The 3rd & 4th Mediterranean International Conference of Pure & Applied Mathematics and Related Areas (MICOPAM 2020-2021)

Antalya, TURKEY
November 11th-12th, 2021

PROCEEDINGS BOOK OF MICOPAM 2020-2021

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Dedicated to our colleagues and all humanity who died or was affected due to the coronavirus pandemic

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OPENING CEREMONY TALK of MICOPAM 2020-2021

Dear distinguished participants of the 3rd and 4th Mediterranean International Conference of Pure & Applied Mathematics and Related Areas (MICOPAM 2020-2021), held at Faculty of Science Department of Mathematics Antalya-TURKEY, on November 11-12, 2021.

Due to the coronavirus pandemic, this conference was organized as a hybrid event with a limited number of physical participants as well as researchers participating or presenting virtually (with a videoconferencing/webinar platform).

Let me start my talk with the following meaningful sentence that will reflect the frightening and excruciating impact of the coronavirus disease, which reminds us of our deepest sorrows:

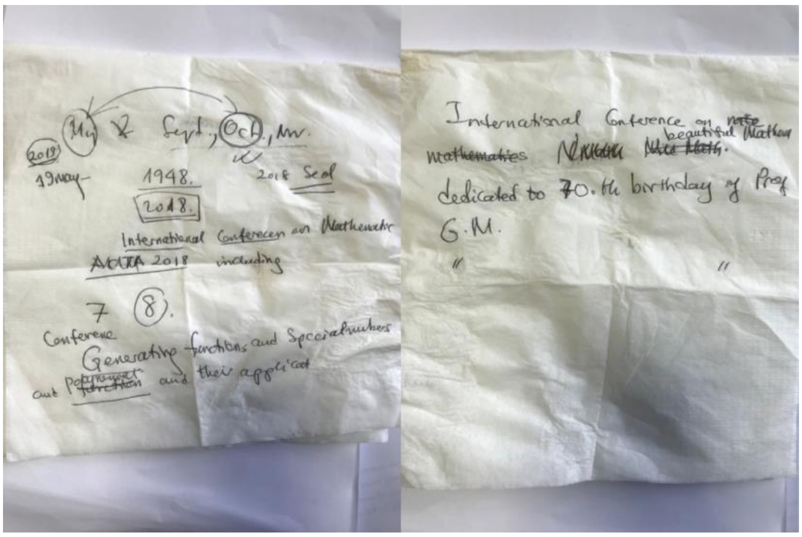
As the committee-in-chief of the organizing committee, I wish “Healthy Days” to all the people who are suffering due to the coronavirus pandemic in our world. In addition, we wish that God would rest their souls for the people who died due to the coronavirus pandemic, and also we hope the other effected people will recover as soon as possible.

Due to the dedication of the conference to people died by reason of the coronavirus pandemic, it has been decided to transform the cover page, which was previously designed in yellow and dark blue, to the colors of BLACK and WHITE.

On behalf of the Scientific and Organizing Committees, I would like to say "Welcome to our conference (with physical, presenting virtually (with a videoconferencing/webinar platform), participants form all of the world.

MICOPAM conference series was started in Antalya-Turkey in 2018. The first conference was dedicated to the 70th birthday of Professor Gradimir V. Milovanović, and then the latter has been held in Paris, France in 2019.

The construction of the MICOPAM conference has been firstly appeared in 2017 in Belgrade, Serbia, while speaking with Professor Gradimir V. Milovanović.



Brainstorming for conference name on napkin with Professor Gradimir V. Milovanović in 2017 at Belgrade, Serbia.

Our dreams came true in 2018 at Antalya and also MICOPAM 2018 was successfully held over four days, with presentations made by not only researchers coming from the international communities, but also distinguished keynote speakers.

With the same passion, in 2019, this conference carried out as MICOPAM 2019 at University of d'Evry Val d'Essonne in Paris France by the Professor Abdelmejid Bayad, me, and Professor Mustafa Alkan.

Today, we are very happy to make the opening ceremony of the 3rd and 4th of the conference MICOPAM together. Thus, dear distinguished participants, you have given honor to us by attending our conference: MICOPAM 2020-2021. Due to the big effect of the Coronavirus pandemic, in 2020 this conference was not organized and postponed to this year to be combine with MICOPAM 2021.

I would like to remind you that MICOPAM conference will be held regularly every year. In particular, "The 5th Mediterranean International Conference of Pure & Applied Mathematics and Related Areas" is scheduled to be held in another beautiful, touristic and historical city of the world to be announced soon on the MICOPAM web site, in 2022!

From Left to Right: *Professor Yilmaz Simsek, Professor Walter Gautschi, Professor Gradimir V. Milovanović*

I would like to thank to the following my colleagues and students who helped me at every stage of the 3rd and 4th Mediterranean International Conference of Pure & Applied Mathematics and Related Areas (MICOPAM 2020-2021):

My sincerely thanks go to not only all my colleagues including all speakers, who sent abstracts of their talks, in all over the world who supported me without hesitation even during the most difficult days of this pandemic period, and also to the Local Organizing Committee: (including especially Professor Dr. Mustafa Alkan, Associate Professor Dr. Irem Kucukoglu, Assistant Professor Dr. Ortaç Öneş, Assistant Professor Dr. Neslihan Kilar, Assistant Professor Dr. Rahime Dere, Dr. Damla Gün, Dr. Büşra Al, Dr. Buket Simsek); my precious family: (my wife Saniye, my daughters Burcin and Buket); and also other friends whose names that I did not mention here.

Thanks also to the Editors (Professor Dr. Mustafa Alkan, Associate Professor Dr. Irem Kucukoglu, and Assistant Professor Dr. Ortaç Öneş) for their most valuable contribution on preparing the Proceedings Book of MICOPAM 2020-2021.

I would like to state few words about Mathematics. Mathematics is not only the common heritage of each people in the world, but also the common language of the world. That is always passed from generation to generation by refreshing.

It would also be appropriate to say the following:

In addition to the poetic and artistic aspect of mathematics, mathematics has such a spiritual, magical and logical power, all natural science and social science cannot breathe and survive without mathematics.

Mathematics is such a branch of science that other sciences cannot develop without it. Therefore, Mathematics, which is the oldest of science, has contributed fundamentally to the development of our world civilizations. Thus, we can enter into the science and technology centers using the power of mathematics and its branches. So, mathematics and its branches create the possibility of bridgework and communication between the Natural Sciences and the Engineering Sciences as well as the Economic and also Social Science.

The aim of the MICOPAM conference is to bring together leading scientists of the pure and applied mathematics and related areas to present their researchers, to exchange new ideas, to discuss challenging issues, to foster future collaborations and to interact with each other. In fact, the main purpose of this conference is to bring to the fore the best of research and applications that will help our world humanity and society. Due to the valuable idea of the MICOPAM, this conference welcomes speakers whose talk or poster contents are mainly related to the following areas: Mathematical Analysis, Algebra and Analytic Number Theory, Combinatorics and Probability, Pure and Applied Mathematics & Statistics, Recent Advances in General Inequalities on Pure & Applied Mathematics and Related Areas, Mathematical Physics, Fractional Calculus and Its Applications, Polynomials and Orthogonal systems, Special numbers and Special functions, Q-theory and Its Applications, Approximation Theory and Optimization, Extremal problems and Inequalities, Integral Transformations, Equations and Operational Calculus, Partial Differential Equations, Geometry and Its Applications, Numerical Methods and Algorithms, Scientific Computation, Mathematical Methods and computation in Engineering, Mathematical Geosciences.

To summarize my speech, this conference has provided a novel opportunity to our distinguished participants to meet each other and share their scientific works and friendships in the above areas.

I am delighted to note that all participants have free and active involvement and meaningful discussion with other participants during the conference by the "Microsoft Teams" platform and other online connections.

By the way, I would like to point out that the Special Issue in Montes Taurus Journal of Pure and Applied Mathematics ISSN: 2687-4814 (MTJPAM) (https://mtjpamjournal.com/special-issues/) of this conference will be dedicated to our colleagues who died and are still suffering from the Coronavirus, and to all humanity in the world.

Due to the Coronavirus Pandemic, I wish you and your family healthy days.

We look forward to your support and participation in our MICOPAM2022 conference next year.

It is my great pleasure to sincerely thank again Local Organizing Committee.

Consequently, I send my sincerely thanks to all valuable participants of the conference MICOPAM 2020-2021.

On behalf of the Organizing Committee of MICOPAM 2020-2021

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FOREWORD

Why we call the name of the conference as "Mediterranean International Conference of Pure & Applied Mathematics and Related Areas (MICOPAM)". Because by the Mediterranean Sea, almost all the countries, the sea and the oceans are connected. For this reason, just like Mediterranean Sea, the main aim of MICOPAM is to give connection between many areas of sciences including physical mathematics and engineering, especially all branches of mathematics. A few of these areas can be given as follows: Pure and Computational and Applied Mathematics, Statistics, Mathematical Physics (related to *p*-adic Analysis, Umbral Algebra and Their Applications). Another important purpose of MICOPAM is to bring together leading scientists of the pure and applied mathematics and related areas to present their researches, to exchange new ideas, to discuss challenging issues, to foster future collaborations and to interact with each other in the following areas: Mathematical Analysis, Algebra and Analytic Number Theory, Combinatorics and Probability, Pure and Applied Mathematics & Statistics, Recent Advances in General Inequalities on Pure & Applied Mathematics and Related Areas, Mathematical Physics, Fractional Calculus and Its Applications, Polynomials and Orthogonal systems, Special numbers and Special functions, Q-theory and Its Applications, Approximation Theory and Optimization, Extremal problems and Inequalities, Integral Transformations, Equations and Operational Calculus, Partial Differential Equations, Geometry and Its Applications, Numerical Methods and Algorithms, Scientific Computation, Mathematical Methods and computation in Engineering, Mathematical Geosciences.

A brief description about the contents of "Proceedings Book of the 3rd and 4th Mediterranean International Conference of Pure & Applied Mathematics and Related Areas (MICOPAM 2020-2021)" is given as follows:

The first section of the Proceedings Book of MICOPAM 2020-2021 includes opening ceremony talk, foreword, and some information about MICOPAM including the meaning of its name and the name of conference committee members. The rest of this book includes all the contributed talks and their related manuscripts.

In this regard, we would like to thank to all speakers and participants for their valuable contributions.

Finally, we express our sincere thanks to all members of the scientific committee and all members of the organizing committee because of their efforts to the success of this conference and this book.

Editors of MICOPAM 2020-2021

Prof. Dr. Yilmaz Simsek
Prof. Dr. Mustafa Alkan
Assoc. Prof. Dr. Irem Kucukoglu
Asst. Prof. Dr. Ortaç Öneş

ABOUT CONFERENCE

The MICOPAM conference series has been started by organizing it in Antalya-Turkey in 2018, and then the latter has been held in Paris, France in 2019. Over the last two years; this conference series has gather together researchers, who work on pure & applied mathematics and related areas, from all over the world. This conference series continues this year by the organization of “The 3rd & 4th Mediterranean International Conference of Pure & Applied Mathematics and Related Areas (MICOPAM 2020-2021)”, under the chairmanship of Professor Yilmaz Simsek, (Akdeniz University, Turkey), as a hybrid event with a limited number of physical participants as well as researchers participating or presenting virtually (with a videoconferencing/webinar platform).

The 3rd and 4th Mediterranean International Conference of Pure & Applied Mathematics and Related Areas (MICOPAM 2020-2021) has been held at Faculty of Science (Block B), Akdeniz University in Antalya, TURKEY for two days from November 11 to November 12, 2021.

During two dates of MICOPAM 2020-2021, a great number of excellent oral presentations was made by 37 participants from 17 different countries (Canada, France, Germany, India, Italy, Jordan, Nigeria, North Cyprus, Pakistan, Portugal, Russia, Serbia, South Korea, Spain, Turkey, USA, Vietnam).

Contents of oral presentations are mainly related to the following areas: Mathematical Analysis, Algebra and Analytic Number Theory, Combinatorics and Probability, Pure and Applied Mathematics & Statistics, Recent Advances in General Inequalities on Pure & Applied Mathematics and Related Areas, Mathematical Physics, Fractional Calculus and Its Applications, Polynomials and Orthogonal systems, Special numbers and Special functions, Q-theory and Its Applications, Approximation Theory and Optimization, Extremal problems and Inequalities, Integral Transformations, Equations and Operational Calculus, Partial Differential Equations, Geometry and Its Applications, Numerical Methods and Algorithms, Scientific Computation, Mathematical Methods and computation in Engineering, Mathematical Geosciences.

Further details about MICOPAM 2020-2021 are given as follows:

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- Yilmaz Simsek, (Akdeniz University, Turkey)

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(Sorted list alphabetically by speaker's last name)

- Ravi P. Agarwal, (Texas A&M University-Kingsville, USA)
- Abdelmejid Bayad, (Université d'Evry Val d'Essonne, France)
- Daeyeoul Kim, (Jeonbuk National University, Korea)
- Min-Soo Kim, (Kyungnam University, South Korea)
- Dmitry Kruchinin, (Tomsk State University of Control Systems and Radioelectronics, Russia)
- Helmuth Robert Malonek, (Universidade de Aveiro, Portugal)
- Sabadini Irene Maria, (Politecnico di Milano, Italy)
- Gradimir V. Milovanović, (Serbian Academy of Sciences and Arts, Serbia)
- Wolfgang Sprößig, (TU Bergakademie Freiberg, Institute of Applied Analysis, Germany)

Proceedings Book of the 3rd & 4th Mediterranean International Conference of Pure & Applied Mathematics and Related Areas (MICOPAM 2020-2021)

December 1, 2021

Contents

1 INVITED TALKS

Extended Eulerian numbers and prime zeta functions type

Abdelmejid Bayad [1]

Let λ be a complex number. An *extended Eulerian number* $H(n,\lambda)$ is defined by means of its Dirichlet series

$$\frac{\lambda-1}{\lambda-\zeta(s)}=\sum_{n\geq 1}\frac{H(n,\lambda)}{n^s}, \tag{1}$$

where $\zeta(s)=\sum_{n\geq 1}\frac{1}{n^s}$ is the Riemann Zeta function defined for $\Re(s)>1$.

Firstly, we extend and improve some results obtained by Kalmár [12], Hille [9], Erdös [5], Evans [6], Klazar and Luca [14], Deleglise, Hernane, and Nicolas [7], concerning the maximal order of extended Eulerian numbers.

Our motivation comes from the following important particular cases:

1) If $\lambda=0$, then $H(n,0)=\mu(n)$ is a Möbius number.

2) If $n=p_1p_2\cdots p_r$ is square-free, then $H(n,\lambda)=H_r(\lambda)$, where $H_r(\lambda)$ is an Eulerian number. In addition, if $\lambda=-1$, thus

   $$H(n,-1)=H_r(-1)=E_r$$

   is the so-called Euler number. It's well known in [18] a combinatorial aspect of these numbers is the fact that the kth Betti number of the overall cohomology is the $(k+1)$th Eulerian number. Moreover, well known also in geometry ([8]), statistical applications ([13, 4]), and spline theory ([17]).

3) If $\lambda=2$, then $H(n,2)=K(n)$, where K is the Kalmár arithmetic function which counts the number of ordered factorizations of a positive integer n in factors bigger than 1. Various properties of this function were studied by many mathematicians. In fact, Kalmár found the average order of $K(n)$, for $x\to\infty$

   $$\sum_{n\leq x}K(n)=-\frac{x^\rho}{\rho\zeta'(\rho)}\{1+o(1)\}, \tag{2}$$

   where $\rho=1.72864\ldots$ is the positive real solution to $\zeta(s)=2$.

   Recently, Klazar and Luca [14], Deleglise, Hernane, and Nicolas [7] improved the bounds for the maximal order of $K(n)$.

For more details and further results for $H(n,\lambda)$ see [1] where these numbers were studied by A. Bayad, M. O. Hernane and A. Togbé.

Secondly, we introduce and investigate the Möbius-prime zeta function given by

$$\frac{\lambda-1}{\zeta_{\mathcal{P}}(s)-\lambda}=\sum_{n\geq 1}\frac{H_{\mathcal{P}}(n,\lambda)}{n^s},$$

where

$$\zeta_{\mathcal{P}}(s) = 1 + \sum_{p \text{ prime}} p^{-s}$$

is the so-called prime zeta function and $H_{\mathcal{P}}(n, \lambda)$ *are the prime-extended Eulerian numbers.*

Among the results we obtain:

(*i*) For $\lambda = 2$ we have

$$\frac{1}{2 - \zeta_{\mathcal{P}}(s)} = \sum_{n \geq 1} \mu_{\mathcal{P}}(n, 2)/n^s, \tag{3}$$

where

$$\mu_{\mathcal{P}}(n, 2) = \binom{\alpha_1 + \alpha_2 + \cdots + \alpha_k}{\alpha_1, \alpha_2, \ldots, \alpha_k}, \text{ if } n = \prod_{i=1}^{k} p_i^{\alpha_i} \tag{4}$$

which is the prime analogue of the Kalmar function $K(n)$, for more details see [7].

(*ii*) For $\lambda = 0$, we note $\mu_{\mathcal{P}}(n, 0) = \mu_{\mathcal{P}}(n)$. We prove that

$$\mu_{\mathcal{P}}(n) = (-1)^{\Omega(n)} \binom{\alpha_1 + \alpha_2 + \cdots + \alpha_k}{\alpha_1, \alpha_2, \ldots, \alpha_k}, \tag{5}$$

where $\Omega(n) = \sum_{i=1}^{k} \alpha_i$ for $n = \prod_{i=1}^{k} p_i^{\alpha_i}$

(*iii*) For $\lambda > 1$, our study may be seen as a prime analogue of the work done for extended Eulerian numbers [1].

We introduce and evaluate the prime λ-Mertens function defined by

$$M_{\mathcal{P}}(x, \lambda) = \sum_{n \leqslant x} \mu_{\mathcal{P}}(n, \lambda). \tag{6}$$

For $\lambda > 1$, we study the asymptotic behavior and the large value of the function $\mu_{\mathcal{P}}(n, \lambda)$. Then, we define the $\mu_{\mathcal{P}}$-champion number as an integer N such that

$$\forall M < N \Rightarrow \mu_{\mathcal{P}}(M, \lambda) < \mu_{\mathcal{P}}(N, \lambda). \tag{7}$$

Several properties of such numbers will be given. In particular, we investigate the factorization of N into small and large prime factors, the size of prime factors in the stander factorization of N into primes and the number $Q(X)$ of $\mu_{\mathcal{P}}$-champion numbers $N \leq X$.

Furthermore, we consider its associated Mertens type function. Among others, we study the asymptotic behavior and the large value of the Möbius-prime zeta function. Also, we find the Stieltjes constants associated to Dirichlet series associated to the Möbius-prime zeta function. Finally, concerning these new zeta functions we will cover the following arithmetic questions like: Average and error term; Upper and lower bounds; Estimate of Champions numbers and large values ... etc.

References

[1] A. Bayad, M. O. Hernane and A. Togbé, On extended Eulerian numbers, Funct. Approx. Comment. Math. 55(1) (2016), 113–130.

[2] L. Carlitz, V. E. Hoggart, Generalized Eulerian numbers and polynomials, Math. Magazine 32 (1959), 247–260.

[3] L. Carlitz, Extended Bernoulli and Eulerian numbers, Duke Math. J. 31(4) (1964), 667–689.

[4] L. Carlitz, D. C. Kurtz, R. Scoviele and O. P. Stackelberg, Asymptotic properties of Eulerian numbers, Z. Wahrsch. verw. Gebiete 23 (1972), 47–54.

[5] P. Erdős, On some asymptotic formulas in the theory of the "factorisatio numerorum", Ann. of Math. 42 (1941), 989–993. Corrections to two of my papers, Ann. of Math. 44 (1943), 647–651.

[6] R. Evans, An asymptotic formula for extended Eulerian numbers, Duke Math. J. 41 (1974), 161–175.

[7] M. Deleglise, M. O. Hernane, J.-L. Nicolas, Grandes valeurs et nombres champions de la fonction arithmtique de Kalmár, J. Number Theory 128 (2008), 1676–1716.

[8] D. Henseey, Eulerian numbers and the unit cube, Fibonacci Quart. 20 (1982), 344–348.

[9] E. Hille, A problem in "factorisatio numerorum", Acta Arith. 2 (1936), 134–144.

[10] H.-K. Hwang, Distribution of the number of factors in random ordered factorizations of integers, J. Number Theory 81 (2000), 61–92.

[11] S. Ikehara, On Kalmár's problem in " factorisation numerorum", Proc. Phys.-Math. Soc. Japan, 21 (1939), 208–219; II, 23 (1941), 767–774.

[12] L. Kalmár, A "factorisatio numerorum" problémàjàròl, Mathematikai ès Fisikai Lapok 38 (1931), 1–15.

[13] A. C . Kimber, Eulerian numbers and links with some statistical procedures, Utilitas Math. 31 (1987), 57–65.

[14] M. Klazar and F. Luca, On the maximal order of numbers in the "factorizatio numerorum" problem, J. Number Theory 124 (2007), 470–490.

[15] D.-H. Lehmer, Generalized Eulerian numbers, J. Comb. Theory, Ser. A 32 (1982), 195–215.

[16] N. P. Nörlund, Vorlesungen über Differenrechnungen, Berlin: Springer; 1924.

[17] I. J. Schoenberg, Cardinal spline interpolation, CBMS 12, SIAM, Philadelphia: 1973.

[18] R. Stanley, Enumerative combinatorics, Vol. I, Monterey: Wadsworth and Brooks/Cole; 1986.

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Numerical computation of the two dimensional exponential integrals

Gradimir V. Milovanović [1,2]

In 2015 Ş. Yardımcı, M. Olgun and Ç. Can [6] investigated a problem of convergence and asymptotic behaviour, as well as numerical computations, of the two-dimensional exponential integral (TDEI) functions arising in the study of the radiative transfer in a multi-dimensional medium. In [6] they started with the following definition of TDEI:

$$\varepsilon_n(\tau,\beta) = \frac{\tau^{n-1}}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\frac{\mathrm{e}^{-r}}{r^{n+1}}\,\mathrm{e}^{-\mathrm{i}\beta x}\mathrm{d}x\mathrm{d}y, \tag{1}$$

where $r^2 = x^2 + y^2 + \tau^2$ and $n = 1, 2, \ldots$ and proved that the function $(\tau,\beta) \mapsto \varepsilon_n(\tau,\beta)$ is uniformly convergent on $D(\varepsilon) = \big\{(\tau,\beta) \in \Omega \,|\, \tau \in [\varepsilon,+\infty),\ \beta \in \mathbb{R}\big\}$, and nonuniformly convergent on Ω, where $\Omega = [0,+\infty)\times\mathbb{R}$. Furthermore, using these facts, they proved that for $(\tau,\beta) \in D(\varepsilon)$ the function $(\tau,\beta) \mapsto \varepsilon_n(\tau,\beta)$ satisfies certain asymptotic formulas. Note that these functions (1) are two-dimensional analogs of the well-known exponential integral

$$E_n(\tau) = \int_1^{+\infty}\frac{\mathrm{e}^{-\tau t}}{t^n}\,\mathrm{d}t, \quad n \in \mathbb{N}. \tag{2}$$

As an application of the obtained results in [6], the authors constructed an algorithm for the calculation of TDEI function.

For numerical computation of the values of TDEI functions we can give some alternative (and very simple) methods using some quadrature processes (cf. [2]). We present a very efficient method based on some integral representations and the application of numerical integration.

