), \tag{4}$$

where λ, a, b, $x \in \mathbb{R}$ and

$$E^a[f](x) = f(x+a)$$

and

$$T_\lambda[f;a,b](x) = \frac{\mathcal{O}_\lambda[f;a,b](x)}{a+b+1} \tag{5}$$

(*cf.* [12]).

Many special cases of this operator were given in [12].

In [14], we modify the operators $\mathcal{O}_\lambda[f;a,b]$ as follows:

$$\mathbb{Y}_{\lambda,\beta}[f;a,b](x) = \lambda E^a[f](x) + \beta E^b[f](x), \tag{6}$$

where λ and β are complex or real parameters, a and b are real parameters.(*cf.* [14]).

Using some special values of a, b, λ, and β, in [12], we investigated some properties of the opeartor $\mathbb{Y}_{\lambda,\beta}[f;a,b]$. For instance, substituting $a = \lambda = 1$, $\beta = -1$ and $b = 0$ into (6), we have

$$\mathbb{Y}_{1,-1}[f;1,0](x) = f(x+1) - f(x) = \Delta[f](x),$$

which is known asl a finite difference operator, which can be used in the theory of approximation of derivatives (*cf.* [12]). Thus this operator plays certainly a central role infinite difference methods for the numerical solution of differential equations, boundary value problems, other mathematical and other science problems. Especially, in numerical analysis, finite difference operator is widely used for approximating derivatives.

By using the operator $\mathbb{Y}_{\lambda,\beta}[f;a,b]$, we have

$$\mathbb{Y}_{\lambda,\beta}[f;a,b](x) = \beta\mathcal{O}_{\frac{\lambda}{\beta}}[f;a,b](x) = \beta(a+b+1)T_{\frac{\lambda}{\beta}}[f;a,b](x).$$

By the aid of the operator $\mathbb{Y}_{\lambda,\beta}[f;a,b]$, we defined the following special polynomials and numbers:

$$Q_n(x;a,b;\lambda,\beta) = \sum_{j=0}^{n}\binom{x}{j}\mathbb{Y}^{j}_{\lambda,\beta}[f;a,b](0) \tag{7}$$

where $\mathbb{Y}^{j}_{\lambda,\beta}[f;a,b](0)$ means that

$$\mathbb{Y}^{j}_{\lambda,\beta}[f;a,b](0) = \mathbb{Y}^{j}_{\lambda,\beta}[f;a,b](y)\,|_{y=0}$$

(*cf.* [14]).

$$y_5(n,k;a,b;\lambda,\beta) = \sum_{j=0}^{k}\binom{k}{j}\lambda^{k-j}\beta^{j}\sum_{l=1}^{n}d_l(jb+(k-j)a)^l \tag{8}$$

(*cf.* [14]).

Identities arised from p-adic bosonic integral representations of the polynomials $Q_n(x;a,b;\lambda,\beta)$

In this section, by applying p-adic bosonic integral on $\mathbb{Z}_p$ to the polynomials $Q_n(x;a,b;\lambda,\beta)$, some identities and formulas associated with combinatorial sums, the Bernoulli polynomials, the Stirling numbers and the Daehee numbers are given.

Let $\mathbb{Z}_p$ be a set of p-adic integers. Let f be a uniformly differential function on $\mathbb{Z}_p$.

The p-adic bosonic (Volkenborn) integral of the uniformly differential function f is defined by

$$\int_{\mathbb{Z}_p} f(x)\,d\mu_1(x) = \lim_{N\to\infty}\frac{1}{p^N}\sum_{x=0}^{p^N-1}f(x), \tag{9}$$

where $\mu_1(x)$ denotes the Haar distribution which is given by

$$\mu_1(x) = \frac{1}{p^N}$$

(*cf.* [5], [7], [8], [10], [13], [16]).

We need the following well-known p-adic integral formulas:

$$B_n(x) = \int_{\mathbb{Z}_p}(x+y)^n\,d\mu_1(y), \tag{10}$$

(*cf.* [5], [6], [10], [13]).

By applying the equation (9) to the equation (7), and combining the equations (3), and (10) with the following formulas which was proved by the author [13, Theorem 3. 6, Eqs. (3.8)-(3,9)]:

$$\int_{\mathbb{Z}_p}\int_{\mathbb{Z}_p}\binom{x+y}{n}d\mu_1(x)\,d\mu_1(y)=\sum_{k=0}^{n}\frac{(-1)^n}{(k+1)(n-k+1)},$$

and

$$\int_{\mathbb{Z}_p}\int_{\mathbb{Z}_p}\binom{x+y}{n}d\mu_1(x)\,d\mu_1(y)=\sum_{k=0}^{n}\sum_{v=0}^{k}\binom{k}{v}\frac{S_1(n,k)B_vB_{k-v}}{n!}.$$

after algebraic operations, we get the following novel results:

Lemma 1. *Let λ and β are complex or real parameters, a and b are real parameters. Let $n\in\mathbb{N}_0$. Then we have*

$$\int_{\mathbb{Z}_p}\int_{\mathbb{Z}_p}Q_n(x+y;a,b;\lambda,\beta)\,d\mu_1(x)\,d\mu_1(y)=\sum_{j=0}^{n}\sum_{k=0}^{n}(-1)^n\frac{\mathbb{Y}_{\lambda,\beta}^{j}[f;a,b](0)}{(k+1)(n-k+1)}. \tag{11}$$

Lemma 2. *Let λ and β are complex or real parameters, a and b are real parameters. Let $n\in\mathbb{N}_0$. Then we have*

$$\int_{\mathbb{Z}_p}\int_{\mathbb{Z}_p}Q_n(x+y;a,b;\lambda,\beta)\,d\mu_1(x)\,d\mu_1(y)=\sum_{j=0}^{n}\sum_{k=0}^{n}\sum_{v=0}^{k}\binom{k}{v}\frac{S_1(n,k)B_vB_{k-v}\mathbb{Y}_{\lambda,\beta}^{j}[f;a,b](0)}{n!}. \tag{12}$$

Combining (11) with (12), we get the following theorem.

Theorem 3. *Let λ and β are complex or real parameters, a and b are real parameters. Let $n\in\mathbb{N}_0$. Then we have*

$$\sum_{j=0}^{n}\sum_{k=0}^{n}\left((-1)^n\frac{1}{(k+1)(n-k+1)}-\sum_{v=0}^{k}\binom{k}{v}\frac{S_1(n,k)B_vB_{k-v}}{n!}\right)\mathbb{Y}_{\lambda,\beta}^{j}[f;a,b](0)=0.$$

Conclusion

In the light of what is given in this presentation and results in [12] and [14] , in the future, the operator $\mathbb{Y}_{\lambda,\beta}[f;a,b](x)$ and the special numbers and polynomials found as a result of it are planned to be examined and investigated in a way that includes different fields.

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On Bohr's Theorem for stable functions

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We consider the class of stable harmonic mappings $f = h + \overline{g}$ intruduced by Martin, Hernandez, and the class of stable logharmonic mappings $f = zh\overline{g}$ introduced by AbdulHadi, El-Hajj. We determine Bohr's radius for the classes of stable univalent harmonic mappings, stable convex harmonic mappings and stable univalent logharmonic mappings. We also consider improved and refined versions of Bohr's inequality and discuss the Bohr's Rogonsiski radius for these family of mappings

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