For $\tau > 0$, $\beta > 0$, and $n \in \mathbb{N}$, we can prove that

$$\varepsilon_n(\tau,\beta) = \int_0^{+\infty} J_0(\tau\beta\sqrt{\mathrm{e}^{2x}-1})\mathrm{e}^{-(n-1)x-\tau\mathrm{e}^x}\,\mathrm{d}x, \tag{3}$$

and

$$\varepsilon_n(\tau,\beta) = \int_0^{+\infty}\frac{J_0(\tau\beta\sinh t)}{\cosh^n t}\,\mathrm{e}^{-\tau\cosh t}\sinh t\,\mathrm{d}t. \tag{4}$$

These expressions (3) and (4) are very appropriate for numerical calculation of the values of $\varepsilon_n(\tau,\beta)$. For example, the integrand $x \mapsto F_n(\tau,\beta,x) = J_0(\tau\beta\sqrt{\mathrm{e}^{2x}-1})\mathrm{e}^{-(n-1)x-\tau\mathrm{e}^x}$ in (3) is an analytic function with expansion

$$F_n(\tau,\beta,x) = \mathrm{e}^{-\tau}\sum_{\nu=0}^{+\infty}A_\nu x^\nu, \quad x \geq 0,$$

where $A_0 = 1$, $A_1 = -\frac{1}{2}\left[\beta^2\tau^2 + 2\tau + 2(n-1)\right]$, etc. Moreover, since $|J_0(t)| \leq 1$ for $t \geq 0$, we see that

$$|F_n(\tau,\beta,x)| \leq \left|J_0(\tau\beta\sqrt{\mathrm{e}^{2x}-1})\right|\mathrm{e}^{-(n-1)x}\,\mathrm{e}^{-\tau\mathrm{e}^x} \leq \mathrm{e}^{-\tau\mathrm{e}^x},$$

i.e., the analytic function $F_n(\tau,\beta,x)$ decays double exponentially when $x \to +\infty$. It is known that the trapezoidal formula with an equal mesh size gives an optimal formula for integrals of such functions (cf. [1, 3, 4, 5]).

An analysis of different methods for numerical calculation of the function $\varepsilon_n(\tau,\beta)$ will be presented in details in the extended version of this paper, including several numerical examples.

References

[1] M. Iri, S. Moriguti and Y. Takasawa, On certain quadrature formula, J. Comput. Appl. Math. 17 (1987), 3–20.

[2] G. Mastraianni and G. V. Milovanović, Interpolation processes – Basic theory and applications, Berlin-Heidelberg-New York: Springer Monographs in Mathematics, Springer Verlag; 2008.

[3] G. V. Milovanović, Expansions of the Kurepa function, Publ. Inst. Math. (Beograd) (N.S.) 57(71) (1995), 81–90.

[4] M. Mori, Quadrature formulas obtained by variable transformation and DE-rule, J. Comput. Appl. Math. 12&13 (1985), 119–130.

[5] H. Takahasi and M. Mori, Quadrature formulas obtained by variable transformation, Numer. Math. 21 (1973), 206–219.

[6] Ş. Yardımcı, M. Olgun, Ç. Can, Numerical computation and properties of the two dimensional exponential integrals, Hacet. J. Math. Stat. 45(4) (2015), 975–990.

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An overview on Aharonov-Berry superoscillating functions and their interplay with complex analysis

Irene Maria Sabadini [1]

In the past 50 years, physicists have discovered, and experimentally demonstrated, a phenomenon which they termed *superoscillations.* Aharonov and his collaborators showed that superoscillations naturally arise when dealing with weak values, a notion that provides a fundamentally different way to regard measurements in quantum physics.

From a mathematical point of view, superoscillating functions are sequences of suitable linear combinations of exponentials functions with frequencies bounded by 1 that, in the limit, can oscillate with frequencies much larger then 1.

The prototypical example of superoscillatory function is the following: let $a > 1$ be a real number we define the sequence of complex valued functions $F_n(x, a)$ defined on $\mathbb{R}$ by

$$F_n(x,a) = \Big(\cos\Big(\frac{x}{n}\Big) + ia\sin\Big(\frac{x}{n}\Big)\Big)^n = \sum_{k=0}^{n} C_k(n,a)e^{i(1-2k/n)x} \tag{1}$$

where

$$C_k(n,a) = \binom{n}{k}\left(\frac{1+a}{2}\right)^{n-k}\left(\frac{1-a}{2}\right)^k, \tag{2}$$

and $\binom{n}{k}$ denotes the binomial coefficients. The first thing one notices is that if we fix $x \in \mathbb{R}$, and we let n go to infinity, we immediately obtain that

$$\lim_{n\to\infty} F_n(x,a) = e^{iax}.$$

Moreover, the convergence is uniform on all compact sets in $\mathbb{R}$ but it is not uniform on all of $\mathbb{R}$, see [1]. The representation in terms of $e^{i(1-2k/n)x}$, together with the calculation of the limit of $F_n(x, a)$ when n goes to infinity, explains why such a sequence is called superoscillatory.

A detailed study of superoscillatory functions has been done in [2] where it is shown how the use of infinite order differential operators and spaces of holomorphic functions with suitable growth conditions are useful mathematical tools, see [6, 7, 4].

In this talk we shall discuss the interplay between complex analysis and superoscillations, also discussing some new methods to generate superoscillating functions, see [3, 5]. In particular, in [5] we rewrite the sequence in (1) as

$$F_n(x,a) := \sum_{j=0}^{n} C_j(n,a)e^{ik_j(n)x}, \quad x \in \mathbb{R}, \tag{3}$$

and we show how to use an operation inspired by the realtivist sum of velocities to construct a superoscillating sequence starting with two sequences of the form (3).

References

[1] Y. Aharonov, F. Colombo, I. Sabadini, D. C. Struppa and J. Tollaksen, Some mathematical properties of superoscillations, J. Phys. A: Math. Theor. 44(36) 365304 (2011), 16 pp.

[2] Y. Aharonov, F. Colombo, I. Sabadini, D. C. Struppa and J. Tollaksen, The mathematics of superoscillations, Mem. Amer. Math. Soc. 247(1174) (2017), 107 pp.

[3] Y. Aharonov, F. Colombo, I. Sabadini, T. Shushi, D. C. Struppa and J. Tollaksen, A new method to generate superoscillating functions and supershifts, Proc. A. 477(2249) (2021), Paper No. 20210020, 12 pp.

[4] Y. Aharonov, I. Sabadini, J. Tollaksen and A. Yger, Classes of superoscillating functions, Quantum Stud. Math. Found. 5(3) (2018), 439–454.

[5] D. Alpay, F. Colombo, S. Pinton and I. Sabadini, Holomorphic functions, relativistic sum, Blaschke products and superoscillations, Anal. Math. Phys. 11(3) (2021), Paper No. 139, 28 pp.

[6] T. Aoki, F. Colombo, I. Sabadini and D.C. Struppa, Continuity of some operators arising in the theory of superoscillations, Quantum Stud. Math. Found. 5(3) (2018), 463–476.

[7] T. Aoki, F. Colombo, I. Sabadini and D. C. Struppa, Continuity theorems for a class of convolution operators and applications to superoscillations, Ann. Mat. Pura Appl. 4(5) 197 (2018), 1533–1545.

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Quaternionic approach for special fluid flow equations in porous media

Wolfgang Sproessig [1]

One of the most relevant equations for a saturated non-slow flow in porous media are the time-dependent *Brinkman-Forchheimer equations* which read as follows

$$\begin{aligned} \partial_t \mathbf{u} - \lambda\Delta\mathbf{u}) + \mu^2\mathbf{u} - b|\mathbf{u}|\mathbf{u} + \nabla p &= -\mathbf{f} \quad \text{on} \quad G, \\ \operatorname{div}\mathbf{u} &= 0 \quad \text{on} \quad G, \\ \mathbf{u} &= 0 \quad \text{on} \quad \partial G. \end{aligned}$$

Here are λ the *Brinkman coefficient*, b the *Forchheimer coefficient* and $\mu^2 = \nu/k\lambda$ with ν the *kinematic viscosity* and k the *permeability*, p is *(pore) pressure*.

Set $T = n\tau$, where τ is the meshwidth of the discretisation. We further set $\mathbf{u}_k := \mathbf{u}(k\tau,)$, $\mathbf{f}_k := \frac{1}{\tau}\int\limits_{k\tau}^{(k+1)\tau} \mathbf{f}(t,\mathbf{x})dt$. Using finite forward differences $(\mathbf{u}_{k+1} - \mathbf{u}_k)/\tau$ and quaternionic multiplication we get the system

$$\begin{aligned} -(D - iA)(D + iA)\mathbf{u}_{k+1} + (1/\lambda)Dp_{k+1} &= \mathbf{F}_k \\ &:= -(1/\lambda)\mathbf{f}_k + (1/\tau\lambda)\mathbf{u}_k \\ &\quad +(b/\lambda)|\mathbf{u}_k|\mathbf{u}_k \\ \\ ScD\mathbf{u}_{k+1} &= 0. \end{aligned}$$

with $\mathbf{u}_{k+1} = 0$ on the boundary ∂G. Sc means *scalar part* of the corresponding quaternion, D is the *Dirac operator* with $\Delta = -D^2$ (quaternionic mulitplication!). i is complex unit and $A^2 = (1 + \mu^2\tau)/\lambda\tau$.

The following result can be fomulated in terms of an quaternionic operator calculus with $\alpha := iA$

$$\mathbf{u}_{k+1} = T_{-\alpha}\mathbf{Q}_\alpha[p_{k+1} + \alpha T_\alpha p_{k+1}] - T_{-\alpha}Q_\alpha T_\alpha \mathbf{F}_k$$

where $\mathbf{Q}_\alpha = I - \mathbf{P}_\alpha$ (*Pompeiu transform*) and $\mathbf{P}_\alpha$ is the projection onto $ker(D+\alpha)$ in the quaternionic Hilbert modul decomposition

$$L^2(G) = ker(D+\alpha) \oplus (D+\alpha)W_0^{2,1}(G).$$

T_α decribes the generalized *Teodorescu transform.*

A solution is given by

$$\begin{aligned} \mathbf{u}_{k+1,spec} &= T_{-\alpha}Q_\alpha Vec T_\alpha \mathbf{F}_k, \\ \\ \tilde{p}_{k+1} := p_{k+1} + \alpha T_\alpha p_{k+1} &= ScT_\alpha\mathbf{F}_k + \Phi_{k+1} \quad (\Phi_{k+1} \in kerD_\alpha). \end{aligned}$$

It can be shown that $||\mathbf{u}_{k+1}||_{2,1}$ is bounded and the truncation error tends to zero.

For more details we refer to Guerlebeck/Habetha/Sproessig: *Application of Holomorhic Functions in Two and Higher Dimensions*, Birkhauser, 2016.

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A note of arithmetical functions

Daeyeoul Kim [1]

This presentation first introduces the definition of an arithmetic function and its basic properties. Arithmetic functions are also introduced to the natural property of being a ring for addition and dirichlet convolution. Briefly, we define a convolution extended from an arithmetic function.

Let f and g be arithmetical functions and $A(n)$ subset of the set of divisors of n. Define an A-convolution of f and g, as follows:

$$(f *_A g)(n) = \sum_{d \in A(n)} f(d)g(n/d), ^{\forall} n$$

In 1963, W. Narkiewicz [1] defined a regular convolution A.

The binary operation $*_A$ is regular convolution if the following conditions hold:

(1) The ring $(F, +, *_A)$ is a commutative ring with unity δ.

(2) The convolution preserves multiplicativity.

(3) The "Möbius-function" of the convolution A, defined by the eqaution $1 *_A \mu_A = \delta$.

Here, $\delta(n) = [\frac{1}{n}] = \begin{cases} 1 & n = 1, \\ 0 & \text{otherwise.} \end{cases}$

In 1978, V. Sita Ramaiah [2] introduced a regular convolution A_k.
For a given regular convolution A and k a fixed positive integer, define sets, for each $n \in \mathbb{N}$,

$$A_k(n) := \{d \,|\, d^k \in A(n^k)\}.$$

Then the covolution A_k defined by the sets $A_k(n)$ is regular.

Using the definitions above, I would like to introduce some important properties. Finally, I would like to point out that this presentation is a survey presentation.

References

[1] W. Narkiewicz, On a class of arithmetical convolutions, Colloq. Math. 10(1) (1963), 81–94.

[2] V. S. Ramaiah, Arithmetical sums in regular convolutions, Journal fr die reine und angewandte Mathematik 0303_0304 (1978), 265–283.

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Using hypergeometric functions in hypercomplex analysis

Helmuth R. Malonek [1]

The survey is concerned with series representations of some Clifford algebra-valued functions in Hypercomplex Analysis and the use of hypergeometric functions as generating functions for corresponding sets of real coefficients. The approach relates to generalized Appell polynomials which for the first time have been introduced in 2006. An important connection with Vietoris' numbers in problems of real and complex analysis is mentioned. A special pair of non-monogenic variables is considered, and problems related to an adequate choice of variables in general are discussed.

Introduction

In general, hypercomplex function theory as part of Clifford Analysis (cf. [3]) is considered as refinement of Harmonic Analysis. It can heavily rely on representation theoretic and algebraic tools, particularly motivated by the seminal paper [9]. A big part of research in Clifford Analysis in the last 50 years seems to confirm this observation, but of course, meanwhile in Clifford Analysis other modern trends in connecting Algebra and Analysis also have been applied.

In the 90ies the papers (cf. [5, 6, 7]) contributed to a radical change of this more algebraic perspective, describing regular functions by using several hypercomplex variables (Fueter variables) as a new starting point for hypercomplex function theory. As one of several examples, applications to number theory or methods related to combinatorial questions became accessible via a more classical analytic point of view. It allows to show that a great number of real and complex analytic results can naturally and sometimes immediately obtained as special cases of hypercomplex analysis results. More about this approach and its applications the reader can find in the very recent paper [1].

Main Result

With a pair of two non-monogenic variables with the algebraic advantage of being commutative, namely $x = x_0e_0 + x_1e_1 + x_2e_2 + \cdots + x_e e_n$ and $\bar{x} = x_0e_0 - x_1e_1 - x_2e_2 - \cdots - x_n e_n$, one can prove the following

Theorem 1. *The differential of a monogenic function $f = f(x, \bar{x})$ has the form*

$$df = \partial_x f dx + \partial_{\bar{x}} f d\bar{x},$$

with the corresponding hypercomplex gradient

$$\nabla_f = (\partial_x, \partial_{\bar{x}}) = \left(\frac{1}{2}(\partial_0 - \frac{1}{n}\partial_{\underline{x}}), \frac{1}{2}(\partial_0 + \frac{1}{n}\partial_{\underline{x}})\right).$$

Moreover, the Taylor series expansion of a paravector valued function f depending on the paravector x and its conjugate $\bar{x}$ around the origin is given by

$$\begin{aligned} f(x,\bar{x}) &= \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{\partial}{\partial x} x + \frac{\partial}{\partial \bar{x}} \bar{x} \right)^k f(0) \\ &= \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{s=0}^{k} \binom{k}{s} \frac{\partial^k f(0)}{\partial x^{k-s} \partial \bar{x}^s} \, x^{k-s} \bar{x}^s \qquad (1) \\ &= \sum_{k=0}^{\infty} \sum_{s=0}^{k} \frac{1}{(k-s)!s!} \frac{\partial^k f(0)}{\partial x^{k-s} \partial \bar{x}^s} \, x^{k-s} \bar{x}^s. \end{aligned}$$

The last expression in formula (1) is nothing else than the expansion of the given function f in a series of polynomials of the form

$$\mathcal{P}_k^n(x,\bar{x}) = \sum_{s=0}^{k} T_s^k(n) \, x^{k-s} \, \bar{x}^s, \quad k = 0,1,2,\ldots. \qquad (2)$$

with real coefficients

$$T_s^k(n) = \binom{k}{s} \frac{\left(\frac{n+1}{2}\right)_{k-s} \left(\frac{n-1}{2}\right)_s}{(n)_k} \qquad (3)$$

which constitute a *generalized Appell sequence with respect to* the hypercomplex derivative ∂ (cf. [5]) introduced in [4].

With $a = 1, b = \frac{n+1}{2}, b' = \frac{n-1}{2}, c = n, u = x$, and $v = \bar{x}$, the corresponding function $f = f(x,\bar{x})$ in (1) is recognized as Appell's function F_1 of two variables (see [8], 139).

Corollary 2. *The generating function of the generalized Appell sequence with respect to ∂ in the form* (2) *is Appell's hypergeometric function $F_1(x,\bar{x})$, i.e.*

$$\begin{aligned} F_1(a;b,b';c;x,\bar{x}) &= \sum_{r,t=0}^{\infty} \frac{(a)_{r+t}(b)_t(b')_r u^t v^r}{t!r!(c)_{r+t}} \\ &= \sum_{k=0}^{\infty} \sum_{s=0}^{k} \binom{k}{s} \frac{\left(\frac{n+1}{2}\right)_{k-s} \left(\frac{n-1}{2}\right)_s}{(n)_k} \, x^{k-s} \bar{x}^s \qquad (4) \\ &= \sum_{k=0}^{\infty} \mathcal{P}_k^n(x,\bar{x}). \end{aligned}$$

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References

[1] D. Alpay, I. L. Paiva, and D. Struppa , A general setting for functions of Fueter variables: differentiability, rational functions, Fock module and related topics, Israel J. Math. 236 (2020), 207–246.

[2] P. Appell, Sur une classe de polynômes, Ann. Sci. École Norm. Sup. 9(2) (1880), 119–144.

[3] F. Brackx, R.Delanghe and F. Sommen, Clifford analysis, Boston-London-Melbourne: Pitman Advanced Publishing; 1982.

[4] M. I. Falcão, J. Cruz and H. R. Malonek, Remarks on the generation of monogenic functions, In: K. Gürlebeck and C. Könke, (eds.), Proc. of the 17-th Inter. Conf. on the Application of Computer Science and Mathematics in Architecture and Civil Engineering (ISSN 1611- 4086), Bauhaus-University Weimar, 2006.

[5] K. Gürlebeck and H. R. Malonek, A hypercomplex derivative of monogenic functions in $\mathbb{R}^{n+1}$ and its applications, Complex Variables Theory Appl. 39, (1999), 199–228.

[6] H. Malonek, A new hypercomplex structure of the Euclidean space $\mathbb{R}^{m+1}$ and the concept of hypercomplex differentiability, Complex Variables Theory Appl. 14(4) (1990), 25–33.

[7] H. Malonek, Power series representation for monogenic functions in $\mathbb{R}^{n+1}$ based on a permutational product. Complex Variables, Theory Appl. 15(3) (1990), 181–191.

[8] E. D. Rainville, Special functions, New York: Macmillan; 1960.

[9] E. M. Stein and G. Weiss, Generalization of the Cauchy-Riemann equations and representations of the rotation group. Am. J. Math. 90(1) (1968), 163–196.

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On the approach of solving iterative functional equations with generating functions

Dmitry Kruchinin [1]

Using properties of compositae we present an approach of solving the following iterative functional equation $A(A(x)) = F(x)$, where the expression for coefficients of a generating function $A(x) = \sum_{n>0} a(n)x^n$ is not known, but its composition with itself is known. We construct some recurrent identities. Also we give an example of applying the obtained results

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Introduction

In [6, 10] and on the last "Mediterranean International Conference of Pure & Applied Mathematics and Related Areas" in 2019 we had studied the following type of functional equations for the different cases of the power:

1. for a natural number m
$$B(x) = H(xB(x)^m), \tag{1}$$
coefficients of powers of generating function $xB(x)$ (or composita of $xB(x)$) are defined by
$$B_x^\Delta(n,k,m) = \frac{k}{i_{m-1}} H_x^\Delta(i_m, i_{m-1}), \tag{2}$$
where $i_m = (m+1)n - mk$ and $H_x^\Delta(n,k)$ and $B_x^\Delta(n,k)$ are the compositae of the generating functions $xH(x)$ and $xB(x)$, respectively;

2. for a rational number m
$$B(x) = H(xB(x)^r), \tag{3}$$
the composita of $x[H(x)]^r$ is defined by
$$H_x^\Delta(n,m,r) = \begin{cases} H^\Delta(1,1)^{m\,r}, & n = m \\ \sum\limits_{k=1}^{n-m} H_m^\Delta(n-m,k)\binom{mr}{k} H^\Delta(1,1)^{mr-k} & n > m, \end{cases} \tag{4}$$
where $H_m^\Delta(n,k)$ is the coefficients of generating function $[H(x) - h(0)]^k$ that is defined by the composita $H^\Delta(n,k)$ of the generating function $xH(x)$.

In this paper, we study application of the composita for solving the iterative functional equation with respect to selfcomposition.

For a given $A(x)$ the n-th iteration is the function that is composed with itself n times

$$A^0(x) = x, \quad A^1(x) = A(x), \quad A^2(x) = A(A^1(x)), \ldots, A^n(x) = A(A^{n-1}(x))$$

and denoted by $A^n(x)$.

By an iterative functional equation, we mean the equation, where the analytical form of a function $A(x)$ is not known, but its composition with itself is known.

In the literature, you can find extensive investigations concerning iterates and functional equations, classical books [1, 11, 12] and recent surveys and papers [2, 5, 4, 3, 13].

Iterative functional equations arise in various fields such as fractal theory, computer science, dynamical systems, and maps. Despite their prevalence, they are very difficult to solve, and few mathematical tools exist to analyse them. In the general case, solving the equations involve deep mathematical insight and experimentation with different substitutions and reformulations.

In this paper, we propose using the notion of the composita to identify and solve such iterative functional equations, where $F(x) = \sum_{n>0} f(n)x^n$, $f(1) \neq 0$. By the composita introduced in the papers [8, 7, 9], we mean the following notion:

The composita is the function of two variables defined by

$$F^{\Delta}(n,k) = \sum_{\pi_k \in C_n} f(\lambda_1) f(\lambda_2) \ldots f(\lambda_k), \tag{5}$$

where C_n is a set of all compositions of an integer n, π_k is the composition $\sum_{i=1}^{k} \lambda_i = n$ into k parts exactly.

Suppose $F(x) = \sum_{n>0} f(n)x^n$ is the generating function with the free term equals 0, i.e. $f(0) = 0$. From this generating function we can write the following equation:

$$[F(x)]^k = \sum_{n>0} F(n,k)x^n. \tag{6}$$

The expression $F(n,k)$ is the composita, and it is denoted by $F^{\Delta}(n,k)$. In this case, the composita is an expression for the coefficients of powers of the generating function with the free term equals 0.

Main Results

The simplest case of an iterative functional equation for an unknown function $A(x)$ with a given function $F(x)$

$$A(A(x)) = F(x)$$

appeared in 1820 (Babbage, with $F(x) = x$).

Let us consider the case, when the expression for coefficients of a generating function $A(x) = \sum_{n>0} a(n)x^n$ is not known, but its composition with itself is known.

Theorem 1.

Suppose $F(x) = \sum_{n>0} f(n)x^n$ is a generating function, where $f(1) \neq 0$, $F^{\Delta}(n,k)$ is the composita of $F(x)$, $A(x) = \sum_{n>0} a(n)x^n$ is the generating function, which is obtained from the functional equation $A(A(x)) = F(x)$. Then, for the composita of the generating function $A(x)$, we have the following recurrent formula:

$$A^{\Delta}(n,k) = \begin{cases} f(1)^{\frac{n}{2}}, & n = k; \\ \dfrac{F^{\Delta}(n,k) - \sum\limits_{m=k+1}^{n-1} A^{\Delta}(n,m) A^{\Delta}(m,k)}{f(1)^{\frac{n}{2}} + f(1)^{\frac{k}{2}}}, & n > k. \end{cases}$$

For applications of Theorem 1, we give the following example.

Example 1.

Let us consider the iterative functional equation $A(A(x)) = \sin(x)$ (for details, see A048602, A048603 in [14]). The composita of $F(x) = \sin(x)$ is

$$F^{\Delta}(n,k) = \frac{1+(-1)^{n-k}}{2^k n!} \sum_{m=0}^{\lfloor \frac{k}{2} \rfloor} \binom{k}{m} (2m-k)^n (-1)^{\frac{n+k}{2}-m}.$$

Using Theorem 1, we obtain the following recurrent formula

$$A^{\Delta}(n,k) = \begin{cases} 1, & n = k; \\ \frac{1}{2}\left(F^{\Delta}(n,k) - \sum\limits_{m=k+1}^{n-1} A^{\Delta}(n,m)A^{\Delta}(m,k)\right), & n > k. \end{cases}$$

Therefore, the expression for coefficients of $A(x) = \sum_{n>0} a(n)x^n$ is

$$a(n) = A^{\Delta}(n,1).$$

Conclusion

Using the properties of compositae, we have showed an approach of solving the iterative functional equation $A(A(x)) = F(x)$, where the expression for coefficients of a generating function $A(x) = \sum_{n>0} a(n)x^n$ is not known, but its composition with itself is known. Those results can contribute to the development of methods for solving functional and iterative equations in terms of generating functions.

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References

[1] J. Aczel, On applications and theory of functional equations, Basel: Birkhauser; 1969.

[2] K. Baron and W. Jarczyk, Recent results on functional equations in a single variable, perspectives and open problems, Aequationes Math. 61(1-2) (2001), 1–48.

[3] L. Berg, Iterative functional equations, Rostock. Math. Kolloq. 64 (2009), 3–10.

[4] B. A. Brown, A. R. Brown and M. F. Shlesinger, Solutions of doubly and higher order iterated equations, J. Stat. Phys. 110 (2003), 1087–1097.

[5] J. Kobza, Iterative functional equation $x(x(t)) = f(t)$ with $f(t)$ piecewise linear, J. Comput. Appl. Math. 115(1-2) (2000), 331–347.

[6] D. V. Kruchinin, On solving some functional equations, Adv. Difference Equ. 2015(1) (2015), Article number: 17.

[7] D. Kruchinin and V. Kruchinin, A method for obtaining generating functions for central coefficients of triangles, J. Integer Seq. 15(9) (2012), Article ID: 12.9.3.

[8] D. V. Kruchinin and V. V. Kruchininm, Application of a composition of generating functions for obtaining explicit formulas of polynomials, J. Math. Anal. Appl. 404(1) (2013), 161–171.

[9] D. Kruchinin, V. Kruchinin and Y. Shablya, Method for obtaining coefficients of powers of bivariate generating functions, Mathematics 9(4) (2021), Article ID: 428.

[10] D. V. Kruchinin and M. Y. Perminova, About solving some functional equations related to the Lagrange inversion theorem, Montes Taurus J. Pure Appl. Math. 3(1) (2021), 62–69.

[11] M. Kuczma, Functional equations in a single variable, Warszawa: PWN-Polish Scientific Publishers; 1968.

[12] R. Kuczma, M. Choczewski and B. Ger, Iterative functional equations, Cambridge: Cambridge Univ. Press; 1990.

[13] L. Liu, Existence of analytic solutions of an iterative functional equation, Appl. Math. Comput. 217(17) (2011), 7245–7257.

[14] N. J. A. Sloane, The on-line encyclopedia of integer sequences, (https://oeis.org/).

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2 CONTRIBUTED TALKS

Baire-type properties in $c_0(\Omega, X)$

Manuel López-Pellicer [1], *Salvador López-Alfonso* [2], *Santiago Moll-López* [3]

Let Ω be a non-empty set, X a locally convex space over the field $\mathbb{K}$ (of the real or complex numbers), $cs(X)$ the family of all continuous seminorms in X and $c_0(\Omega, X)$ the locally convex space over $\mathbb{K}$ of all functions $f : \Omega \to X$ such that for each $\varepsilon > 0$ and $p \in cs(X)$ the set $\{\omega \in \Omega : p(f(\omega)) > \varepsilon\}$ is finite or empty, with the topology defined by the semi-norms $\|f\|_p = \sup\{p(f(\omega)) : \omega \in \Omega\}$, $p \in cs(X)$. In particular, for $\Omega = \mathbb{N}$ or for $X = \mathbb{K}$, we have the well known spaces $c_0(\Omega) := c_0(\Omega, \mathbb{K})$, $c_0(X) := c_0(\mathbb{N}, X)$ and $c_0 := c_0(\mathbb{N}, \mathbb{K})$.

In [15] it was proved that $c_0(X)$ is quasibarrelled if and only if X is quasibarrelled and its strong dual satisfies condition (B) of Pietsch and in [16] it was proved that $c_0(E)$ is barrelled if and only if E is barrelled and its strong dual satisfies condition (B) of Pietsch. Metrizable locally convex spaces as well as dual metric locally convex spaces verify the property (B) of Pietsch ([18]).

Ferrando and Lüdkowsky extended in [9] Mendoza's result of [16]. They showed that $c_0(\Omega, X)$ is barrelled, ultrabornological or unordered Baire-like ([23]) if and only if X is, respectively, barrelled, ultrabornological or unordered Baire-like, providing also concrete examples of vector valued function spaces $c_0(\Omega, X)$ with properties mentioned above. In [4] it was proved that the normed space of all continuous functions vanishing at infinite defined on a locally compact topological space with values in a normed space and endowed with the supremum norm topology is barrelled if and only if X is barrelled; this answered a question posed by J. Horváth.

The linear subspace l_0^∞ of the sequence space l_∞ of finite-valued sequences in the field $\mathbb{K}$ is of the first Baire category [1]. Dieudonné [26, p. 133] and, independently, Saxon [20] proved that l_0^∞ is barrelled. This result was extended by Schachermayer by showing that the linear hull $l_0^\infty(\mathcal{A})$ of the characteristic functions $\mathcal{X}_A$, $A \in \mathcal{A}$, where $\mathcal{A}$ is a ring of subsets of Ω, endowed with the supremum norm topology is barrelled if and only if the vector space of all bounded finitely additive scalar measures defined on $\mathcal{A}$ and equipped with the supremum norm topology verifies the Nikodým boundedness theorem, see [3, p. 80]. Moreover, if $\mathcal{A}$ is a σ-algebra, the space $l_0^\infty(\mathcal{A})$ is barrelled, see [3, p. 80] and [22]. Valdivia [24] proved even more: If $(E_n)_n$ is an increasing sequence of vector subspaces of $l_0^\infty(\mathcal{A})$ covering $l_0^\infty(\mathcal{A})$, then there is an E_n which is barrelled and dense in $l_0^\infty(\mathcal{A})$. This property defines *suprabarrelled* spaces, called (db) spaces in [19] and [21]. Interesting applications of suprabarrelled spaces are in [24], [25] and [17, Chapter 9]. A natural generalization of suprabarrelled spaces are p-barrelled spaces. Recall, see [2], that a p-net in a vector space X is a family $\mathcal{W} = \{X_t : t \in \mathbb{T}_p\}$ of vector subspaces of X, where $\mathbb{T}_p = \bigcup_{k=1}^p \mathbb{N}^k$, such that $X = \cup\{X_n : n \in \mathbb{N}\}$, $X_n \subset X_{n+1}$, $X_t = \bigcup\{X_{t,n} : n \in \mathbb{N}\}$, $X_{t,n} \subset X_{t,n+1}$, for $t \in \mathbb{T}_r$, $1 \leq r < p$ and $n \in \mathbb{N}$.

A locally convex space X is called *p-barrelled* if given a p-net $\mathcal{W} = \{X_t : t \in \mathbb{T}_p\}$ in X there is a $t \in \mathbb{N}^p$ such that X_t is barrelled and dense in X.

Note that suprabarrelled are the 1-barrelled spaces. We refer the reader to [7] and [10] for several applications of p-barrelled spaces; for example for vector measures. More strong properties are considered in [5], [6], [11], [12], [13] and [14].

Notice that if the locally convex space X is metrizable and if $\{\|.\|_n \in \mathbb{N}\}$ is an increasing sequence of semi-norms defining the topology of X, then the locally convex space $c_0(\Omega, X)$ is metrizable, because its topology is defined by the seminorms $\|f\|_n = \sup\left\{\|f(\omega)\|_n : \omega \in \Omega\right\}$, $n \in \mathbb{N}$, and for every $f \in c_0(\Omega, X)$ its support, supp$f := \{w \in \Omega : f(w) \neq 0\}$, is countable, since $\{w \in \Omega : f(w) \neq 0\} = \bigcup_{n,m=1}^{\infty} \left\{\omega \in \Omega : \|f(\omega)\|_n > \frac{1}{m}\right\}$ and, by definition, for each $\varepsilon > 0$ and $n \in \mathbb{N}$ the set $\left\{\omega \in \Omega : \|f(x)\|_n > \varepsilon\right\}$ is finite or empty.

Our aim is to get shorter proofs of main results of [9] for X metrizable showing that $c_0(\Omega, X)$ is barrelled, quasibarrelled or unordered Baire-like if and only if X has the same property, respectively. Moreover we prove that $c_0(\Omega, X)$ is totally barrelled or p-barrelled if and only if X is totally barrelled or p-barrelled, respectively. Last section provides an application to a closed graph theorem.

Recall that a locally convex space X is *barrelled* (*quasibarrelled*) if every closed absolutely convex and absorbing (and bornivorous) subset of E is a neighbourhood of zero. Barrelled spaces are just the locally convex spaces that verify Banach-Steinhaus boundedness theorem. A natural and applicable generalization of Baire spaces to locally convex spaces was discovered by A. Todd and S. Saxon, see [23]: A locally convex space X is called *unordered Baire-like*, if every sequence of absolutely convex and closed subsets of X covering X contains a member which is a neighbourhood of zero. Finally, a locally convex space X is *totally barrelled* if for every sequence of subspaces $(X_n)_{n\in\mathbb{N}}$ of X covering X there is some X_p which is barrelled and its closure is finite-codimensional in X, see [8, Definition 1.4.1] and [27]. Note that Baire $\Rightarrow$ unordered Baire-like$\Rightarrow$ totally barrelled $\Rightarrow$ p-barrelled $\Rightarrow$ Baire-like $\Rightarrow$ barrelled$\Rightarrow$ quasibarrelled.

Several open problems will be discussed.

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References

[1] G. Bennet and N. Kalton, Inclusion theorems for K-spaces, Canad. J. Math 25 (1973), 511-524.

[2] M. de Wilde, Closed graph theorems and webbed spaces, Boston, London: Research Notes in Mathematics 19, Pitman; 1978.

[3] J. Diestel, Sequences and series in Banach spaces, New York; Graduate Texts in Mathematics 92 Springer-Verlag; 1984.

[4] J.C. Ferrando, J. Kakol and M. López-Pellicer, On a problem of Horváth concerning barrelled spaces of vector valued continuous functions vanishing at infinity, Bull. Belg. Math. Soc. Simon Stevin 11 (2004), 127–132.

[5] J.C. Ferrando, S. López-Alfonso and M. López-Pellicer, On Nikodým and rainwater sets for $ba(\mathcal{R})$ and a problem of M. Valdivia, Filomat 33 (2019), 2409–2416.

[6] J.C. Ferrando, S. López-Alfonso and M. López-Pellicer, On Grothendieck Sets, Axioms 9(1), 34 (2020) 1–7.

[7] J.C. Ferrando and M. López Pellicer, A note on a theorem of J. Diestel and B. Faires. Proc. Amer. Math. Soc. 115 (1992), 1077–1081.

[8] J.C. Ferrando, M. López Pellicer and L.M. Sánchez Ruiz, Metrizable barrelled spaces Harlow: Pitman Research Notes in Mathematics Series 332 Longman; 1995.

[9] J.C. Ferrando and S.V. Lüdkovsky, Some barrelledness properties of $c_0(\Omega, X)$. J. Math. Anal. Appl. 274 (2002), 577–585.

[10] J.C. Ferrando and L.M. Sánchez Ruiz, A survey on recent advances on the Nikodým boundedness theorem and spaces of simple functions, Rocky Mountain J. Math. 34 (2004), 139–172.

[11] S. López-Alfonso and M. López-Pellicer, Weak sequential convergence in bounded finitely additive measures, Vietnam J. Math. 48 (2020), 379–389.

[12] S. López-Alfonso, M. López-Pellicer and J. Mas, A Survey on Nikodým and Vitali-Hahn-Saks properties, Montes Taurus J. Pure Appl. Math. 3 (2021), 112–121.

[13] S. López-Alfonso, M. López-Pellicer and S. Moll-López, On four classical measure theorems, Mathematics 9(5), 526 (2021), 1–17.

[14] M. López-Pellicer, Webs and bounded additive measures, J. Math. Anal. Appl. 210 (1997), 257–267.

[15] A. Marquina and J.M. Sanz-Serna, Barrelledness conditions on $c_0(E)$. Arch. Math. (Basel) 31(1) (1978), 589–596.

[16] J. Mendoza, A barrelledness criterion for $C_0(E)$. Arch. Math. (Basel) 40 (1983), 156–158.

[17] P. Pérez Carreras and J. Bonet, Barrelled Locally Convex Spaces, Amsterdam: North-Holland Mathematics Studies 131 North-Holland Punordered Baire-likeishing Co; 1987.

[18] A. Pietsch, Nuclear locally convex spaces, Ergebnisse der Mathematik und ihrer Grenzgebiete, New York-Heidelberg: Band 66 Springer-Verlag; 1972.

[19] W.J. Robertson, I. Tweddle, and F.E. Yeomans, On the stability of barrelled topologies III, Bull. Austral. Math. Soc. 22 (1980), 99–112.

[20] S.A. Saxon, Some normed barrelled spaces which are not Baire, Math. Ann. 209 (1974), 153–170.

[21] S.A. Saxon, and P.P. Narayanaswani, *Metrizable (LF)-spaces,* (db)-spaces and the separable quotient problem, Bull. Austral. Math. Soc. 23 (1981), 65–80.

[22] W. Schachermayer, On some classical measure-theoretic theorems for non σ-complete Boolean algebras, Dissert. Math. 214 (1982), 1–36.

[23] A.R. Todd and S.A. Saxon, A property of locally convex Baire spaces, Math. Ann. 206 (1973), 23-34.

[24] M. Valdivia, On certain barrelled normed spaces, Ann. Inst. Fourier (Grenoble) 29 (1979), 39–56.

[25] M. Valdivia, On suprabarrelled spaces, in functional analysis, holomorphy, and approximation theory (Proc. Sem., Univ. Fed. Rio de Janeiro, Rio de Janeiro, 1978), pp. 572–580, Lecture Notes in Mathematics 843, Berlin: Springer; 1981.

[26] M. Valdivia, Topics in locally convex spaces, North-Holland Mathematics Studies 67, Amsterdam-New York: North-Holland Punordered Baire-likeishing Co.; 1982.

[27] M. Valdivia and P. Pérez Carreras, On totally barrelled spaces, Math. Z. 178 (1981), 263–269.

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Relations between a certain combinatorial numbers and Fibonacci numbers

Yilmaz Simsek [1]

In "Generating functions for finite sums involving higher powers of binomial coefficients: Analysis of hypergeometric functions including new families of polynomials and numbers, J. Math. Anal. Appl. 477, 1328–1352 (2019)", the author introduced a new kind of combinatorial numbers and presented some identities associated with these combinatorial numbers. In this presentation, the author give some relations between Fibonacci numbers and the combinatorial numbers mentioned above.

2010 Mathematics Subject Classifications: 05A10, 05A15, 11B39, 11B83
Keywords: Bernoulli polynomials, Euler polynomials, multiplication formula, Abel binomial identity, Abel polynomials, p-adic integral

Introduction

The main motivation of this presentation is to investigate relations between the Fibonacci numbers and the combinatorial numbers $y_6(m,n;\lambda;p)$, which is given by the following generating function (*cf.* [3, Theorem 1, p. 1347]):

$$\frac{1}{n!}\,{}_pF_{p-1}\left[\begin{array}{c}-n,-n,...,-n\\1,1,...,1\end{array};(-1)^p\lambda e^t\right]=\sum_{m=0}^{\infty}y_6(m,n;\lambda,p)\frac{t^m}{m!}, \tag{1}$$

where $n,p\in\mathbb{N}$ and $\lambda\in\mathbb{R}$ (or $\mathbb{C}$), above series converges for all z if $p<q+1$, and for $|z|<1$ if $p=q+1$ and ${}_pF_p$ denotes the generalized hypergeometric function, given by

$${}_pF_q\left[\begin{array}{c}\alpha_1,...,\alpha_p\\\beta_1,...,\beta_q\end{array};z\right]=\sum_{m=0}^{\infty}\left(\frac{\prod\limits_{j=1}^{p}(\alpha_j)_m}{\prod\limits_{j=1}^{q}(\beta_j)_m}\right)\frac{z^m}{m!},$$

$(\lambda)_v$ denotes the Pochhammer's symbol known as the rising factorial, given by

$$(\lambda)_v=\frac{\Gamma(\lambda+v)}{\Gamma(\lambda)}=\lambda(\lambda+1)\cdots(\lambda+v-1),$$

and

$$(\lambda)_0=1$$

for $\lambda\neq1$, where $v\in\mathbb{N}$, $\lambda\in\mathbb{C}$, $\Gamma(\lambda)$ denotes the Euler gamma function and

$$\binom{z}{v}=\frac{z(z-1)\cdots(z-v+1)}{v!}=\frac{(z)^{\underline{v}}}{v!}\ (v\in\mathbb{N},\ z\in\mathbb{C})$$

and

$$\binom{z}{0}=1.$$

It is easy to see that

$$(-\lambda)_v = (-1)^v (\lambda)^{\underline{v}}.$$

Some special hypergeometric functions are given as follows:

$${}_0F_0(z) = e^z,$$

$${}_2F_1(\alpha_1;\alpha_2;\beta_1;z) = \sum_{k=0}^{\infty} \frac{(\alpha_1)_k (\alpha_2)_k}{(\beta_1)_k} \frac{z^k}{k!},$$

and

$${}_1F_0\left[\begin{array}{c} b \\ - \end{array};x\right] = \frac{1}{(1-x)^b}$$

Therefore, we have

$$\frac{1}{n!}\sum_{k=0}^{n}\binom{n}{k}^p \lambda^k e^{tk} = \sum_{n=0}^{\infty} y_6(m,n;\lambda;p)\, t^n \tag{2}$$

which were recently introduced and studied by the author [3, Theorem 1, p. 1347].

Using (2), the numbers $y_6(m,n;\lambda;p)$ are given by

$$y_6(m,n;\lambda;p) = \frac{1}{n!}\sum_{k=0}^{n}\binom{n}{k}^p k^m \lambda^k \tag{3}$$

in which m, n and p are nonnegative integers and λ is real or complex number (*cf.* [3, Theorem 1, p. 1333]).

It is well-known that the Fibonacci numbers are integers defined by the following recurrence relation:

$$F_n = F_{n-1} + F_{n-2}, \qquad (n \geq 2) \tag{4}$$

with the initial conditions $F_0 = F_1 = 1$ (*cf.* [1], [2]).

In the case of when $\alpha+\beta = 1$, $\alpha-\beta = \sqrt{5}$ and $\alpha\beta = -1$, the Fibonacci numbers satisfy the following formulas:

$$\alpha^n = F_{n-1} + \alpha F_n, \tag{5}$$

(*cf.* [2, Lemma 5.1, p. 78]) and

$$\beta^n = F_{n-1} + \beta F_n, \tag{6}$$

(*cf.* [2, Corollary 5.3, p. 78]).

Main Results

By setting $m = 0$ and $p = 1$ in (2), we have

$$y_6(0,n;\lambda;1) = \frac{(\lambda+1)^n}{n!}. \tag{7}$$

In this section, by combining (7) with the equations (5) and (6), we present some relations between the combinatorial numbers $y_6(m,n;\lambda;p)$ and the Fibonacci numbers by the following theorems:

Theorem 1. *Let $\alpha+\beta = 1$, $\alpha-\beta = \sqrt{5}$ and $\alpha\beta = -1$. Then we have*

$$y_6(0,n;-\beta;1) = \frac{F_{n-1} + \alpha F_n}{n!}. \tag{8}$$

Theorem 2. *Let* $\alpha = \frac{1-\sqrt{5}}{2}$. *Then we have*

$$y_6(0, n; \alpha; 1) = \frac{1}{n!} \sum_{j=0}^{n} \binom{n}{j} (F_{j-1} + \alpha F_j). \tag{9}$$

Theorem 3. *Let* $\alpha + \beta = 1$, $\alpha - \beta = \sqrt{5}$ *and* $\alpha\beta = -1$. *Then we have*

$$y_6(0, n; -\alpha; 1) = \frac{1}{n!} \sum_{j=0}^{n} \binom{n}{j} \frac{1}{F_{j-1} + \beta F_j}. \tag{10}$$

Theorem 4. *Let* $\alpha + \beta = 1$, $\alpha - \beta = \sqrt{5}$ *and* $\alpha\beta = -1$. *Then we have*

$$y_6(0, n; -\beta; 1) = \frac{1}{n!} \sum_{j=0}^{n} \binom{n}{j} \frac{1}{F_{j-1} + \alpha F_j}. \tag{11}$$

Combining (8) and (11) yields a recurrence relation for the sequence $\{F_{n-1} + \alpha F_n\}_{n=0}^{\infty}$ by the following corollary:

Corollary 5. *Let* $\alpha = \frac{1-\sqrt{5}}{2}$. *Then we have*

$$F_{n-1} + \alpha F_n = \sum_{j=0}^{n} \binom{n}{j} \frac{1}{F_{j-1} + \alpha F_j}. \tag{12}$$

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References

[1] L. Comtet, Advanced combinatorics: The art of finite and infinite expansions, Dordrecht and Boston: D. Reidel Publishing Company; 1974.

[2] T. Koshy, Fibonacci and Lucas numbers with applications, New York, Toronto; John Wiley&Sons. Inc: 2001.

[3] Y. Simsek, Generating functions for finite sums involving higher powers of binomial coefficients: Analysis of hypergeometric functions including new families of polynomials and numbers, J. Math. Anal. Appl. 477 (2019), 1328–1352.

[4] H. M. Srivastava and J. Choi, Zeta and q-Zeta Functions and Associated Series and Integrals, Amsterdam, London and New York: Elsevier Science Publishers; 2012.

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Application of the method of compositae to generating functions in three variables

Dmitry Kruchinin [1], *Yuriy Shablya* [2], *Maria Perminova* [3]

In this paper, we consider the problem of obtaining explicit formulas for the coefficients of generating functions. As one of the steps in solving this problem, we propose to generalize the concept of compositae to the case of generating functions in three variables. The application of this approach allows obtaining explicit formulas for the coefficients of a generating function that can be represented as a composition of generating functions. We also tested the proposed ideas with several examples.

2010 Mathematics Subject Classifications: 05A15, 40B05
Keywords: Multivariate generating function, Composition, Composita, Coefficients function, Explicit formula

Introduction

Generating functions play an important role in solving different problems in pure and applied mathematics [5]. There is a large number of research papers devoted to the study of ordinary generating functions. However, studies that deal with complex structures defined by generating functions in several variables are becoming more common [2, 3, 1]. Therefore, there is a need to develop methods for handling such multivariate generating functions.

A multivariate generating function is the following formal power series:

$$F(x,y,\ldots,z)=\sum_{n\geq 0}\sum_{m\geq 0}\cdots\sum_{l\geq 0} f(n,m,\ldots,k)\,x^n y^m\cdots z^l.$$

The composita $F^{\Delta}(n,m,\ldots,l,k)$ of a multivariate generating function $F(x,y,\ldots,z)$ with $ord(F)\geq 1$ is a coefficients function of the k-th power of the generating function $F(x,y,\ldots,z)$ [4]:

$$F(x,y,\ldots,z)^k=\sum_{n\geq 0}\sum_{m\geq 0}\cdots\sum_{l\geq 0} F^{\Delta}(n,m,\ldots,l,k)\,x^n y^m\cdots z^l.$$

For $k=0$ we get $F(x,y,\ldots,z)^0=1$.

In this research, we propose to generalize the concept of compositae to the case of generating functions in three variables.

Main Results

A generating function in three variables is the following formal power series:

$$F(x,y,z)=\sum_{n\geq 0}\sum_{m\geq 0}\sum_{l\geq 0} f(n,m,l)\,x^n y^m z^l.$$

Using the concept of compositae, the k-th power of the generating function $F(x,y,z)$, with $ord(F) \geq 1$, can be presented as follows:

$$F(x,y,z)^k = \sum_{n\geq 0}\sum_{m\geq 0}\sum_{l\geq 0} F^{\Delta}(n,m,l,k)\, x^n y^m z^l.$$

The following theorem shows a possible way to obtain explicit formulas for the coefficients of a generating function that is represented as a composition of generating functions. In this case, the process of obtaining these formulas is associated with the use of compositae of generating functions and the convolution operation.

Theorem 1. *Suppose that:*

$$H(x,y,z) = \sum_{n\geq 0}\sum_{m\geq 0}\sum_{l\geq 0} h(n,m,l)\, x^n y^m z^l,$$

$$A(x,y,z)^k = \sum_{n\geq 0}\sum_{m\geq 0}\sum_{l\geq 0} A^{\Delta}(n,m,l,k)\, x^n y^m z^l, \quad ord(A) \geq 1,$$

$$B(x,y,z)^k = \sum_{n\geq 0}\sum_{m\geq 0}\sum_{l\geq 0} B^{\Delta}(n,m,l,k)\, x^n y^m z^l, \quad ord(B) \geq 1,$$

$$C(x,y,z)^k = \sum_{n\geq 0}\sum_{m\geq 0}\sum_{l\geq 0} C^{\Delta}(n,m,l,k)\, x^n y^m z^l, \quad ord(C) \geq 1.$$

Then, the coefficients $g(n,m,l)$ of the composition of generating functions in three variables

$$G(x,y,z) = H(A(x,y,z), B(x,y,z), C(x,y,z)) = \\ = \sum_{n\geq 0}\sum_{m\geq 0}\sum_{l\geq 0} g(n,m,l)\, x^n y^m z^l$$

are equal to

$$g(n,m,l) = \sum_{k_a=0}^{n+m+l}\ \sum_{k_b=0}^{n+m+l-k_a}\ \sum_{k_c=0}^{n+m+l-k_a-k_b} h(k_a,k_b,k_c)\, d(n,m,l,k_a,k_b,k_c),$$

where

$$d(n,m,l,k_a,k_b,k_c) = [x^n y^m z^l](A(x,y,z)^{k_a} B(x,y,z)^{k_b} C(x,y,z)^{k_c}).$$

Next we consider some special cases of applying Theorem 1.

Corollary 2. *Suppose that:*

$$H(x) = \sum_{n\geq 0} h(n)\, x^n,$$

$$A(x,y,z)^k = \sum_{n\geq 0}\sum_{m\geq 0}\sum_{l\geq 0} A^{\Delta}(n,m,l,k)\, x^n y^m z^l, \quad ord(A) \geq 1.$$

Then, the coefficients $g(n,m,l)$ of the composition of generating functions in three variables

$$G(x,y,z) = H(A(x,y,z)) = \sum_{n\geq 0}\sum_{m\geq 0}\sum_{l\geq 0} g(n,m,l)\, x^n y^m z^l$$

are equal to

$$g(n,m,l) = \sum_{k=0}^{n+m+l} h(k)\, [x^n y^m z^l](A(x,y,z)^k) = \sum_{k=0}^{n+m+l} h(k)\, A^{\Delta}(n,m,l,k).$$

Corollary 3. *Suppose that:*

$$H(x,y) = \sum_{n\geq 0}\sum_{m\geq 0} h(n,m)\, x^n y^m,$$

$$A(x,y,z)^k = \sum_{n\geq 0}\sum_{m\geq 0}\sum_{l\geq 0} A^{\Delta}(n,m,l,k)\, x^n y^m z^l, \quad ord(A) \geq 1.$$

Then, the coefficients $g(n,m,l)$ of the composition of generating functions in three variables

$$G(x,y,z) = H(A(x,y,z),y) = \sum_{n\geq 0}\sum_{m\geq 0}\sum_{l\geq 0} g(n,m,l)\, x^n y^m z^l$$

are equal to

$$g(n,m,l) = \sum_{k_a=0}^{n+m+l} \sum_{k_b=0}^{n+m+l-k_a} h(k_a,k_b)\,[x^n y^m z^l](A(x,y,z)^{k_a} y^{k_b}) =$$

$$= \sum_{k_a=0}^{n+m+l} \sum_{k_b=0}^{n+m+l-k_a} h(k_a,k_b)\, A^{\Delta}(n, m-k_b, l, k).$$

In a similar way, we can consider any other variants of compositions of generating functions, where the result is a generating function in three variables.

Conclusion

In this paper, we consider the method of compositae that allows obtaining explicit formulas for the coefficients of compositions of generating functions. The effectiveness of applying this method is shown in obtaining explicit formulas for many ordinary generating functions. In this paper, we propose to generalize the concept of compositae to the case of generating functions in three variables.

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References

[1] A. Asinowski and C. Banderier, On lattice paths with marked patterns: Generating functions and multivariate Gaussian distribution, LIPIcs. Leibniz Int. Proc. Inform. 159 (2020), Article ID 161077.

[2] J.-L. Baril, A. Burstein and S. Kirgizov, Pattern statistics in faro words and permutations, Discrete Math. 344(8) (2021), Article ID 112464.

[3] S. Elizalde, Symmetric peaks and symmetric valleys in Dyck paths, Discrete Math. 344(6) (2021), Article ID 112364.

[4] D. Kruchinin and V. Kruchinin, Y. Shablya, Method for obtaining coefficients of powers of bivariate generating functions, Mathematics 9(4) (2021), Article ID 428.

[5] H. M. Srivastava and H. L. Manocha, A Treatise on Generating Functions (Mathematics and Its Applications), UK: Ellis Horwood; 1984.

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Some identities derived from the application of the Volkenborn integral to the Abel's binomial theorem

Irem Kucukoglu [1]

In this talk, by applying the Volkenborn (i.e. bosonic p-adic) integral to the Abel's generalization of the binomial theorem, we derive some identities containing the Bernoulli polynomials, the falling factorial and the Abel polynomials. Finally, we end the paper with a conclusion on our results and planned future studies.

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Keywords: Abel's binomial theorem, Abel polynomials, Bernoulli polynomials, Falling factorial, p-adic integral

Introduction

Up to the present, studies on generalizations of the binomial theorem have been conducted by miscellaneous researchers such as Abel [1], Comtet [2, p. 128], Hurwitz [4], Riordan [10, p. 18, Eq.-(13)], Roman [11], Saslaw [12, p. 589] and the others we haven't mentioned here. Among these researchers, Abel [1] introduced a generalization of the binomial theorem (so-called the Abel's binomial theorem) by the following formula:

$$(x+y)^n=\sum_{k=0}^{n}\binom{n}{k}x\left(x-kz\right)^{k-1}\left(y+kz\right)^{n-k} \tag{1}$$

in which x, y and z are arbitrary quantities and n is a nonnegative integer (*cf.* [1], [2, p. 128]).

Observe that the substitution of $z=0$ into (1) yields the well-known binomial theorem.

The Abel polynomials, $a_k(x,z)$, are defined by the following explicit formula:

$$a_k(x,z)=\frac{x\left(x-kz\right)^{k-1}}{k!} \tag{2}$$

where $k\in\mathbb{N}_0$ (*cf.* [10], [11]).

First few values of the polynomials $a_k(x,z)$ are given as follows:

$$a_0(x,z)=1,\qquad a_1(x,z)=x,$$
$$a_2(x,z)=\frac{x\left(x-2z\right)}{2!},\qquad a_3(x,z)=\frac{x\left(x-3z\right)^2}{3!},$$

and so on (*cf.* [10], [11]).

The Abel poynomials satisfy the following higher-order partial differential equation:

$$\frac{\partial^j}{\partial x^j}\left\{a_k(x,z)\right\}=a_k(x-jz,z) \tag{3}$$

(*cf.* [2]).

It is known from the book of Roman [11] that the Abel polynomials is of a connection with geometric probability, such as the random placement of nonoverlapping arcs on a circle. In recent years, the Abel poynomials have been studied in various ways. For instance, Kim et al. [6] used the technique of umbral calculus to derive some identities containing the Bernoulli, the Euler and the Abel polynomials.

In this paper, by applying the p-adic integral to the Abel's binomial theorem, we derive new and interesting results involving not only the Abel polynomials, but also the Bernoulli polynomials.

To prove the results of this paper, we need the following notations and definitions associated with the p-adic integration.

Let $\mathbb{Z}_p$ be the set of p-adic integers and h be a uniformly differentiable function on $\mathbb{Z}_p$. Then, the Volkenborn (i.e. bosonic p-adic) integral of the function h is defined by

$$\int\limits_{\mathbb{Z}_p} h\left(x\right) d\mu_1\left(x\right) = \lim_{N\to\infty} \frac{1}{p^N} \sum_{s=0}^{p^N-1} h\left(s\right) \tag{4}$$

where $\mu_1\left(x\right)$ denotes the Haar distribution defined as

$$\mu_1\left(x\right) = \mu_1\left(x + p^N\mathbb{Z}_p\right) = \frac{1}{p^N} \tag{5}$$

(*cf.* [13]; see also [7], [8], [9]).

The Volkenborn (i.e. bosonic p-adic) integral representation of the Bernoulli polynomials $B_n(x)$ is given by

$$B_n\left(x\right) = \int\limits_{\mathbb{Z}_p} \left(x+y\right)^n d\mu_1\left(y\right) \tag{6}$$

(*cf.* [7], [8], [13]), such that the Bernoulli polynomials $B_n(x)$ are defined by means of the following exponential generating function:

$$\frac{te^{tx}}{e^t-1} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!} \tag{7}$$

where $|t| < 2\pi$ (*cf.* [2]-[15]).

For more identities and relations derived from p-adic integration techniques, refer to the works of Kim [7], [8], [9], Schikhof [13], and Simsek [14].

Main Results

In this section, main results of this study are presented.

By applying the Volkenborn (i.e. bosonic p-adic) integral to (1), we get

$$\int\limits_{\mathbb{Z}_p} \left(x+y\right)^n d\mu_1\left(y\right) = \sum_{k=0}^{n} \binom{n}{k} x\left(x-kz\right)^{k-1} \int\limits_{\mathbb{Z}_p} \left(y+kz\right)^{n-k} d\mu_1\left(y\right). \tag{8}$$

Combining the above equation with (2) and (6) yields a relation, between the Bernoulli polynomials and the Abel polynomials, given by the following theorem:

Theorem 1. *Let n be a nonnegative integer. Then we have*

$$B_n(x) = \sum_{k=0}^{n} (n)_k \, a_k(x,z) \, B_{n-k}(kz), \tag{9}$$

in which $(n)_k$ corresponds to the falling factorial defined by

$$(n)_k = n(n-1)\dots(n-k+1)$$

with $(n)_0 = 1$.

Substituting (2) in (9) yields the following corollary:

Corollary 2. *Let n be a nonnegative integer. Then we have*

$$B_n(x) = \sum_{k=0}^{n} \binom{n}{k} x(x-kz)^{k-1} B_{n-k}(kz). \tag{10}$$

Taking m-times derivative of both sides of (9), with respect to x, we get the following partial differential equation:

$$\frac{d^m}{dx^m}\{B_n(x)\} = \sum_{k=0}^{n} (n)_k \, B_{n-k}(kz) \frac{d^m}{dx^m}\{a_k(x,z)\}.$$

Using the fact that Bernoulli polynomials constitute an Appell sequence, and by (3), we arrive at the following theorem:

Theorem 3. *Let n be a nonnegative integer and m be a positive integer. Then we have*

$$B_n(x) = \frac{1}{(n+m)_m} \sum_{k=0}^{n+m} (n+m)_k \, B_{n+m-k}(kz) \, a_k(x-mz,z). \tag{11}$$

Substituting (2) in (11) yields the following corollary:

Corollary 4. *Let n be a nonnegative integer and m be a positive integer. Then we have*

$$B_n(x) = \frac{x-mz}{(n+m)_m} \sum_{k=0}^{n+m} \binom{n+m}{k} B_{n+m-k}(kz) \, (x-z(m+k))^{k-1}. \tag{12}$$

Conclusion

In this study, by applying the Volkenborn (i.e. bosonic p-adic) integral to the Abel's generalization of the binomial theorem, some identities, involving the Bernoulli polynomials, the falling factorial and the Abel polynomials, have been obtained. In future studies, it is planned to investigate the relations of these identities with other special sequences and polynomials and to obtain new results that may affect scientists working in the relevant fields.

References

[1] N. H. Abel, Beweis eines Ausdruckes, von welchem die Binomial-Formel ein einzelner Fall ist, J. reine angew. Math (Crelles Journal) 1 (1826), 159–160.

[2] L. Comtet, Advanced Combinatorics: The art of finite and infinite expansions (revised and enlarge edition), Dordrecht, Holland: D. Reidel Publishing Company; 1974.

[3] R. Dere, Y. Simsek and H. M. Srivastava, A unified presentation of three families of generalized Apostol type polynomials based upon the theory of the umbral calculus and the umbral algebra, J. Number Theory 133(10) (2013), 3245–3263.

[4] A. Hurwitz, ber Abel's Verallgemeinerung der binomischen Formel, Acta Math. 26 (1902), 199–203.

[5] L. C. Jang and T. Kim, A new approach to q-Euler numbers and polynomials, J. Concr. Appl. Math. 6 (2008), 159–168.

[6] D. S. Kim, T. Kim, S.-H. Lee and S.-H. Rim, Some identities of Bernoulli, Euler and Abel polynomials arising from umbral calculus, Adv. Differ. Equ. (2013), 2013:15, 1–8.

[7] T. Kim, q-Volkenborn integration, Russ. J. Math. Phys. 19 (2002), 288–299.

[8] T. Kim, q-Euler numbers and polynomials associated with p-adic q-integral and basic q-zeta function, Trends Int. Math. Sci. Study 9 (2006), 7–12.

[9] T. Kim, An invariant p-adic q-integral on $\mathbb{Z}_p$, Appl. Math. Lett. 21 (2008), 105–108.

[10] J. Riordan, Combinatorial identities, New York: Wiley; 1968.

[11] S. Roman, The umbral calculus, New York: Academic Press; 1984.

[12] W. C. Saslaw, Some properties of a statistical distribution function for galaxy clustering, Astrophys. J. 341 (1989), 588-598.

[13] W. H. Schikhof, Ultrametric calculus: An introduction to p-adic analysis, Cambridge Stud. Adv. Math. 4, Cambridge: Cambridge University Press; 1984.

[14] Y. Simsek, Explicit formulas for p-adic integrals: Approach to p-adic distributions and some families of special numbers and polynomials, Montes Taurus J. Pure Appl. Math. 1(1) (2019), 1–76.

[15] H. M. Srivastava, and J. Choi, Zeta and q-zeta functions and associated series and integrals, Amsterdam, London and New York: Elsevier Science Publishers; 2012.

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An analog of Lagrange's theorem for enveloping algebras

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The aim of this paper is to use Schreier's formula for free Lie algebras to obtain an analouge of Lagrange's theorem for the universal enveloping algebras of free Lie algebras in term's of characters.

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Introduction

Let F be a ground field. A Lie algebra over the field F is a vector space L, together with a F-bilinear mapping $L \times L \to L : (x, y) \mapsto [x, y]$ called a Lie product (or Lie bracket), and satisfying the following two identities, for any $x, y, z \in L$:

(i) $[x, x] = 0$ (this identity implies antisymmetry, that is $[x, y] = -[y, x]$).

(ii) $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$ (Jacobi identity).

Subalgebras of Lie algebras and homomorphisms (or isomorphisms) between Lie algebras are defined as usual.
Recall that any associative algebra A over F, with a bracket $[x, y] = xy - yx$ for $x, y \in X$, is a Lie algebra; it will be denoted by A^-. The universal enveloping algebra of a Lie algebra L is a pair (U, ϵ), where U is an associative unital algebra and $\epsilon : L \to U^-$ is a homomorphism of Lie algebras, and for every associative unital algebra B and for every homomorphism $f : L \to B^-$ there exists a unique homomorphism of unital algebras $F : U \to B$ such that $F \circ \epsilon = f$. Also, a well known theorem, Poincare-Birkhoff-Witt (PBW) theorem, states that if $\{e_i : i \in I\}$ is any basis of a Lie algebra L with ordered index set I, then the set of monomials $\left\{e_{i_1}^{k_1} \dots e_{i_n}^{k_n} : i_1 < \dots < i_n,\ k_j \in \mathbb{N}\ \forall j = 1, \dots, n\right\}$ forms a basis of the enveloping algebra $U(L)$ ([8]).
Given a nonempty set X, a free Lie algebra on X over F is a Lie algebra L over F, together with a mapping $i : X \to L$, with the following universal property: for any Lie algebra K and any mapping $f : X \to K$, there exists a unique Lie algebra homomorphism $F : L \to K$ such that $f = F \circ i$ ([7]).
Lagrange's theorem states that if G is a group, and K is any subgroup of G, then the group algebra FG is free as an (left) FK-module ([6]). Indeed this follows from the fact if G is a finite group, then $|K|$ divides $|G|$.
Suppose G is an abelian semigroup written multiplicatively. A *G-graded algebra* is an algebra R together with a direct sum decomposition of the form $R = \bigoplus_{g \in G} R_g$, where each R_g is a subspace and $R_g R_h \subseteq R_{gh}$ for all $g, h \in G$. A nonzero element $r \in R$ is called *homogeneous* if there is $g \in G$ such that $r \in R_g$ (we write $d(r) = g$). A subspace $H \subseteq R$ is said to be homogeneous if $H = \bigoplus_{g \in G} H_g$ where $H_g = H \cap R_g$.

By a *G-graded vector space* we mean a vector space V together with a direct sum decomposition $V = \bigoplus_{g \in G} V_g$. Let V and W be G-graded vector spaces. A linear map $f : V \to W$ is called *homogeneous of degree* $h \in G$ if for all $g \in G$, we have $f(V_g) \subseteq W_{hg}$. In particular, a homogeneous linear map of degree $1_G \in G$ is called a *degree preserving map*.
A well-known theorem, due to Nielsen and Schreier, states that every subgroup of a free group is again free. A corresponding result for Lie algebras was later obtained independently by Shirshov [9] and Witt [10]. The Schreier index formula states that if G is a free group of rank n, and K is a subgroup of finite index in G, then the rank of K is given by $\text{rank}(K) = (n-1)[G : K] + 1$. There are no straightforward analogues of Schreier index formula in the case of free Lie algebra, even with finitely many generators. To obtain the desired formulas, one can replace numbers with characters, as follows. Let Λ be a countable additive abelian semigroup satisfying the following conditions:

1. finiteness condition: each element $\lambda \in \Lambda$ can be written as a sum of other elements only in finitely many ways,
2. Λ is well ordered by $\leq$ such that if $\lambda < \mu$, then $\lambda + \gamma < \mu + \gamma$ for all $\lambda, \mu, \gamma \in \Lambda$, and also $\lambda + \mu > \lambda$ for all $\lambda, \mu \in \Lambda$ (such semigroups are called positive).

Let $U = \bigoplus_{\lambda \in \Lambda} U_\lambda$ be a Λ-graded space. The character of U is defined by $\text{ch}_\Lambda U = \sum_{\lambda \in \Lambda} (\text{dim} U_\lambda) e^\lambda$. It is an element in $\mathbb{Q}[[\Lambda]]$ whose basis consists of symbols e^λ, $\lambda \in \Lambda$ with the multiplication $e^\lambda e^\mu = e^{\lambda+\mu}$ for all $\lambda, \mu \in \Lambda$. By a Λ-graded set we mean a disjoint union $X = \bigcup_{\lambda \in \Lambda} X_\lambda$. If in addition, we have $|X_\lambda| < \infty$ for all $\lambda \in \Lambda$, then we define its character $\text{ch}_\Lambda X = \sum_{\lambda \in \Lambda} |X_\lambda| e^\lambda \in \mathbb{Q}[[\Lambda]]$. For an element $x \in X_\lambda \subseteq X$, we say Λ-weight of x is λ, and we write $\text{wt}_\Lambda x = \lambda$. We call such a set Λ-*finitely graded* (if $\Lambda = \mathbb{N}$, then we say X is a finitely graded set). For any monomial $y = x_1 \ldots x_n$, where $x_j \in X$, we set $\text{wt}_\Lambda y = \text{wt}_\Lambda x_1 + \cdots + \text{wt}_\Lambda x_n$. Suppose Y is a set of all monomials (associative, Lie, ...) in X. We denote $Y_\lambda = \{y \in Y \mid \text{wt}_\Lambda y = \lambda\}$. Also, the Λ-generating function of Y is $\text{ch}_\Lambda(Y) = \sum_{\lambda \in \Lambda} |Y_\lambda| e^\lambda \in \mathbb{Q}[[\Lambda]]$.

Lemma 1 ([4]). *Let $X = \bigcup_{\lambda \in \Lambda} X_\lambda$ be a Λ-graded set with $|X_\lambda| < \infty, \lambda \in \Lambda$, and let $F\langle X \rangle$ be the free associative algebra generated by X. Then*

$$\text{ch}_{\bar{\Lambda}} F\langle X \rangle = \sum_{n=0}^{\infty} (\text{ch}_\Lambda X)^n = \frac{1}{1 - \text{ch}_\Lambda X}.$$

Gradings $U = \bigoplus_{\lambda \in \Lambda} U_\lambda$ and $V = \bigoplus_{\lambda \in \Lambda} V_\lambda$ induce gradings on the spaces $U \oplus V$ and $U \otimes V$: $(U \oplus V)_\lambda = U_\lambda \oplus V_\lambda$; $(U \otimes V)_\lambda = \sum_{\lambda = \mu + \nu} (U_\mu \otimes V_\nu)$ (by finiteness condition the sum is finite). The following theorem holds.

Theorem 2. $\text{ch}_\Lambda(U \oplus V) = \text{ch}_\Lambda U + \text{ch}_\Lambda V$, *and* $\text{ch}_\Lambda(U \otimes V) = \text{ch}_\Lambda U \text{ch}_\Lambda V$.

We consider Λ-graded Lie algebra $L = \bigoplus_{\lambda \in \Lambda} L_\lambda$ with $\text{dim} L_\lambda < \infty$ $\forall \lambda \in \Lambda$. Let $K \subseteq L$ be a not necessarily homogeneous subalgebra of L. For $k \in K$, we consider its decomposition into homogeneous components $k = k_{\lambda_1} + \cdots + k_{\lambda_n}$, $\lambda_i \in \Lambda$, $\lambda_1 \leq \cdots \leq \lambda_n$, $k_{\lambda_n} \neq 0$. In this case we write $\deg(k) = \lambda_n$. Also, K has a filtration (as an algebra) $\bigcup_{\lambda \in \Lambda} K^\lambda$ where $K^\lambda = K \cap (\bigoplus_{\theta \leq \lambda} L_\theta), \lambda \in \Lambda$. Also, the factor-space L/K acquires a factor-filtration given by $(L/K)^\lambda = (L^\lambda + K)/K \cong L^\lambda/(K \cap L^\lambda)$. Denote $K^{\lambda-} = K \cap (\bigoplus_{\theta < \lambda} L_\theta), \lambda \in \Lambda$, then, one can construct a graded algebra $\text{gr} K$ as follows: set $\text{gr} K = \bigoplus_{\lambda \in \Lambda \cup \{0\}} K^\lambda / K^{\lambda-}$ as a vector space (set $K^0 = \{0\}$), and define multiplication as:

$$K^\lambda / K^{\lambda-} \times K^\theta / K^{\theta-} \to K^{\lambda+\theta} / K^{(\lambda+\theta)-} : \ (a + K^{\lambda-})(b + K^{\theta-}) \mapsto ab + K^{(\lambda+\theta)-}.$$

Then, in the nonhomogeneous case we define characters as: $\mathrm{ch}_\Lambda(K) = \mathrm{ch}_\Lambda(\mathrm{gr}K)$ and $\mathrm{ch}_\Lambda(L/K) = \mathrm{ch}_\Lambda(L/\mathrm{gr}K)$.

Theorem 3 ([4])**.** *Let L be a free Lie algebra generated by a Λ-graded set $X = \bigcup_{\lambda\in\Lambda} X_\lambda$ where $|X_\lambda| < \infty$ for all $\lambda \in \Lambda$. If K is a subalgebra of L, then there exists a set of free generators Y of K such that*

$$\mathrm{ch}_\Lambda Y = (\mathrm{ch}_\Lambda X - 1)\mathcal{E}(\mathrm{ch}_\Lambda L/K) + 1,$$

where

$$\mathcal{E} : \sum_{\lambda\in\Lambda} h_\lambda e^\lambda \mapsto \frac{1}{\prod_{\lambda\in\Lambda}(1-e^\lambda)^{h_\lambda}},$$

Moreover, if K is homogeneous, then any homogeneous set of free generators satisfies this equality.

An important special case is $\Lambda = \mathbb{N}$ where $\mathbb{Q}[[\Lambda]]$ is the algebra of formal power series in one variable (without constant term), and we denote here t by e^t ([3]).

Main Results

We start with the following lemma that will be used in the proof of the main theorem.

Lemma 4 ([2, p.47])**.** *The universal enveloping algebra of the free Lie algebra $L(X)$ is the free associative algebra of X.*

Proposition 5. *Suppose that $A = \bigoplus_{\lambda\in\Lambda} A_\lambda$ is a Λ-graded algebra with no homogeneous zero divisors, and B is a subalgebra of A. If A is a free (left) B module with a Λ-homogeneous basis C, then* $\mathrm{dim}A_\lambda = \sum_{\lambda=\mu+\nu} |C_\mu||B_\nu|$.

Proof. For each $\lambda \in \Lambda$, $A_\lambda = \mathrm{span}\,\{c.b \mid c \in C, b \in B, \text{and } \deg(c) + \deg(b) = \lambda\}$. Therefore, $\mathrm{dim}A_\lambda = \sum_{\lambda=\mu+\nu} |C_\mu||B_\nu|$. □

The argument used in the proof of the proposition above is not true in general. For example, if $A = F < x >$ with the standard grading, and $B = F < x^2 >$, then $\{x-1, x+1\}$ is a basis of A as a B-module. But A_0 is not spanned by the set in the formula given in the proof.

Corollary 6. *Suppose that $A = \bigoplus_{\lambda\in\Lambda} A_\lambda$ is a Λ-graded algebra with no homogeneous zero divisors, and B is a subalgebra of A. If A is a free (left) B module with a Λ-homogeneous basis C, then* $\mathrm{ch}_\Lambda A = \mathrm{ch}_\Lambda C \mathrm{ch}_\Lambda B$.

Let L be a Lie algebra, and let K be any subalgebra of L. Then, using the proof of PBW-theorem ([1]), one can see that the universal enveloping algebra $U(L)$ is free as an (left) $U(K)$-module with a homogeneous basis.

Theorem 7. *Let L be a free Lie algebra generated by a Λ-graded set $X = \bigcup_{\lambda\in\Lambda} X_\lambda$ where $|X_\lambda| < \infty$ for all $\lambda \in \Lambda$, K is a subalgebra of L, and C is a basis of the universal enveloping algebra of L, $U(L)$, as a free (left) $U(K)$-module. Then,* $\mathrm{ch}_\Lambda C = \mathcal{E}(\mathrm{ch}_\Lambda L/K)$.

Proof. By Theorem 3, let Y be a basis of K such that

$$\mathrm{ch}_\Lambda Y = (\mathrm{ch}_\Lambda X - 1)\mathcal{E}(\mathrm{ch}_\Lambda L/K) + 1.$$

Also, by Lemma 4 , the universal enveloping algebras of L and K are the free associative algebras on X and Y , respectively. Denote the free associative algebras of X and Y by $F\langle X\rangle$, and $F\langle Y\rangle$, respectively. By Lemma 1,

$$\mathrm{ch}_{\bar{\Lambda}} F\langle X\rangle = \frac{1}{1-\mathrm{ch}_{\Lambda} X} \text{ and } \mathrm{ch}_{\bar{\Lambda}} F\langle Y\rangle = \frac{1}{1-\mathrm{ch}_{\Lambda} Y}.$$

Therefore

$$\frac{1}{1-\mathrm{ch}_{\Lambda} X} = \frac{\mathrm{ch}_{\Lambda} C}{1-((\mathrm{ch}_{\Lambda} X-1)\mathcal{E}\left(\mathrm{ch}_{\Lambda} L/K\right)+1)}.$$

From the previous equation, we get

$$\mathrm{ch}_{\Lambda} C = \mathcal{E}\left(\mathrm{ch}_{\Lambda} L/K\right).$$

□

Conclusion

If H is a subgroup of a finite group G, then the number of (left) cosets of H in G (index) is $|G|/|H|$. We obtain similar formula for enveloping algebras of Lie algebras by replacing the index with characters.

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References

[1] Bahturin, Y., Identical relations in Lie algebras, Utrecht: VNU Science Press; 1987.

[2] Bahturin, Y., Mikhlev, A., Petrogradsky, V. and Zaicev, V., Infinite dimensional Lie superalgebras, Berlin: de Gruyter Exp. Math. 7; 1992.

[3] Petrogradsky, V., Schreier's formula for free Lie algebras, Arch Math (Basel), 75 (2000), 16–28.

[4] Petrogradsky, V., On generating functions for subalgebras of free Lie superalgebras, Discrete Math. 246 (2002), 269–284

[5] Radford, D. E., On an analog of Lagrange's theorem for commutative Hopf algebras, Proc. Amer. Math. Soc. 79 (1980), 164–166.

[6] Radford, D. E., Pointed Hopf algebras are free over Hopf subalgebras, J. Algebra 45 (1977), 266–273.

[7] Rentenauer, C., Free Lie algebras, Oxford University Press; 1993.

[8] Shaqaqha, S., Hilbert series for free Lie superalgebras and related topics, Ph. D thesis: Memorial University; 2015.

[9] Shirshov, A. I., Subalgebras of free Lie algbras, Mat. Sb, 33 (1953), 441–452.

[10] Witt, E., Die Unterringe der freien Lieschen Ringe, Math. Z. 64 (1956), 195–216.

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On the generalized Tribonacci sequences

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In this study, a general recurrence relation for some well-known numbers and polynomials is defined. Sequences of some numbers and polynomials are examined by means of this new definition.

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Keywords: Integral domain, Tribonacci Numbers and Polynomials

Introduction

Horadam defined Horadam numbers (Horadam polynomials) as follows:

$$h_n = ph_{n-1} - qh_{n-2} \ (n \geq 3)$$

with the initial conditions $w_0 = a$, $w_1 = b$ ($h_1 = a$, and $h_2 = bx$) ([7]).

The Tribonacci sequence in [10] is defined as $T_n = T_{n-1} + T_{n-2} + T_{n-3}$ for $n \geq 4$ with the initial conditions $T_1 = T_2 = 1$ and $T_3 = 2$. Then, B. Rybolowicz and A. Tereszkiewicz generalized this concept to Tribonacci polynomial, Tricobstal numbers and polynomials, separately and they gave the generalized Binet formula for these sequences ([12]).

An integral domain plays an important role in ring theory. In an integral domain, every nonzero element has the cancellation property and so an integral domain provides a natural setting to study divisibility. Polynomial rings over integral domain is integral domain. Thus this study deals with the sequences in an integral domain.

In this study, we give a general recurrence relation for these numbers and polynomials. Therefore, we observe that all definitions of sequences of well-known numbers and polynomials such as Lucas numbers and polynomials, Pell numbers and polynomials, Pell-Lucas numbers and polynomials, Jacobsthal numbers and polynomials, Jacobsthal-Lucas numbers and polynomials, Chebyshev numbers and polynomials coincide each other at a suitable initial condition in a domain.

Tribonacci squences

Definition 1. *Let p, q, r, a and b be elements of an integral domain R. Then, for a positive integer $n \geqslant 2$, the recurrence relation is defined as*

$$w_{n+1} = pw_n + qw_{n-1} + rw_{n-2},$$

where $w_0 = a, w_1 = b, w_2 = pw_1 + qw_0$. It is said that

$$w = \{w_{n+1} = pw_n + qw_{n-1} + rw_{n-2} : n \in \mathbb{Z}^+, n \geqslant 2\}$$

is the sequence of p, q, r with a, b on an integral domain R denoted by $w_n(a, b, p, q, r)$ or briefly $\{w_n\}$.

In this study, we fix notations in Definition 1. This definition also generalizes to all the Horadam numbers (Horadam polynomials) in [7], Tribonacci and Tricobstal numbers and polynomials.

If any two of p, q, r are zero, then the sequence $\{w_{n+1} = p^n w_1 : w_1 = b,\ n \in \mathbb{Z}^+\}$ is a geometric sequence. For example, if $q = r = 0$, then $w = \{w_{n+1} = p^n w_1 : w_1 = b,$ $n \in \mathbb{Z}^+\}$ is a geometric sequence.

If $p \neq 0$, $q \neq 0$ and $r = 0$, then the Definition 1 coincides with the definition in [1, Definition 1.1, p. 61].

In this study, we assume that r is non zero. Then for an integer n, we have the relation as follows:

$$rw_n = (w_{n+3} - pw_{n+2} - qw_{n+1}).$$

If r^{-1} is in R, then w_{-1} is in R and so the sequence is extended to $w = \{w_n : n \in \mathbb{Z}\}$ on R. Thus we assume that r is invertible in the domain R.

By Definition 1, it is computed that

$$\begin{aligned}
w_{-2} &= \frac{b - pa}{r},\ w_{-1} = 0,\ w_2 = pb + qa, \\
w_3 &= (p^2 + q)b + (pq + r)a, \\
w_4 &= (p^3 + 2pq + r)b + (p^2 q + pr + q^2)a, \\
w_5 &= (p^4 + q^2 + 3p^2 q + 2pr)b + (p^3 q + 2pq^2 + p^2 r + 2qr)a
\end{aligned}$$

For the initial values $a = 0$ and $b = 1$, we use the notation k_n instead of w_n. Thus we have the relation as follows:

$$\begin{aligned}
k_{-2} &= \frac{1}{r},\ k_{-1} = 0,\ k_0 = 0, \\
k_1 &= 1,\ k_2 = p, \\
k_3 &= p^2 + q, \\
k_4 &= p^3 + 2pq + r, \\
k_5 &= p^4 + q^2 + 3p^2 q + 2pr
\end{aligned}$$

Substituting the parameters, we attain some well-known sequences in the literature such as Pell, Jacobsthal, Fermat, Chebyshev, Morgan-Voyce, Delannoy, Gegenbauer, Humbert, Kinney, Legendre, Liouville, Pincherle and so on.

Remark 1. *Let R be the ring of integers ($R = \mathbb{Z}$) and $r = 0$. Then we have*

p	q	a	b	w_n *is sequence of numbers*
1	1	0	1	*the Fibonacci number*
1	1	2	1	*the Lucas number*
2	1	0	1	*the Pell number*
1	1	2	2	*Pell-Lucas number*
1	2	2	1	*the Jacobsthal-Lucas number*

Remark 2. *Let R be the polynomial ring of integers ($R = \mathbb{Z}[x]$) and $r = 0$. Then we have*

p	q	a	b	w_n *is sequence of polynomials*
x	1	0	1	*the Fibonacci polynomial*
x	1	2	1	*the Lucas polynomial*
$3x$	-2	2	1	*the Jacobstal polynomial*
$2x$	-1	0	1	*the Chebyshev 2nd kind polynomial*
$x+2$	-1	0	1	*the Morgan-Voyce polynomial*
$x+1$	-2	0	1	*the Delannoy polynomial*

Remark 3. *Let R be the ring of integers ($R=\mathbb{Z}$). Then we have*

p	q	r	a	b	w_n *is sequence of numbers*
1	1	1	0	1	*the Tribonacci number*
1	1	1	1	1	*the Tricobsthal number*

Remark 4. *Let R be the polynomial ring of integers ($R=\mathbb{Z}[x]$). Then we have*

p	q	r	a	b	w_n *is sequence of polynomials*
x^2	x	1	0	1	*the Tribonacci polynomial*
1	x	x^2	1	1	*the Tricobsthal polynomial*

Let us define the following matrix

$$W(n)=\begin{bmatrix} w_{n+1} & w_n & w_{n-1} \\ w_n & w_{n-1} & w_{n-2} \\ w_{n-1} & w_{n-2} & w_{n-3} \end{bmatrix}.$$

Hence, we have that $W(0)=\begin{bmatrix} w_1 & w_0 & w_{-1} \\ w_0 & w_{-1} & w_{-2} \\ w_{-1} & w_{-2} & w_{-3} \end{bmatrix}$.

Lemma 2. *For any integer n, we have the following relation*

$$KW(n-1)=W(n),$$

where $K=\begin{bmatrix} p & q & r \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$.

Proof. By Definition 1, we get that $w_{n+1}=pw_n+qw_{n-1}+rw_{n-2}$ and $w_n=pw_{n-1}+qw_{n-2}+rw_{n-3}$ for $n\geqslant 3$. Then it follows that $KW(n-1)=W(n)$. $\square$

References

[1] M. Alkan, The generalized Fibonacci sequences on an integral domain, Montes Taurus J. Pure Appl. Math. 3(2) (2021), 60–69.

[2] G. E. Bergum and V. E. Hoggatt, Sums and products for recurring sequences, Fibonacci Quart. 13(2) (1975), 115–120.

[3] G. S. Cheon, H. Kim and L. W. Shapiro, A generalization of Lucas polynomial sequence, Discrete Appl. Math. 157 (5) (2009), 920–927.

[4] K. A. Dilcher, Generalization of Fibonacci polynomials and a representation of Gegenbauer polynomials of integer order, Fibonacci Quart. 25 (4) (1987), 300–303.

[5] S. Falcón, On the k-Lucas numbers, Int. J. Contemp Math. Sci. 6 (21-24)(2011), 1039–1050.

[6] S. Falcón and Á. Plaza, On the Fibonacci k-numbers, Chaos Solitons Fractals 32 (5) (2007), 1615–1624.

[7] A. F. Horadam, Basic properties of a certain generalized sequence of numbers, Fibonacci Quart. 3 (2) (1965), 161–176.

[8] L. C. Hsu, On Stirling-type pairs and extended Gegenbauer-Humbert-Fibonacci polynomials, In: G. E. Bergum, A. N. Philippou, A. F. Horadam, editors. Applications of Fibonacci numbers, Vol. 5., Kluwer Academic Publishers, 367–377, Dordrecht, 1993.

[9] M. Janjic, Hessenberg matrices and integer sequences, J. Integer Seq. 13 (7) ,10 pp, Article 10.7.8, 2010.

[10] T. Koshy, Fibonacci and Lucas Numbers with applications, New York: Wiley-Interscience Publications; 2001.

[11] J. L. Ramírez, Some properties of convolved k-Fibonacci numbers, ISRN Combin. Art. ID 759641 (2013), 1–5.

[12] B. Rybolowicz and A. Tereszkiewicz, Generalized tricobsthal and tribonacci and polynomials, Appl. Math and Compt. 325 (2018), 297–308.

[13] G. Ozdemir and Y. Simsek, Generating functions for two-variable polynomials related to a family of Fibonacci type polynomials and numbers, Filomat 30 (4) (2016), 969–975.

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Some new summation formulas obtained with the Christoffeel-Darboux identity for orthogonal polynomials and the using of fractional operator ${}_zO_\beta^\alpha$

Richard Tremblay [1]

In the field of special functions, the theory related to orthogonal polynomials and multiple orthogonal polynomials is fundamental. There are several formulas and many useful applications in mathematical physics, numerical analysis, statistics and probability and many other disciplines. For example, Tygert obtains two new similar identities for the Bessel functions of the well-known formula of Christoffel-Darboux. In this paper, we obtain several summation formulas involving orthogonal polynomials using the well-poised fractional operator defined in terms of the fractional derivative,

$$ {}_zO_\beta^\alpha f(z) = \frac{\Gamma(\beta)}{\Gamma(\alpha)} z^{1-\beta} D_z^{\alpha-\beta} z^{\alpha-1} f(z), $$

which has been used previously in several papers involving special functions.

2010 Mathematics Subject Classifications: 26A33, 42C05, 33C45, 33C20, 33C10

Keywords: Fractional derivatives, Fractional Operator O , Special Functions, Orthogonal Polynomials, Christoffel-Darboux Inentity

References

[1] M. Tygert, Analogues for Bessel Functions of Christoffel-Darboux Identity, Research Report YALEU / DCS / RR-1351 (2006), 8 pages.

[2] R. Tremblay, Une contribution à la théorie de la dérivée fractionnaire, Doctoral Thesis, Laval University, Québec, 1974.

[3] R. Tremblay, S. Gaboury, Well-posed fractional calculus: obtaining new transformations formulas involving Gauss hypergeometric functions with rational quadratic, cubic and higher degree arguments, Math. Meth. Appl. Sc. 13 (2018), 4967–4985.

[4] R. Tremblay, New quadratic transformations of hypergeometric functions and associated summation formulas Oobtained with the Well-Poised Fractional Calculus Operator, Montes Taurus J. Pure Appl. Math. 2 (1) (2020), 36–62.

[5] R. Tremblay, Using the Well-Poised Fractional Calculus Operator ${}_{g(z)}O_\beta^\alpha$ to obtain transformations of the Gauss hypergeometric function with higher level arguments,Montes Taurus J. Pure Appl. Math. 3 (3) (2021), 260–283.

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Remarks and some formulas on Fubini type numbers and polynomials

Neslihan Kilar [1]

The main motivation of this paper is to give some formulas and identities for the Fubini type numbers and polynomials. By using generating functions and their functional equations technique, we derive not only interesting relations but also some other identities associated with the Fubini type numbers and polynomials, the Frobenius-Euler polynomials, and Apostol type numbers and polynomials. Consequently, the results of this paper may be potentially used in many fields such as mathematics and mathematical physics.

2010 Mathematics Subject Classifications: 05A15, 05A19, 11B73
Keywords: Apostol-Euler numbers and polynomials, Apostol-Genocchi numbers and polynomials, Frobenius-Euler numbers and polynomials, Fubini type numbers and polynomials, Generating function

Introduction

Recently, the special numbers and polynomials, especially the Fubini type numbers and polynomials, have wide been used to solve combinatorial problems. These type numbers which may be computed via a summation formula including binomial coefficients, or a recurrence relation are count the number of weak orderings, the possible outcomes of a selection with multiple candidates, and others. Thus, in this paper, our motivation is to give some formulas and relations by many applications of the Fubini type numbers and polynomials. In order to give main results of this paper, here, we first remind some definitions and notations related to special numbers and polynomials and their generating functions.

Let $\mathbb{N}=\{1,2,3,...\}$, $\mathbb{N}_0=\mathbb{N}\cup\{0\}$, $\mathbb{Z}$ denote the set of integers, $\mathbb{C}$ denote the set of complex numbers. For $n\in\mathbb{N}$ and $v\in\mathbb{C}$, $(v)_n=\binom{v}{n}n!=v\left(v-1\right)\left(v-2\right)...\left(v-n+1\right)$ with $(v)_0=1$ (*cf.* [1]-[16]).

The Apostol-Euler polynomials of higher order are defined by

$$F_{AE}\left(t,k,x;\lambda\right)=\left(\frac{2}{\lambda e^{t}+1}\right)^{k}e^{xt}=\sum_{n=0}^{\infty}\mathcal{E}_{n}^{(k)}\left(x;\lambda\right)\frac{t^{n}}{n!}, \tag{1}$$

where $|t|<\pi$ when $\lambda=1$; $|t|<|\log\left(-\lambda\right)|$ when $\lambda\neq 1$, $k\in\mathbb{N}_0$ and $\lambda\in\mathbb{C}$, and also for $x=0$, the Apostol-Euler polynomials of higher order reduce to the Apostol-Euler numbers of higher order:

$$\mathcal{E}_{n}^{(k)}\left(0;\lambda\right)=\mathcal{E}_{n}^{(k)}\left(\lambda\right)$$

(*cf.* [9], [10], [12], [15]; see also the references cited therein).

The Apostol-Genocchi polynomials of higher order are defined by

$$F_{AG}\left(t,k,x;\lambda\right)=\left(\frac{2t}{\lambda e^{t}+1}\right)^{k}e^{xt}=\sum_{n=0}^{\infty}\mathcal{G}_{n}^{(k)}\left(x;\lambda\right)\frac{t^{n}}{n!}, \tag{2}$$

where $|t| < \pi$ when $\lambda = 1$; $|t| < |\log(-\lambda)|$ when $\lambda \neq 1$, $k \in \mathbb{N}_0$ and $\lambda \in \mathbb{C}$, and also for $x = 0$, the Apostol-Genocchi polynomials of higher order reduce to the Apostol-Genocchi numbers of higher order:

$$\mathcal{G}_n^{(k)}(0; \lambda) = \mathcal{G}_n^{(k)}(\lambda)$$

(*cf.* [10], [15], [16]; see also the references cited therein).

The Frobenius-Euler polynomials of higher order are defined by

$$F_E(t, k, x; u) = \left(\frac{1-u}{e^t - u}\right)^k e^{xt} = \sum_{n=0}^{\infty} H_n^{(k)}(x; u) \frac{t^n}{n!}, \tag{3}$$

where $u \in \mathbb{C} \setminus \{1\}$, $k \in \mathbb{N}_0$ and for $x = 0$, the Frobenius-Euler polynomials of higher order reduce to the Frobenius-Euler numbers of higher order:

$$H_n^{(k)}(0; u) = H_n^{(k)}(u)$$

(*cf.* [2], [7], [8], [11], [13], [14], [15]; see also the references cited therein).

By using (1) and (3), we have the following well-known relation:

$$H_n^{(k)}(x; 2) = 2^{-2k} \mathcal{E}_n^{(k)}\left(x; -\frac{1}{2}\right). \tag{4}$$

The Fubini type polynomials of higher order are defined by

$$F_a(t, k, x) = \frac{2^k}{(2 - e^t)^{2k}} e^{xt} = \sum_{n=0}^{\infty} a_n^{(k)}(x) \frac{t^n}{n!}, \tag{5}$$

where $|t| < \ln 2$, $k \in \mathbb{N}_0$ and for $x = 0$, the Fubini type polynomials of higher order reduce to the Fubini type numbers of higher order:

$$a_n^{(k)}(0) = a_n^{(k)}$$

(*cf.* [4]; see also [3], [5], [6]).

By using (5), we have

$$a_n^{(k)}(x) = \sum_{v=0}^{n} \binom{n}{v} a_v^{(k)} x^{n-v} \tag{6}$$

(*cf.* [4]; see also [3], [5], [6]).

Main Results

In this section, by using generating functions and their functional equations, we obtain some formulas and relations including the Fubini type numbers and polynomials, the Apostol-Euler numbers and polynomials, the Apostol-Genocchi numbers and polynomials, and the Frobenius-Euler numbers and polynomials.

Theorem 1. *Let $n \in \mathbb{N}_0$. Then we have*

$$a_n^{(k)}(w + y) = 2^{-3k} \sum_{j=0}^{n} \binom{n}{j} \mathcal{E}_j^{(k)}\left(w; -\frac{1}{2}\right) \mathcal{E}_{n-j}^{(k)}\left(y; -\frac{1}{2}\right). \tag{7}$$

Proof. Using (1) and (5), we derive the following functional equation:

$$F_a(t,k,w+y)=\frac{1}{2^{3k}}F_{AE}\left(t,k,w;-\frac{1}{2}\right)F_{AE}\left(t,k,y;-\frac{1}{2}\right).$$

From the above equation, we get

$$\sum_{n=0}^{\infty}a_n^{(k)}(x+y)\frac{t^n}{n!}=\frac{1}{2^{3k}}\sum_{n=0}^{\infty}\mathcal{E}_n^{(k)}\left(w;-\frac{1}{2}\right)\frac{t^n}{n!}\sum_{n=0}^{\infty}\mathcal{E}_n^{(k)}\left(y;-\frac{1}{2}\right)\frac{t^n}{n!}.$$

Therefore

$$\sum_{n=0}^{\infty}a_n^{(k)}(x+y)\frac{t^n}{n!}=\frac{1}{2^{3k}}\sum_{n=0}^{\infty}\sum_{j=0}^{n}\binom{n}{j}\mathcal{E}_j^{(k)}\left(w;-\frac{1}{2}\right)\mathcal{E}_{n-j}^{(k)}\left(y;-\frac{1}{2}\right)\frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we arrive at the desired result. □

Combining (7) with (4), we have the following theorem:

Theorem 2. (*cf.* [3]) *Let $n\in\mathbb{N}_0$. Then we have*

$$a_n^{(k)}(w+y)=2^k\sum_{j=0}^{n}\binom{n}{j}H_j^{(k)}(w;2)\,H_{n-j}^{(k)}(y;2). \tag{8}$$

Remark 5. *Substituting $y=0$ into* (8)*, we obtain the following formula for the Fubini type polynomials of order k:*

$$a_n^{(k)}(w)=2^k\sum_{j=0}^{n}\binom{n}{j}H_j^{(k)}(w,2)\,H_{n-j}^{(k)}(2),$$

where $n\in\mathbb{N}_0$ (*cf.* [3])*.*

By using (7), after some elementary calculations, we arrive at the following theorem:

Theorem 3. *Let $n\in\mathbb{N}_0$. Then we have*

$$a_n^{(k)}(w+y)=\frac{1}{2^{3k}}\mathcal{E}_n^{(2k)}\left(w+y;-\frac{1}{2}\right). \tag{9}$$

Substituting $w=0$ into (9), we have the following corollary:

Corollary 4. *Let $n\in\mathbb{N}_0$. Then we have*

$$a_n^{(k)}(y)=\frac{1}{2^{3k}}\mathcal{E}_n^{(2k)}\left(y;-\frac{1}{2}\right). \tag{10}$$

Corollary 5. *If substitute $y=0$ into* (10)*, then we get the Fubini type numbers of order k:*

$$a_n^{(k)}=\frac{1}{2^{3k}}\mathcal{E}_n^{(2k)}\left(-\frac{1}{2}\right),$$

where $n\in\mathbb{N}_0$ (*cf.* [3])*.*

By using (2) and (5), we arrive at the following corollary:

Corollary 6. (*cf.* [3]) *Let $n\in\mathbb{N}_0$. Then we have*

$$a_n^{(k)}(y)=\frac{\mathcal{G}_{n+2k}^{(2k)}\left(y;-\frac{1}{2}\right)}{2^{3k}\left(n+2k\right)_{2k}}.$$

References

[1] L. Comtet, Advanced combinatorics: The art of finite and infinite expansions, Dordrecht and Boston: D. Reidel Publishing; 1974.

[2] L. Carlitz, Eulerian numbers and polynomials, Math. Mag. 32(5) (1959), 247–260.

[3] N. Kilar, Fubini type numbers and their generating functions, MSc Thesis in Mathematics, Antalya: Akdeniz University; 2017.

[4] N. Kilar and Y. Simsek, A new family of Fubini numbers and polynomials associated with Apostol-Bernoulli numbers and polynomials, J. Korean Math. Soc. 54(5) (2017), 1605–1621.

[5] N. Kilar and Y. Simsek, Some relationships between Fubini type polynomials and other special numbers and polynomials, AIP Conf. Proc. 2116(100017) (2019), 100017.1–100017.4.

[6] N. Kilar and Y. Simsek, Identities and relations for Fubini type numbers and polynomials via generating functions and p-adic integral approach, Publ. Inst. Math., Nouv. srie 106(120) (2019), 113–123.

[7] T. Kim, Identities involving Frobenius-Euler polynomials arising from non-linear differential equations, J. Number Theory 132(12) (2012), 2854–2865.

[8] I. Kucukoglu and Y. Simsek, Identities and relations on the q-Apostol type Frobenius-Euler numbers and polynomials, J. Korean Math. Soc. 56(1) (2019), 265–284.

[9] Q-M. Luo, Apostol-Euler polynomials of higher order and gaussian hypergeometric functions, Taiwanese J. Math. 10(4) (2006), 917–925.

[10] Q-M. Luo and H. M. Srivastava, Some generalizations of the Apostol-Genocchi polynomials and the Stirling numbers of the second kind, Appl. Math. Comput. 217(12) (2011), 5702–5728.

[11] Y. Simsek, Generating functions for generalized Stirling type numbers, Array type polynomials, Eulerian type polynomials and their applications, Fixed Point Theory Appl. 87(2013) (2013), 1–28.

[12] Y. Simsek, New families of special numbers for computing negative order Euler numbers and related numbers and polynomials, Appl. Anal. Discrete Math. 12 (2018), 1–35.

[13] Y. Simsek, O. Yurekli and V. Kurt, On interpolation functions of the twisted generalized Frobenius-Euler numbers, Adv. Stud. Contemp. Math. 2 (2007), 187–194.

[14] H. M. Srivastava, M. A. Boutiche and M. Rahmani, Some explicit formulas for the Frobenius-Euler polynomials of higher order, Applied Mathematics and Information Sciences 11(2) (2017), 621–626.

[15] H. M. Srivastava and J. Choi, Zeta and q-zeta functions and associated series and integrals, Amsterdam: Elsevier Science Publishers; 2012.

[16] H. M. Srivastava, B. Kurt and Y. Simsek, Some families of Genocchi type polynomials and their interpolation functions, Integral Transforms Spec. Funct. 23(12) (2012), 919–938.

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Direct and inverse Fueter-Sce-Qian mapping theorem and spectral theories

Fabrizio Colombo [1]

In classical complex operator theory, the Cauchy formula of holomorphic functions is a fundamental tool for defining functions of operators. Moreover, the Cauchy-Riemann operator factorizes the Laplace operator, so holomorphic functions play also a crucial role in harmonic analysis and in boundary value problems. In higher dimensions, for quaternion-valued functions or more in general for Clifford-algebra-valued functions, there appear two different notions of hyper-holomorphicity. The first one is called slice hyperholomorphicity and the second one is known under different names, depending on the dimension of the algebra and the range of the functions: Cauchy-Fueter regularity for quaternion-valued functions and monogenicity for Clifford-algebra-valued functions. The Fueter-Sce-Qian mapping theorem reveals a fundamental relation between the different notions of hyperholomorphicity and it can be illustrated by the following two maps

$$F_1 : Hol(\Omega) \mapsto \mathcal{N}(U) \quad \text{and} \quad F_2 : \mathcal{N}(U) \mapsto \mathcal{AM}(U).$$

The map F_1 transforms holomorphic functions in $Hol(\Omega)$, where Ω is a suitable open set Ω in $\mathbb{C}$, into intrinsic slice hyperholomorphic functions in $\mathcal{N}(U)$ defined on the open set U in $\mathbb{H}$. Applying the second transformation F_2 to intrinsic slice hyperholomorphic functions, we get axially Fueter-regular resp. axially monogenic functions. Roughly speaking the map F_1 is defined as follows:

1. We consider a holomorphic function $f(z)$ that depends on a complex variable $z = u + \iota v$ in an open set of the upper complex halfplane. (In order to distinguish the imaginary unit of $\mathbb{C}$ from the quaternionic imaginary units, we denote it by ι). We write
$$f(z) = f_0(u, v) + \iota f_1(u, v),$$
where f_0 and f_1 are $\mathbb{R}$-valued functions that satisfy the Cauchy-Riemann system.
2. For suitable quaternions q, we replace the complex imaginary unit ι in $f(z) = f_0(u, v) + \iota f_1(u, v)$ by the quaternionic imaginary unit $\frac{\Im(q)}{|\Im(q)|}$ and we set $u = \Re(q) = q_0$ and $v = |\Im(q)|$. We then define
$$f(q) = f_0(q_0, |\Im(q)|) + \frac{\Im(q)}{|\Im(q)|} f_1(q_0, |\Im(q)|).$$

The function $f(q)$ turns out to be slice hyperholomorphic by construction.

When considering quaternion-valued functions, the map F_2 is the the Laplace operator, i.e. $F_2 = \Delta$. When we work with Clifford-algebra-valued functions then $F_2 = \Delta_{n+1}^{(n-1)/2}$, where n is the number of generating units of the Clifford algebra and Δ_{n+1} is the Laplace operator in dimension $n + 1$. The Fueter-Sce-Qian mapping theorem can be adapted to the more general case in which $\mathcal{N}(U)$ is replaced by slice hyperholomorphic functions and the axially regular (or axially monogenic) functions $\mathcal{AM}(U)$ are replaced by monogenic functions. The generalization of holomorphicity to quaternion or Clifford-algebra-valued functions produces two different notions of hyper-holomorphicity that are useful for different purposes. Precisely, we have that:

(I) The Cauchy formula of slice hyperholomorphic functions leads to the definition of the S-spectrum and the S-functional calculus for quaternionic linear operators. Moreover, the spectral theorem for quaternionic linear operators is based on the S-spectrum.

(II) The Cauchy formula associated with Cauchy-Fueter regularity resp. monogenicity leads to the notion of monogenic spectrum and produces the Cauchy-Fueter functional calculus for quaternion-valued functions and the monogenic functional calculus for Clifford-algebra-valued functions. This theory has applications in harmonic analysis in higher dimension and in boundary value problems.

The main references for these topics are the following books.

References

[1] D. Alpay, F. Colombo, I. Sabadini, *Slice Hyperholomorphic Schur analysis,* Operator Theory, Advances and Applications, 256. Birkhäuser/Springer, Cham, (2016). xii+362.

[2] D. Alpay, F. Colombo, I. Sabadini, *Quaternionic de Branges spaces and characteristic operator function*, Briefs in Mathematics, Springer, Cham, 2020.

[3] F. Colombo, J. Gantner, *Quaternionic closed operators, fractional powers and fractional diffusion process,* Operator Theory: Advances and Applications, 274. Birkhäuser/Springer, Cham, 2019, VIII, vi+322 pp.

[4] F. Colombo, J. Gantner, D. P. Kimsey, *Spectral Theory on the S-spectrum for quaternionic operators,* Operator Theory: Advances and Applications, 270. Birkhäuser/Springer, Cham, (2018), ix+356 pp.

[5] F. Colombo, I. Sabadini, D. C. Struppa, *noncommutative functional calculus. theory and applications of Slice hyperholomorphic functions*, Progress in Mathematics, Vol. 289, Birkhäuser, (2011), VI, 222 pp.

[6] F. Colombo, I. Sabadini, D. C. Struppa, *Michele Sce's works in hypercomplex analysis. a translation with commentaries*, Birkhäuser/Springer, Cham, (2020), 122 pp.

[7] B. Jefferies, *Spectral properties of noncommuting operators*, Lecture Notes in Mathematics, 1843, Berlin: Springer-Verlag; 2004.

[8] T. Qian, P. Li, *Singular integrals and Fourier theory on Lipschitz boundaries*, Science Press Beijing, Beijing; Springer, Singapore, (2019), xv+306 pp.

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Remarks on infinite series representations of the certain family of the combinatorial numbers

Damla Gun [1], *Yilmaz Simsek* [2]

This paper is particularly motivated by the recent paper "Construction of some new families of Apostol-type numbers and polynomials via Dirichlet character and p-adic q-integrals, Turkish J. Math. 42 (2018), 557–577" in which, with aid of generating function method, the following explicit formula for the numbers $Y_n(\lambda)$ was given:

$$Y_n(\lambda) = \frac{2(n!)}{\lambda - 1}\left(\frac{\lambda^2}{1-\lambda}\right)^n.$$

The aim of this paper is to investigate and study not only he numbers $Y_n(\lambda)$, but also Fibonacci type numbers and polynomials. Some infinite series representations of the certain family of the combinatorial numbers involving the numbers $Y_n(\lambda)$, the Changhee numbers, the Humbert polynomials and Fibonacci type numbers and polynomials.
In this presentation, we will give a blend of both survay, some certain families special numbers and polynomials, and their some known results, including their properties. Infinite series representations of these special numbers are studied not only with the help of their generating functions, but also with the aid of their explicit formulas. Moreover, some new interesting infinite series representations of these special numbers are given.

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Keywords: Generating functions, Humbert polynomials, Fibonacci type numbers and polynomials, Combinatorial numbers

Introduction

In this paper, we briefly give the most used generating functions for special numbers and polynomials. Using these functions, we also give some properties of these special numbers and polynomials.

The second author [9] gave the following generating function for the numbers $Y_n(\lambda)$:

$$\frac{2}{\lambda^2 t + \lambda - 1} = \sum_{n=0}^{\infty} Y_n(\lambda)\frac{t^n}{n!}. \tag{1}$$

Recently many authors have studied on these numbers (*cf.* [2], [3], [10], [9]). With the aid of (1), the following explicit formula is given:

$$Y_n(\lambda) = 2(-1)^n \frac{n!\lambda^{2n}}{(\lambda-1)^{n+1}} \tag{2}$$

(*cf.* [10]).

Substituting $\lambda = -1$ into (2), we have the following well-known certain family of special numbers:

$$Y_n(-1) = (-1)^{n+1} Ch_n \tag{3}$$

where Ch_n denotes the Changhee numbers, which are defined by means of the following generating function:

$$F_{Ch}(t) = \frac{2}{t+2} = \sum_{n=0}^{\infty} Ch_n \frac{t^n}{n!} \tag{4}$$

(*cf.* [4]).

These numbers have been studied by many authors (*cf.* [7], [8], [10]).

By using (4), one has the following the explicit formula for the Changhee numbers Ch_n. That is, assuming that $|\frac{t}{2}| < 1$, we have

$$\sum_{n=0}^{\infty} Ch_n \frac{t^n}{n!} = \sum_{n=0}^{\infty} (-1)^n \left(\frac{t}{2}\right)^n.$$

Therefore

$$Ch_n = \frac{(-1)^n n!}{2^n} \tag{5}$$

(*cf.* [4]).

The Humbert polynomials $\Pi_{n,m}^{(\alpha)}(x)$ are defined by

$$H(t,x,m;\alpha) = \frac{1}{(1 - mxt + t^m)^{\alpha}} = \sum_{n=0}^{\infty} \Pi_{n,m}^{(\alpha)}(x)\, t^n \tag{6}$$

(*cf.* [1]).

The generating function in (7) provides us with a generalization and unification of the Fibonacci polynomials and numbers, the Lucas polynomials and numbers, the Pell polynomials and numbers, the Chebyshev polynomials, the Jacobsthal polynomials, the Vieta-Lucas polynomials, the Humbert polynomials, the Geganbauer polynomials, etc., which were considered in many earlier investigations by (among others) Ozdemir and Simsek [6] and also Ozdemir, Simsek, and Milovanović [5]

$$\frac{1}{1 - x^k t - y^m t^{m+n}} = \sum_{n=0}^{\infty} \mathcal{G}_n(x,y,k,m,v)\, t^n. \tag{7}$$

Infinite series representation of the ratios of the numbers $Y_n(\lambda)$ and Ch_n

In this section, we try to give one of the our results involving infinite series representation of the ratios of the numbers $Y_n(\lambda)$ and Ch_n. That is, by using definitions of the numbers $Y_n(\lambda)$ and Ch_n, we have

$$\sum_{n=0}^{\infty} \frac{Y_n(\lambda)}{Ch_n} = \frac{2(-1)^n \frac{n!}{\lambda-1}\left(\frac{\lambda^2}{\lambda-1}\right)^n}{\frac{(-1)^n n!}{2^n}}.$$

Thus

$$\sum_{n=0}^{\infty} \frac{Y_n(\lambda)}{Ch_n} = \sum_{n=0}^{\infty} \frac{2}{\lambda-1}\left(\frac{2\lambda^2}{\lambda-1}\right)^n.$$

After some elementary calculations, we get

$$\sum_{n=0}^{\infty}\frac{Y_n(\lambda)}{Ch_n}=-2H\left(\sqrt{2}\lambda,\frac{1}{2\sqrt{2}},2;1\right).$$

Combining the above equation, we get

$$\sum_{n=0}^{\infty}\frac{Y_n(\lambda)}{Ch_n}=-2\sum_{n=0}^{\infty}\Pi_{n,2}\left(\frac{1}{2\sqrt{2}}\right)\left(\sqrt{2}\lambda\right)^n,$$

where $\Pi_{n,2}^{(1)}(x)=\Pi_{n,2}(x)$.

Therefore, we arrive at the following theorem:

Theorem 1.

$$\sum_{n=0}^{\infty}\frac{Y_n(\lambda)}{Ch_n}=-\sum_{n=0}^{\infty}2^{\frac{n+2}{2}}\Pi_{n,2}\left(\frac{1}{2\sqrt{2}}\right)\lambda^n.$$

Substituting $k=0$, $m=1$, $v=1$, $y=-2$, $x=1$, $t=\lambda$ into (7), we also get the following result:

Corollary 2.

$$\sum_{n=0}^{\infty}\frac{Y_n(\lambda)}{Ch_n}=\sum_{n=0}^{\infty}\mathcal{G}_n(1,-2,0,1,1)\lambda^n.$$

Conclusion

In this study, infinite series containing the ratios of special combinatorial numbers and new formulas for them are given.

On the other hand, with the aid of the method used here and with the help of the explicit formulas for the certain family of special combinatorial numbers, we will also be attempted to obtain many interesting infinite series representations involving the related numbers and polynomials.

References

[1] P. Humbert, Some extensions of Pincherle's polynomials, Proc. Edinb. Math. Soc. 39 (1920), 21–24.

[2] S. Khan, T. Nahid and M. Riyasat, Partial derivative formulas and identities involving 2-variable Simsek polynomials, Bol. Soc. Mat. Mex. 26 (2020), 1–13.

[3] S. Khan, T. Nahid and M. Riyasat, Properties and graphical representations of the 2-Variable form of the Simsek polynomials, Vietnam J. Math. 2021. https://doi.org/10.1007/s10013-020-00472-6.

[4] D. S. Kim, T. Kim and J. Seo, A note on Changhee numbers and polynomials, Adv. Stud. Theoret. Phys. 7 (2013), 993–1003.

[5] G. Ozdemir and Y. Simsek, G. V. Milovanović, Generating functions for special polynomials and numbers including Apostol-type and Humbert-type polynomials, Mediterr. J. Math. 14 (2017) 1–17, Article ID 117.

[6] G. Özdemir and Y. Simsek, Generating functions for two-variable polynomials related to a family of Fibonacci type polynomials and numbers, Filomat 30(4) (2016), 969–975.

[7] Y. Simsek, Apostol type Daehee numbers and polynomials, Adv. Stud. Contemp. Math. 26(3) (2016), 1–12.

[8] Y. Simsek, Identities on the Changhee numbers and Apostol–Daehee polynomials, Adv. Stud. Con-temp. Math. 27(2) (2017), 199–212.

[9] Y. Simsek, Construction of some new families of Apostol-type numbers and polynomials via Dirichlet character and p-adic q-integrals, Turkish J. Math. 42(2) (2018), 557–577.

[10] H. M. Srivastava, I. Kucukoglu and Y. Simsek, Partial differential equations for a new family of numbers and polynomials unifying the Apostol-type numbers and the Apostol-type polynomials, J. Number Theory 181 (2017) 117-146.

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Some fundamental differences between octonionic analysis and Clifford analysis

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The aim of this presentation is to shed some light on fundamental differences between octonionic function theory and Clifford analysis.

On the very first glance one might get the impression that despite of the lack of associativity in octonions, there are still many similarities between octonionic function theory on the one hand and function theories treated in the context of an associative Clifford algebra on the other hand. Both settings apparently offer a very similar looking generalization of a Cauchy integral formula, Fueter polynomials, related Taylor series representations and even an argument principle together with a Rouche's and Hurwitz theorem addressing some special situations. For details we refer the reader for instance to our recent paper [2].

However, the lack of associativity in the octonions completely destroys the modular structure of the set of octonionic monogenic functions. If f is an octonionic mononogenic function, then neither $\alpha \cdot f$ or $f \cdot \alpha$ turns out to be in the kernel of the octonionic Cauchy-Riemann operator if α is a general octonion. In general, octonionic monogenic functions only exhibit the algebraic structure of an $\mathbb{R}$-module. It is not an $\mathbb{O}$-module. This seems to represent a serious obstacle in the development of a theory of octonionic reproducing kernel Hilbert modules, because all the classical theorems explicitly use the Cauchy-Schwarz type inequality in their classical proofs.

Since we do not have that, one has to be extremely careful in the consideration of inner products. Notice that even the most fundamental theorems like the Riesz representation theorem or the existence of an adjoint operator all do rely on a Cauchy-Schwarz inequality. The latter however does not hold for granted in the context of octonion-valued inner products.

This lack needs to be carefully taken into account when we want to introduce meaningful generalizations of octonionic monogenic Bergman and Hardy modules. One effective possibility to overcome this serious problem is to work with real-valued inner products on the sets of L^2-integrable monogenic functions instead. This treatment namely allows us to apply the corresponding standard functional analytic results on the real components of the octonionic functions. Then the deal is to find a suitable way how to lift all the component functions to one octonionic function. This is a highly non-trivial problem and it has not yet been solved completely by now. In this presentation we give some explicit formulas both for the Bergman and Szegö kernels of some special domains, *cf.* also our recent paper [4]. In this case we have a global lifting to one explicitly given octonionic function. Furthermore, we introduce an adjoint of the Cauchy transform together with some octonionic Kerzman-Stein operators, where orthogonality is understood in the sense of a properly chosen real-valued inner product, see our new paper [1] for details.

In the second part of this paper we also present an application of octonionic monogenic functions to class field theory that cannot be deduced with methods from Clifford monogenic functions, either. This represents another really fundamental example showing that octonionic function theory really has essentially different features im comparison to Clifford analysis. The closed multiplicative structure of octonions namely admit the construction of lattices in $\mathbb{R}^8$ that are closed under multiplication in $\mathbb{R}^8$. So, one can consider corresponding left and right ideals. They provide us with the natural analogue of lattices with complex multiplications to the eight dimensional setting. Classical number theory now tells us that the division values of the normalized classical complex analytic Weierstraß $\wp$-function are elements from algebraic Galois field extensions of imaginary quadratic number fields when taking as period lattice a lattice with complex multiplication. Now the octonionic monogenic analogues of the Weierstraß $\wp$-function turns out to play a rather analogues canonical role for tri-quadratic number fields instead, *cf.* [3]. This is another essentially different feature between octonionic monogenic functions and Clifford monogenic functions in $\mathbb{R}^8$ and shows that octonionic function theory definitely provides us with important applications that cannot be proved with methods from associative Clifford analysis.

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References

[1] D. Constales and R. S. Kraußhar, Octonionic Kerzman-Stein operators, Complex Anal. Oper. Theory 15(104) (2021), 23pp.

[2] R. S. Kraußhar, Differential topological aspects in octonionic monogenic function theory, Adv. Appl. Clifford Algebr. 30(51) (2020), 25pp.

[3] R. S. Kraußhar, Function theories in Cayley-Dickson algebras and number theory, Milan J. Math. 89(1) (2021), DOI: 10.1007/s00032-021-00325-y, 26pp.

[4] R. S. Kraußhar, Recent and new results on octonionic Bergman and Szegö kernels, Math. Methods Appl. Sci. (2021), https://doi.org/10.1002/mma.7316.

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Approximation by Szász type operators involving Bernoulli type polynomials

Mine Menekşe Yılmaz [1]

The goal of this study is to investigate the possibility of approximation of the Szász-type operators ([1]) defined using Bernoulli type polynomials. We benefit from the Korovkin theorem ([2]) when examining the convergence conditions of our operator. In the case of convergence, we give some theorems about the quality of the approximation.

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Keywords: Szász operator, Bernoulli type polynomials, Generating functions, Rate of convergence, Korovkin Theorem

Introduction

The possibility of approximation of operators is one of the problems of approximation theory. Korovkin's theorem is a well-known result in this field. The other problem involves the study of orders of approximation, it is about the quality of the approximation. Using generalized Appell polynomials and their generating functions, linear positive operators may be possible to construct under certain conditions. We can list the authors working in this field recently as follows: [3], [4], [5], [6]. Appell polynomials $S_n(x)$ with the help of generating functions are expressed as follows:

$$G(x,t) = A(t)e^{xt} = \sum_{n=0}^{\infty} \frac{S_n(x)}{n!} t^n,$$

where $A(t) = \sum_{k=0}^{\infty} \frac{a_k}{k!} t^k$, $A(0) \neq 0$, is an analytic function at $t = 0$, and $a_k := S_n(0)$. It is easy to see that, if $A(t) = \frac{t}{e^t - 1}$, $S_n(x)$ polynomials are Bernoulli polynomials. In paper [7], a new definition of Bernoulli type polynomials was given using umbral calculus (see [8]). By the same motivation, we construct a Szász type operator involving Bernoulli type polynomials at the following equation:

$$A_n(f;x) = (e-1)^{-1} e^{-nx} \sum_{k=0}^{\infty} \frac{\tilde{\beta}_k(nx)}{k!} f\left(\frac{k}{n}\right), \tag{1}$$

where $\tilde{\beta}_k(x)$ is called adjoint Appell-Bernoulli of the first kind.

Lemma 1. *For the operators A_n, we have*

$$A_n(1;x) = 1 \tag{2}$$

$$A_n(s;x) = x + \frac{1}{n(e-1)} \tag{3}$$

$$A_n(s^2;x) = x^2 + \frac{x}{n}\left(\frac{e+1}{e-1}\right) + \frac{1}{n^2} \tag{4}$$

Proof. With (1) it is possible to write the following equalities:

$$\sum_{k=0}^{\infty}\frac{\tilde{\beta}_k\left(nx\right)}{k!}=(e-1)e^{nx}, \tag{5}$$

$$\sum_{k=0}^{\infty}k\frac{\tilde{\beta}_k\left(nx\right)}{k!}=\left\{1+\left(e-1\right)nx\right\}e^{nx} \tag{6}$$

$$\sum_{k=0}^{\infty}k^2\frac{\tilde{\beta}_k\left(nx\right)}{k!}=\left\{\left(e-1\right)n^2x^2+\left(e+1\right)nx+e-1\right\}e^{nx} \tag{7}$$

In view of these equalities, we obtain (3)–(5). □

In view of Lemma 1, we have the following result.

Remark 6. *For* $x\in[0,\infty)$

$$A_n\left(s-x;x\right)=\frac{1}{n(e-1)}\quad \textit{and}\quad A_n\left(\left(s-x\right)^2;x\right)=\frac{x}{n}+\frac{1}{n^2}$$

Proof. Since the operator is linear, we can write at the following equations:

$$A_n\left(s-x;x\right)=A_n\left(s;x\right)-xA_n\left(1;x\right)$$

and

$$A_n\left(\left(s-x\right)^2;x\right)=A_n\left(s^2;x\right)-2xA_n\left(s;x\right)+x^2A_n\left(1;x\right).$$

If we then use Lemma 1, we get the desired results. □

Main Results

We begin this section by introducing some of the spaces we use and the norms on them. Let's denote the space of continuous and bounded functions with $C_B\left[0,\infty\right)$ and it is equipped with $\|f\|=\sup\limits_{0\leq x<\infty}\left|f\left(x\right)\right|.$ Suppose function f belongs to uniformly continuous space on $\left[0,\infty\right).$

$$\omega\left(f,\delta\right)=\sup\left\{\left|f\left(s\right)-f\left(t\right)\right|,\ s,t\in\left[a,b\right],\ \left|s-t\right|\leq\delta\right\} \tag{8}$$

is called modulus of continuity of f. From (8), for any $\delta>0$, and each $x\in[0,\infty)$,

$$\left|f\left(x\right)-f\left(y\right)\right|\leq\omega\left(f,\delta\right)\left(\frac{\left|x-y\right|}{\delta}+1\right) \tag{9}$$

holds. Let $C_B^2\left[0,\infty\right):=\left\{h\in C_B\left[0,\infty\right):h',h''\in C_B\left[0,\infty\right)\right\}$ equipped with

$$\|h\|_{C_B^2\left[0,\infty\right)}=\|h\|+\|h'\|+\|h''\|. \tag{10}$$

$$\mathcal{K}\left(f,\delta\right)=\inf\left\{\|f-h\|+\delta\|h\|_{C_B^2\left[0,\infty\right)}\right\}, \tag{11}$$

is known as the Peetre' s $\mathcal{K}$-functional.

Finally, we define $E=\left\{f:f\in\left[0,\infty\right),\ \lim\limits_{x\to\infty}\frac{f(x)}{1+x^2}\text{ exist}\right\}.$

Theorem 2. *Let* $f\in C\left[0,\infty\right)\cap E$. *Then the sequence* $\left\{A_n\left(s;x\right)\right\}$ *converges uniformly to* $f(x)$ *on every closed interval* $\left[0,a\right)$ *as* n *tends to infinity.*

Proof. We have the following fact from (2), (3), (4):

$$\lim_{n\to\infty} A_n\left(j^i;x\right) = x^i; \quad i = 0,1,2. \tag{12}$$

Now, thanks to (12), we can use the Korovkin's theorem, and that gets us the desired result. □

Theorem 3. *Suppose f is uniformly continuous function on $[0,1)$ and belongs to set E. Then, we have*

$$|A_n(f;x) - f| \leq 2\omega\left(f;\sqrt{\frac{x}{n}+\frac{1}{n^2}}\right), \tag{13}$$

where ω is given as in (8).

Proof. Thanks to $A_n(f;x)$ is monotonic operator and Lemma 1 , we can write the following:

$$|A_n(f;x) - f(x)| \leq A_n(|f(s) - f(x)|;x) \tag{14}$$

When we apply the (9) over the (14), we obtain the following form:

$$|A_n(f;x) - f(x)| \leq \omega(f,\delta)\left(1+\frac{1}{\delta}A_n(|x-y|;x)\right) \tag{15}$$

If we can apply the Cauchy-Schwarz property to (15), we get the following inequality:

$$|A_n(f;x) - f(x)| \leq \omega(f,\delta)\left(1+\frac{1}{\delta}\sqrt{A_n\left((x-y)^2;x\right)}\right) \tag{16}$$

by choosing $\delta := \delta_n(x) = \sqrt{\frac{x}{n}+\frac{1}{n^2}}$ in (16), we completed the proof. □

In the next theorem, order of approximation will be estimated by Peetre's $\mathcal{K}$-functional.

Theorem 4. *Let $f \in C_B[0,\infty)$, then for every $x \geq 0$, the following inequality holds*

$$|A_n(f;x) - f(x)| \leq 2K(f;\gamma_n(x)),$$

where $\gamma_n(x) = \frac{x}{4n}+\frac{1}{4n^2}+\frac{1}{2n(e-1)}$.

Proof. Let $h \in C_B^2[0,\infty)$ and $t,s \in [0,\infty)$. Using Taylor's expansion, we obtain

$$A_n(h;x) - h(x) = h'(x)A_n(s-x;x) + \frac{h''(\upsilon)}{2}A_n\left((s-x)^2;x\right), \upsilon \in (x,s). \tag{17}$$

Using Remark 2 and (10), we can write

$$\begin{aligned} A_n(h;x) - h(x) &= \|h'\|\frac{1}{n(e-1)} + \frac{\|h''\|}{2}\left(\frac{x}{n}+\frac{1}{n^2}\right) \\ &\leq \left(\frac{x}{2n}+\frac{1}{2n^2}+\frac{1}{2n(e-1)}\right)\|h\|_{C_B^2[0,\infty)}. \end{aligned}$$

On the other side, we have the following:

$$\begin{aligned} |A_n(h;x) - h(x)| &\leq |A_n(f-h;x) - h(x)| + |A_n(h;x) - h(x)| + |f(x) - h(x)| \\ &\leq 2\|f-h\| + |A_n(h;x) - h(x)| \\ &\leq 2\left(\|f-h\| + \left(\frac{1}{4n^2}+\frac{x}{4n}+\frac{1}{4n(e-1)}\right)\|h\|_{C_B^2[0,\infty)}\right) \end{aligned}$$

$$\leq 2\left(\|f-h\|+\gamma_n(x)\|h\|_{C_B^2[0,\infty)}\right) \tag{18}$$

We obtain the following inequality, when we take the infimum overall $h \in C_B^2[0,\infty)$ in (18):

$$|A_n(f;x)-f(x)| \leq 2\mathcal{K}(f;\gamma_n(x)).$$

□

Theorem 5. *Suppose f belongs to Lipschitz class of α order. If $x \in [0,\infty)$, then*

$$|A_n(f;x)-f(x)| \leq C\left[A_n\left((s-x)^2;x\right)\right]^{\frac{\alpha}{2}}.$$

Proof. Taking advantage of the fact that $A_n(f;x)$ is a monotonic operator, we can obtain the following

$$|A_n(f;x)-f(x)| \leq CA_n\left((s-x)^{\alpha};x\right) \tag{19}$$

Applying the Hölder inequality to equation (19), helps us to get the following equality:

$$|A_n(f;x)-f(x)| \leq C\left[A_n\left((s-x)^2;x\right)\right]^{\frac{\alpha}{2}}.$$

Hence, the proof is completed. □

References

[1] O. Szasz, Generalization of S. Bernstein's polynomials to the infinite interval, Journal of Research of the National Bureau of Standards. 45(3), (1950), 239–245.

[2] P. P. Korovkin, Linear operators and approximation theory, Delhi: Hindustan Publishing; 1960.

[3] G. İçöz, S. Varma and S. Sucu, Approximation by operators including generalized Appell polynomials, Filomat. 30(2) (2016), 429–440.

[4] S. Varma, S. Sucu and G. İçöz, Generalization of Szasz operators involving Brenke type polynomials, Computers & Mathematics with Applications. 64(2), (2012), 121–127.

[5] S. Sucu Approximation by Sequence of operators including Dunkl Appell polynomials, Bull. Malays. Math. Sci. Soc. 43 (2020), 2455–2464 . https://doi.org/10.1007/s40840-019-00813-w

[6] Karateke S., Atakut and Ç., Büyükyazıcı, İ., Approximation by generalized integral Favard-Szasz type operators involving Sheffer polynomials, Filomat. 33(7) (2019), 1921–1935. https://doi.org/10.2298/FIL1907921K

[7] N. Pierpaolo and P. Ricci, Adjoint Appell-Euler and First Kind Appell-Bernoulli Polynomials, Appl.Appl. Math. 14(2) (2019), 1112–1122.

[8] S. Roman, The Umbral Calculus, Pure and Applied Mathematics, Academic Press, Inc.: New York, NY, USA; 1984.

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Analysis on copper(I) oxide using ve-degree and ev-degree indices

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Copper oxide is well known p-type semiconductor and inorganic compound. It is a chemical element having formula as Cu_2O which is used in chemical sensors and solar orientated cells. A compound's molecular structure includes all the information needed to determine it's chemical, biological and physical properties. At the start of the topological index study, there were three diffrent types of index that were based on degree, distance and neighbourhood.$ve-$ degree and $ev-$degree concepts were newly defined in the theory of graphs. The generalizations of descriptors can not only reduce the number of descriptors based on molecular graphs, but also enhance existing results and provide better correlation with several molecular properties. This type of study can be helpful in understanding the atomic mechanism of corrosion and stress-corrosion cracking of copper.Topological indexes are essential devices for studying chemical compounds in order to understand the basic topology of chemical structures. Topological descriptors are significant or invariant numerical amounts within the domains of chemical graph theory. The topological indices $ev-$degree and $ve-$degree were defined using their respective traditional topological indices.Topological indexes were the critical tools for analyzing these chemicals to consider the essential topology of chemical structures.The standard definition of degrees has been converted into $ev-$degree and $ve-$degree.

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References

[1] M. O. Albertson, The irregularity of a graph, Ars Combin. 46 (1997), 219–225.

[2] B. Chaluvaraju, V. Lokesha, S. Jain and T. Deepika, General extremal degree based indices of a graph, Palest. J. Math. 8(1) (2019), 217–228.

[3] F. Chaudhry, I. Shoukat, D. Afzal, C. Park, M. Cancan and M. R. Farahani, M-polynomials and degree-based topological indices of the molecule copper(I) oxide, Journal of Chemistry 2021, 1–12.

[4] M. Chellali, T. W. Haynes, S. T. Hedetniemi and T. M. Lewis, On ve-degrees and ev-degrees in graphs, Discrete Math. 340(2) (2017), 31–38.

[5] S.-B. Chen, A. Rauf, M. Ishtiaq, M. Naeem and A. Aslam, On ve-degree- and ev-degree-based topological properties of crystallographic structure of cuprite, Open Chemistry 19 (2021), 576–585.

[6] T. Deepika, An emphatic study on VL index and their bounds on tensor product of F sum graph, TWMS J. Appl. Eng. Math. 11(2) (2021), 374–385.

[7] J. Devillers and A. T. Balaban, Topological indices and related descriptors in QSAR and QSPR, Gordon and Breach, Amsterdam: CRC Press; 1999.

[8] S. Ediz, Predicting some physicochemical properties of octane isomers: a topological approach using ev-degree and vedegree zagreb indices, International Journal of Systems Science and Applied Mathematics 2(5) (2017), 87–92.

[9] I. Gutman, Geometric approach to degree based topological indices, MATCH Commun. Math. Comput. Chem. 86(1) (2021), 11–16.

[10] I. Gutman and N. Trinajsti, Graph theory and molecular orbitals. total ϕ-electron energy of alternant hydrocarbons, Chemical Physics Letters 17(4) (1972), 535–538.

[11] V. R. Kulli, On K- Indices of graphs, Intern. J. Fuzzy Mathematical Archive 10(2) (2016), 105–109.

[12] V. R. Kulli, Nirmala index, International Journal of Mathematics Trends and Technology 2021, 8–12.

[13] V. Lokesha and S. Jain, New results on the F-index of graphs based on a corona type products of graphs, Proc. Jangjeon Math. Soc. 23(2) (2020), 141–147.

[14] V. Lokesha, S. Jain, T. Deepika and A. S. Cevik, Operation on topological indices of Dutch windmill graph, Proc. Jangjeon Math. Soc. 21(3) (2018), 525–534.

[15] V. Lokesha, S. Jain, T. Deepika and K. M. Devendraiah, Some computational aspects of polycyclic aromatic hydrocarbons, Gen. Math. 25(1-2) (2017), 175–190.

[16] M. Manjunath, V. Lokesha, Suvarna and S. Jain, Bounds for the topological indices of φ graphs, Eur. J. Pure Appl. Math. 14(2) (2021), 340–350.

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Convergence by Szász type operators based on Euler type polynomials

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In this work, we give a Szász type operator with related to Euler type polynomials by the aid of generating functions. For obtaining approximation properties of our operator, we construct moments and central moments. We obtain a uniformly convergence theorem for our operator with the help of moments and Korovkins theorem. We show rate of convergence of our new operator by using modulus of continuity notation.

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Keywords: Euler polynomials, Generating function, Szász operator, Modulus of continuity

Introduction

The Euler polynomials were studied by many authors (see for details [1-4]). In [1], Simsek constructed the generating function for q-Eulerian type polynomials and obtained some identities by the aid of generating function. Ozdemir et.al. derived generating functions and demonstrated several properties for new polynomials family of special polynomials such as Euler and Bernoulli polynomials [2]. Natalini and Ricci introduced to the adjunction property for Appell polynomials and applied to special Appell type polynomials family such as Appell-Euler polynomials [3]. In [4], Prvost derived some recurrence relations between Apostol-Bernoulli and Apostol-Euler polynomials by using the Padé approximation.

By the same motivation of the above studies, in this paper, we construct an operator which including Euler type polynomials and its generating functions. In addition, we give some convergence properties of our operator by the aid of Korovkin's theorem and modulus of continuity.

The adjoint-Euler polynomials are defined by the aid of generating function at the following equation:

$$\sum_{k=0}^{\infty} \tilde{\varepsilon}_k(x) \frac{t^k}{k!} = \frac{e^t+1}{2} e^{xt}. \tag{1}$$

Our operator are given by using generating function of adjoint-Euler polynomials for $t=1$ at the following equation:

$$E_n^*(f,x) = \left(\frac{2}{e+1}\right) e^{-nx} \sum_{k=0}^{\infty} \frac{\tilde{\varepsilon}_k(x)}{k!} f\left(\frac{k}{n}\right) \tag{2}$$

Main Results

In this part, firstly, we give moments and central moments functions for constructing the Korovkin's theorem for $E_n^*(f,x)$. And then, we present convergence properties of $E_n^*(f,x)$ by the aid of Korovkin's theorem and modulus of continuity.

Lemma 2.1.1 For all $x \in [0,\infty)$, Eq.(2) satisfies at the following equalities:

$$E_n^*(1,x) = 1 \tag{3}$$

$$E_n^*(s,x) = x + \frac{e}{n(e+1)} \tag{4}$$

$$E_n^*(s^2,x) = x^2 + \left(\frac{3e+1}{e+1}\right)\frac{x}{n} + \left(\frac{2e}{e+1}\right)\frac{1}{n^2}. \tag{5}$$

Proof Using the generating functions of the adjoint Euler polynomials given by (1), we get

$$\sum_{k=0}^{\infty} \frac{\tilde{\varepsilon}_k(x)}{k!} = \frac{e^t+1}{2}e^{xt}. \tag{6}$$

$$\sum_{k=0}^{\infty} k\frac{\tilde{\varepsilon}_k(x)}{k!} = \frac{e^{nx}}{2}\left[e + \frac{nx(e+1)}{2}\right]. \tag{7}$$

$$\sum_{k=0}^{\infty} k^2\frac{\tilde{\varepsilon}_k(x)}{k!} = \frac{e^t+1}{2}e^{xt}. \tag{8}$$

By applying these equalities, the desired results are obtained.

Lemma 2.1.2 Let $x \in [0,\infty)$. Eq.(2) satisfies at the following equalities.

$$E_n^*((s-x),x) = \frac{e}{n(e+1)}, \tag{9}$$

$$E_n^*((s-x)^2,x) = \frac{5e+1}{(e+1)n}x + \frac{2e}{(e+1)n^2}. \tag{10}$$

Proof By using the linearity property of E_n^*, we have

$$E_n^*((s-x),x) = L_n^*(s,x) - xL_n^*(1,x). \tag{11}$$

$$E_n^*((s-x)^2,x) = E_n^*(s^2,x) - 2xE_n^*(s,x) + x^2E_n^*(1,x) \tag{12}$$

By applying Eq.(3)-Eq.(5) on Eq.(10) and Eq.(11), the desire results are obtained.

Now we define a set for investigating convergence properties for $E_n^*(f,x)$.

Let the set E is defined as:

$$E = \{f \mid f \in [0,\infty), \lim_{x\to-\infty} \frac{f(x)}{1+x^2} \ exist\}$$

Theorem 2.2.1 For $f \in C[0,\infty) \cap E$, we have

$$\lim_{n\to\infty} E_n^*(s^i,x) = x^i \tag{13}$$

where Eq.(3), Eq.(4) and Eq.(5) is uniformly on each compact subset of $[0,\infty)$

Proof By applying Lemma 1 and Korovkin's theorem, we obtain

$$\lim_{n\to\infty} E_n^*(s^i,x) = x^i, \tag{14}$$

where $i = 1,2,3$. The desired result is completed by using the property (vii) of Theorem 4.1.4 in [*cf.* 5].

Let f be uniformly continuous function on $[0,\infty)$ and $\delta > 0$. The modulus of continuity $\omega(f,\delta)$ of the function f is defined as follows:

$$\omega(f,\delta) := sup|f(x) - f(y)| \tag{15}$$

where $x, y \in [0,\infty)$ and $|x-y| \leq \delta$ [*cf.* 12].

Then, for any $\delta > 0$ and each $x \in [0,\infty)$ the following relation holds:

$$|f(x) - f(y)| = \omega(f,\delta_n)\left(\frac{|x-y|}{\delta} + 1; x\right). \tag{16}$$

Theorem 2.3.1 Let f is uniformly continuous function on $[0,1)$ and also belongs to set E. Then, we have

$$|E_n(f;x) - f| \leq 2\omega\left(f; \sqrt{E_n\left((s-x)^2; x\right)}\right), \tag{17}$$

where ω is the modulus of continuity of the function f.

Proof. It follows from Lemma 1 and monotonicity of operators $E_n(f;x)$ that

$$|E_n(f;x) - f(x)| \leq E_n(|f(s) - f(x)|; x) \tag{18}$$

Using the (16), we obtain at the following inequality from (18)

$$|E_n(f;x) - f(x)| \leq \omega(f,\delta)\left(1 + \frac{1}{\delta}E_n(|x-y|; x)\right) \tag{19}$$

Applying the Cauchy-Schwarz inequality to the right side of (19), we get

$$|E_n(f;x) - f(x)| \leq \omega(f,\delta)\left(1 + \frac{1}{\delta}\sqrt{E_n\left((x-y)^2; x\right)}\right) \tag{20}$$

by choosing $\delta := \delta_n(x) = \sqrt{\frac{5e+1}{(e+1)n}x + \frac{2e}{(e+1)n^2}}$ in (19), the proof is completed.

Conclusion

In this study, we constructed a Szász type operator including adjoint-Euler polynomials. In addition, we investigated convergence properties of $E_n^*(f,x)$ such as uniform convergence and rate of convergence with the aid of Korovkin theorem and modulus of continuity.

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References

[1] F. Altomare and M. Campiti, Korovkin-type approximation theory and its applications,Berlin New york: Walter de Gruyter; 2011.

[2] G. Ozdemir, Y. Simsek and G. V. Milovanovi, Generating functions for special polynomials and numbers including Apostol-type and Humbert-type polynomials. Mediterr. J. Math. 14(3) (2017), Article number:117.

[3] P. Natalini and P. E. Ricci, Adjoint Appell-Euler and first kind Appell-Bernoulli polynomials. Appl. Appl. Math. 14(2) (2019), 1112–1122.

[4] M. Prvost, Pad approximation and ApostolBernoulli and ApostolEuler polynomials. J. Comput. Appl. Math. 233(11) (2010), 3005–3017.

[5] Y. Simsek, Generating functions for q-Apostol type FrobeniusEuler numbers and polynomials, Axioms 1(3) (2012), 395–403.

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A new quantum linear operator defined by the Airy differential formula acting on a normalized analytic function in the open unit disk

Rabha Waell Ibrahim [1]

Fundamental series of special functions and polynomials and their consequences are documented to have wide performances, in the number theory including the theory of partitions. These functions are appreciated additionally in an all-encompassing variety of fields connecting, for instance, finite vector spaces, combinator investigation, lie philosophy, nonlinear electric circuit notion, particle physics, optical investigations, fluid model, mechanical engineering, quantum mechanics, cosmology, theory of thermal conduction and measurements. The theory of geometric function is ironic with altered kinds of these special functions formulating convolution operators acting on different types of classes of analytic functions in the open unit disk $\bigcirc := \{\xi \in \mathbb{C} : |\xi| < 1|\}$.

In this work, we express a generalization of Airy functions utilizing Jackson calculus (quantum calculus or q-calculus) in a complex domain. Airy functions are special functions dominated by the hypergeometric function of a complex variable fulfilling the Airy equation

$$\varphi''(\xi) - \xi\,\varphi(\xi) = 0, \quad \xi \in \mathbb{C},$$

which can be expressed by the formula

$$\left(\frac{\xi\varphi''(\xi)}{\varphi'(\xi)}\right) - \frac{\xi^3}{\left(\dfrac{\xi\varphi'(\xi)}{\varphi(\xi)}\right)} = 0, \quad \xi \in \mathbb{C}.$$

More interesting formula can be presented as follows:

$$\left(1 + \frac{\xi\varphi''(\xi)}{\varphi'(\xi)}\right) - \frac{\xi^3}{\left(\dfrac{\xi\varphi'(\xi)}{\varphi(\xi)}\right)} = 1, \quad \xi \in \mathbb{C}.$$

The above formula can be studied geometrically, because it involves the convexity of normalized analytic functions:

$$\mathcal{C} := \left\{\varphi \in \bigcirc : \Re\left(1 + \frac{\xi\varphi''(\xi)}{\varphi'(\xi)}\right) > 0\right\}$$

and starlikeness:

$$\mathcal{S}^* := \left\{\varphi \in \bigcirc : \Re\left(\frac{\xi\varphi'(\xi)}{\varphi(\xi)}\right) > 0\right\},$$

when φ is a normalized analytic function in the open unit disk satisfying $\varphi(0) = \varphi'(0) - 1 = 0$ taking the series

$$\varphi(\xi) = \xi + \sum_{n=2}^{\infty} \varphi_n \xi^n.$$

By using the q-hypergeometric function, we generalize the Airy function, where for a number $\xi \in \mathbb{C}$, the q-shifted factorials is formulated by the formal

$$(\xi;q)_\ell = \prod_{\imath=0}^{\ell-1}(1-q^{\imath}\xi), \quad \ell \in \mathbb{N},\ (\xi;q)_0 = 1. \tag{1}$$

According to (1) and in terms of gamma function, we get the q-shifted formula

$$(q^\xi;q)_\ell = \frac{\Gamma_q(\xi+\ell)(1-q)^\ell}{\Gamma_q(\xi)}, \quad \Gamma_q(\xi) = \frac{(q;q)_\infty(1-q)^{1-\xi}}{(q^\xi;q)_\infty}, \tag{2}$$

where

$$\Gamma_q(\xi+1) = \frac{\Gamma_q(\xi)(1-q^\xi)}{1-q}, \quad q \in (0,1).$$

and

$$(\xi;q)_\infty = \prod_{\imath=0}^{\infty}(1-q^{\imath}\xi). \tag{3}$$

Based on this generalization, we formulate the q-Airy equation and study its behavior in view of the geometric function theory, where the q-Airy fractional operator is defined on $\varphi \in \Delta$ (the class of normalized analytic function in the open unit disk $\bigcirc$)

$$\begin{aligned}\left[\mathbb{A}\imath * \varphi\right]_q(\xi) &= \xi + \sum_{n=2}^{\infty}\left(\frac{3^{(n-1)/3}\Gamma_q(\frac{2}{3})\Gamma_q(\frac{n}{3})}{\Gamma_q(n)\Gamma_q\left(1-\dfrac{2n}{3}\right)\Gamma_q\left(\dfrac{2n}{3}\right)}\right)\varphi_n\xi^n \\ &:= \xi + \sum_{n=2}^{\infty}[A_n]_q\varphi_n\xi^n,\end{aligned} \tag{4}$$

where

$$[n]_q = \frac{1-q^n}{1-q}.$$

This operator will be considered in some classes of analytic functions. In this investigation, we act the suggested q-Airy operator on the subclass of normalized analytic functions taking the structure

$$\left(\frac{1-\sigma}{\xi}\right)\ [\mathbb{A}\imath * \varphi(\xi)]_q + \sigma\,[\mathbb{A}\imath * \varphi(\xi)]'_q \prec p(\xi) = \frac{a\xi+1}{b\xi+1}. \tag{5}$$

$$\Big(\xi \in \bigcirc, \nu, \sigma \in [0,1], -1 \le b < a \le 1\Big),$$

where p is convex univalent in $\bigcirc$.
Our method is given by the theory of subordination and superordination. Some examples will be illustrated in the sequel. Moreover, an application is formulated for finding the upper solution of a complex wave diffusive equation using the suggested q-operator

$$(\frac{1-\sigma}{\xi})\,[\mathbb{A}\imath * \varphi(\xi)]_q + \sigma\,[\mathbb{A}\imath * \varphi(\xi)]'_q = \frac{a\xi+1}{b\xi+1}, \tag{6}$$

$$\Big([\mathbb{A}\imath * \varphi(0)]_q = 0,\ q \in (0,1),\ \sigma \in [0,1],\ \xi \in \bigcirc\Big)$$

where φ is the height deviation of the horizontal pressure surface at two-dimensional position $\xi = \rho + i\,\varrho$ and $[\mathbb{A}\imath * \varphi(\xi)]_q$ represents the bed slope.

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Note on non-commutative partition

Busra Al [1], *Mustafa Alkan* [2]

We study non-commutative partitions of an integer. We characterize the set of non-commutative partitions of an integer as union of the sets. Then we give an alternative proof for the well known result in [12].

2010 Mathematics Subject Classifications: 03E02, 11P83, 11P99
Keywords: Partitions of an integer, the generating function of the number of partitions

Introduction

Many researher have interested in partition of numbers and have done a lot of research. In 1674, Leibniz asked that how many different ways a positive integer can be written as the sum of positive integers aroused great interest and many studies have been done on the partition of numbers. These studies also led to the topic of partition of a positive integer. The number of these parts will be denoted by $p(n)$. For a positive integer n, the partition function is the number of ways n can be written as a sum of positive integers n.

Partitions are divided into two as commutative and non-commutative. The displacement of summands in commutative partitions are not important, but the displacement of summands in non commutative partitions are important.

Example 1. *The number* 3 *has* 3 *commutative partitions and the number of non-commutative partitions is* 4.

The set of commutative partition of 3 *is* $\{3,\ (1+2)\,,\ (1+1+1)\}$ *and* 3 *has* 3 *partitions.*

The set of non-commutative partition of 3 *is* $\{3,\ (1+2)\,,\ (2+1)\,,\ (1+1+1)\}$ *and* 3 *has* 4 *partitions.*

We recall the following basis facts from the literature about commutative partition theory.

Euler investigated the generating function of the number of partitions of an integer n, as follows

$$f(x)=\prod_{n=1}^{\infty}\frac{1}{(1-x^n)}=\sum_{n=0}^{\infty}p(n)x^n$$

where $0<x<1$ ([7]).

After Euler's recurrence for the partition of an integer, at recent years recurrence formulas have been improved by J.A.Ewell, M. Merca, M. Alkan and B. Al ([10, Theorem 1.2], [15, Theorem 1], [2, Theorem 2.4 and 2.5]). In this recurance relations, both numbers of steps and larger of numbers founded in each step is main problem since increasing of $p(m)$ is faster than increasing of an integer m and so the recurrence formulas that compute the values of the partition with the help of the smaller integers are much useful and effective.

In the literature, the restricted partitions are substantial as unrestricted partition of an integer ([5], [8], [16], [17], [18]). The generating function of the number of partitions of an integer n into odd part is

$$\prod_{n=1}^{\infty} \frac{1}{1-x^{2n-1}} = \sum_{n=0}^{\infty} Q(n)x^n.$$

The generating function of the number of partitions of an integer n into distinct odd parts is

$$\prod_{n=1}^{\infty} (1+x^{2n-1}) = \sum_{n=0}^{\infty} Z(n)x^n.$$

From [7, page 309], we recall that the number of partitions of k into parts not exceeding m is denoted by $p_m(k)$ for integers m, k. Then $p_m(k) = p(k)$ for $m \geq k$. It is clear that $p_m(k)$ is less than $p(k)$ and the computation of $p_m(k)$ is simpler for integers m, k. The generating function for the number of partitions of k into parts not exceeding m is defined as

$$F_m(x) = \prod_{i=1}^{m} \frac{1}{1-x^i} = 1 + \sum_{i=1}^{\infty} p_m(i)x^i.$$

While there are many studies in the literature about commutative partition in partition theory, there is nor suffient results about non-commutative partitions. In this notes, we give an alternative proof for the well known result in [7].

We recall some fact from [1]. Let n be a positive integer and we define the set

$$P_n = \left\{(a_1, a_2, ..., a_t) : \ a_1 + a_2 + ... + a_t = n, \quad a_i, t \in \mathbb{Z}^+\right\}.$$

In fact, the element of P_n is a partition of integer n and so P_n is the set of a non commutative partition of an integer n.

In [1], we have constructed the set P_{n+1} of non commutative partitian for a positive integer n by using recurance relations on the set P_n. First we recall operations with the a partition $a = (a_1, a_2, ..., a_t)$ of integer n;

$$(1 \odot a) = (1, a_1, a_2, ..., a_t),$$

$$(1 \oplus a) = (a_1 + 1, a_2, ..., a_t).$$

Then $1 \oplus a, 1 \odot a \in P_{n+1}$ and sowe also use the notaions $1 \oplus P_n$, $1 \odot P_n$ for the set of new type elements, i.e.;

$$1 \oplus P_n = \{1 \oplus a : a \in P_n\}$$

$$1 \odot P_n = \{1 \odot a : a \in P_n\}.$$

Theorem 2 (*cf.* [1])**.** *For a positive integer n, we have*

$$P_{n+1} = (1 \oplus P_n) \cup (1 \odot P_n).$$

Proof. If $a \in P_n$ then $b = 1 \oplus a$, $b = 1 \odot a \in P_{n+1}$ and so

$$(1 \oplus P_n) \cup (1 \odot P_n) \subseteq P_{n+1}.$$

Let $b = (b_1, b_2, ..., b_k) \in P_{n+1}$ and so $b_1 + b_2 + ... + b_k = n + 1$.

If $b_1 \neq 1$ then $c_1 = b_1 - 1$, $c_i = b_i$, where $i \in \{2, ..., k\}$ and so $c_1 + c_2 + ... + c_k = n$. It follows that $c = (c_1, c_2, ..., c_k) \in P_n$ and $b = 1 \oplus c \in (1 \oplus P_n)$.

If $b_1 = 1$ then $c_{i-1} = b_i$, where $i \in \{2, ..., k\}$ and so $c_1 + c_2 + ... + c_{k-1} = n$. It follows that $c = (c_1, c_2, ..., c_{k-1}) \in P_n$ and $b = 1 \odot c \in (1 \odot P_n)$.

Theorefore, we have that $P_{n+1} = (1 \oplus P_n) \cup (1 \odot P_n)$ and $(1 \oplus P_n) \cap (1 \odot P_n) = \varnothing$. $\square$

We can prove the well-known result in [12] by the Theorem 2 by using the set defined above.

Theorem 3. *The number of partition of a noncommutative partition of an integer n $(n > 1)$ is 2^n.*

Proof. Proof 1: Let n be a positive integer. It is clear that the number of element of both $(1 \oplus P_n)$ and $(1 \odot P_n)$ are equal, i.e. $(|1 \oplus P_n| = |1 \odot P_n|$ and so $|P_{n+1}| = 2\,|1 \odot P_n|$. Since $|P_2| = 2$, we have that $|P_{n+1}| = 2^n$ by induction method.

Proof 2 in [12]: While there are many studies in the literature about commutative partition in partition theory, there is only formula information about non-commutative partitions that reports their number.

The total number of non commutative partitions of n is, in fact, given by 2^{n-1} . This is easily proved as follows:

The partitions of n can be classified under two heads:

Those in which the first part is 1; and Those in which the first part is > 1.

Removing 1 (the first part), from each partition of n of the first kind, we obtain all the different partitions of $(n-1)$, each once. Reducing by 1 the first part in the partitions of the second kind, we again get all the partitions of $(n-1)$ as before. Hence, the number of partitions of n, is twice the number of partitions of $(n-1)$ and the result follows readily by induction [12]. □

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References

[1] B. Al and M. Alkan, Some relations between partitions and Fibonacci numbers, Proc. Book of 2nd.Micopam 2019.

[2] B. Al and M. Alkan, On Relations for the partitions of numbers, Filomat, 34(2) (2020), 567–574.

[3] G. E. Andrews, The theory of partitions, New York: Addison-Wesley Publishing; 1976.

[4] G. E. Andrews and K. Erikson, Integer partitions, Cambridge: Cambridge University Press; 2004.

[5] G. E. Andrews, M. D. Hirschhorn and J. A. Sellers, Arithmetic properties of partitions with even parts distinct, Ramanujan J. 23 (2010), 169–181.

[6] T. M. Apostol, On the Lerch zeta function, Pacific J. Math. 1 (1951), 161–167.

[7] T. M. Apostol, Introduction to analytic number theory, NewYork: Springer-Verlag; 1976.

[8] S. C. Chen, On the number of partitions With distinct even parts, Discrete Math. 311 (2011), 940–943.

[9] L. Euler, Introduction to analysis of the infinite, vol. 1, New York: Springer-Verlag; 1988 (translation by J.D. Blanton).

[10] J. A. Ewell, Recurrences for the partition function and its relatives, Rocky Mountain J. Math. Number2, Spring 34 (2004), 619–627.

[11] J. A. Ewell, Recurrences for two restricted partition functions, Fibonacci Quart. 18 (1980), 1–2.

[12] H. Gupta, Partitions - a survey, Journal of Research of the Notional Bureau of Standards - B. Mathematical Sciences, 74B(1) (1970), 1–29.

[13] G. H. Hardy and E.M. Wright, An Introduction to the theory of numbers, 4th ed., Oxford: Clarendon Press; 1960.

[14] P. A. MacMahon, Note on the parity of the number which enumerates the partitions of a number, Proc. Cambridge Philos. Soc. 20 (1921), 281–283.

[15] M. Merca, Fast computation of the partition function, J. Number Theory, 164 (2016), 405-416.

[16] M. Merca, A note on the partitions involving parts of k different magnitudes, J. Number Theory, 162 (2016), 23–34.

[17] M. Merca, New relations for the number of partitions with distinct even parts, J. Number Theory, 176 (2017), 1–12.

[18] M. Merca, On the number of partitions into parts of k different, J. Discrete Math. 340 (2017), 644–648.

[19] G. N. Watson, Two tables of partitions, Proc. London Math. Soc. 42 (1937), 550–555.

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A visual proof that $e < a < b \implies e^a b^b > a^b e^b$

Taekyun Kim [1]

In this note, we will give new interesting the following inequalities:

$$e < a < b \implies e^a b^b > a^b e^b.$$

This inequality is interested related to number and mathematical class room.

In particular, we note that $e^a > a^e$ for $e < a$. (see [1,2]). Thus, we have $\pi^e < e^\pi$.

2010 Mathematics Subject Classifications: 11B68, 11B83

Keywords: Degenerate cosine-Euler polynomials, degenerate sine-Euler polynomials, degenerate cosine-Bernoulli polynomials, degenerate sine-Bernoulli polynomials, degenerate cosine-polynomials, degenerate sine-polynomials

References

[1] N. Haque, A visual proof that $e < A \implies e^A > A^e$, Math. Intelligencer 42(74) (2020), https://doi.org/10.1007/s00283-019-09964-x.

[2] T. Kim et al., q-number theory and their application, Kyowoo sa, 2006, ISBN: 8981726361, 9788981726362.

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Some notes on the curve couple of Bézier curve

Ayşe Yılmaz Ceylan [1] , *Merve Kara* [2]

The aim of this work is to examine the geometric stucture of the pedal curves of Bézier curves which has many applications in computer graphics and related areas. Especially, we give the curvature of the pedal curves of Bézier curves in Euclidean 2−space.

2010 Mathematics Subject Classifications: 14H50, 53A04, 65D17
Keywords: Bézier curve, curvature, pedal curve, pedal point

Introduction

Geometry of curves is very important because it has many crucial applications in several different areas. Consequently, different types of curves and surfaces have been studied by many authors for many years. Lately, due to its different structure, Bézier curves have attracted the attention of many researchers. Bézier curves are the most significant mathematical representations of curves which are applied to computer graphics and related areas.

In differential geometry, the plane curves in the Euclidean plane are one of the most important subjects. For that reason, examining the pedal curves is an essential issue. Among them, the pedal curves of regular curves have an importance and are studied by many authors in different areas of mathematics. A pedal curve of a regular plane curve is the locus of the feet of the perpendiculars from a point to the tangents to the curve. In the recent studies, [3] and [6] studied pedal and contrapedal curves of fronts in the Euclidean plane. In the CAGD field, a classical family of sinusoidal spirals was introduced by Ueda [7] and [8] via a pedal-point construction and later identified as belonging to the special subset of rational Bézier curves called p-Bézier curves [5]. In [1], Ceylan and Kara characterized the pedal curve of a planar Bézier curve.

This work is organized as folows. In section 2, some basic notations and definitions are given. In section 3, pedal curve of a planar Bézier curve is characterized and investigated at the starting and the ending points. In section 4, the curvature of the pedal curve of Bézier curve in Euclidean 2−space is calculated. In the final section, we conclude our work.

Preliminaries

A classical Bézier curve of degree n with control points P_j is defined as

$$B(t) = \sum_{j=0}^{n} B_j^n(t) P_j, t \in [0,1] \qquad (1)$$

$$B_{j,n}(t) = \begin{cases} \frac{n!}{(n-j)!j!}(1-t)^{n-j}t^j, & 0 \le j \le n \\ 0, & otherwise \end{cases}$$

are called the Bernstein basis functions of degree n. The polygon formed by joining the control points $P_0, P_1, ..., P_n$ in the specified order is called the Bézier control polygon.

Definition 1. *The first derivative $B'(t)$ of a degree-n Bézier curve $B(t)$ is clearly a degree $n-1$ curve. Such a curve can be written in Bézier form as*

$$B'(t) = \sum_{j=0}^{n-1} B_j^{n-1}(t)\triangle^1 P_j \tag{2}$$

where $\triangle^1 P_j = P_{j+1} - P_j$, $j = 0, 1, ..., n-1$ are the control points of $B^{'}(t)$ [4].

Definition 2. *The second derivative $B''(t)$ of a degree-n Bézier curve $B(t)$ is clearly a degree $n-2$ curve. Such a curve can be written in Bézier form as*

$$B''(t) = n\,(n-1) \sum_{j=0}^{n-2} B_j^{n-2}(t)\triangle^2 P_j \tag{3}$$

where $\triangle^2 P_j = \left(\triangle^1 P_{j+1} - \triangle^1 P_j\right) = \left(P_{j+2} - 2P_{j+1} + P_j\right)$ are the control points of $B^{''}(t)$ [4].

Definition 3. *Let $J : E^2 \to E^2$ be a linear transformation defined by the following equation [2]:*

$$J(P_1, P_2) = (-P_2, P_1).$$

Definition 4. *Let $\alpha : I \to E^2$ be a non-unit speed planar curve. The Serret-Frenet frame $\{T, N\}$ and curvature κ of α for $\forall t \in I$ are defined by the following equations [2]:*

$$T(t) = \frac{\alpha'(t)}{\|\alpha'(t)\|}, N(t) = \frac{J\alpha'(t)}{\|\alpha'(t)\|}, \kappa(t) = \frac{< \alpha''(t), J\alpha'(t) >}{\|\alpha'(t)\|^3}. \tag{4}$$

Definition 5. *The pedal curve of a regular curve $\beta : (a, b) \to R^{^2}$ with respect to a pedal point $p \in R^2$ is defined by*

$$\beta^*[\beta, p](t) = p + \frac{< \beta(t) - p, J\beta'(t) >}{\|\beta'(t)\|^2} J\beta'(t) \tag{5}$$

[2].

Pedal Curve of a planar Bézier curve

In this section, we construct pedal curve of a planar Bézier curve and investigate this curve at the end points.

Theorem 6. *The pedal curve $B^*(t)$ of a Bézier curve defined by (1) for $\forall t \in R$ and pedal point p is*

$$B^*\left[B, p\right](t) = p + \frac{\langle \sum\limits_{j=0}^{n} B_j^n\left(t\right) P_j - p, \sum\limits_{i=0}^{n-1} B_i^{n-1}(t) J\triangle P_i\rangle}{\sum\limits_{j,i=0}^{n-1} B_j^{n-1}(t) B_i^{n-1}(t)\langle\triangle P_j, \triangle P_i\rangle} \sum_{k=0}^{n-1} B_k^{n-1}(t) J\triangle P_k$$

[1].

Remark 7. *The pedal curve couple $B^*(t)$ of a Bézier curve which is defined by (1) and pedal point p is*

$$B^*\left[B,p\right](0) = p + \frac{\langle P_0 - p, J\triangle P_0\rangle}{\langle \triangle P_0, \triangle P_0\rangle} J\triangle P_0$$

at $t = 0$ [1].

Remark 8. *The pedal curve couple $B^*(t)$ of a Bézier curve which is defined by (1) and pedal point p is*

$$B^*\left[B,p\right](1) = p + \frac{\langle P_n - p, J\triangle P_{n-1}\rangle}{\langle \triangle P_{n-1}, \triangle P_{n-1}\rangle} J\triangle P_{n-1}$$

at $t = 1$ [1].

Main Results

In this section, we give the curvature of the pedal curve of a planar Bézier curve.

Theorem 7. *The curvature of the pedal curve $B^*(t)$ of a Bézier curve defined by (1) for $\forall t \in R$ and pedal point p is given by the following equation:*

$$\kappa^*_{B^*[B,p]}(t) = \frac{\begin{gathered} n\langle \sum_{i=0}^{n} B_i^n(t) P_i - p, \sum_{j=0}^{n-1} B_j^{n-1}(t) J\triangle^1 P_j\rangle \Bigg(\sum_{i,j=0}^{n-1} B_i^{n-1}(t) B_j^{n-1}(t) \\ \langle \triangle^1 P_i, \triangle^1 P_j\rangle\Big) + 2(n-1) \sum_{i=0}^{n-2} B_i^{n-2}(t) \sum_{j=0}^{n-1} B_j^{n-1}(t) \langle \triangle^2 P_i, J\triangle^1 P_j\rangle \\ \cdot \left(\sum_{i,j=0}^{n} B_i^n(t) B_j^n(t) \langle P_i, P_j\rangle - 2\sum_{i=0}^{n} B_i^n(t)\langle P_i, p\rangle + \|p\|^2 \right) \end{gathered}}{\begin{gathered} (n-1)\sum_{i=0}^{n-2} B_i^{n-2}(t) \sum_{j=0}^{n-1} B_j^{n-1}(t)\langle \triangle^2 P_i, J\triangle^1 P_j\rangle \\ \cdot \left(\sum_{i,j=0}^{n} B_i^n(t) B_j^n(t) \langle P_i, P_j\rangle - 2\sum_{i=0}^{n} B_i^n(t)\langle P_i, p\rangle + \|p\|^2 \right)^{\frac{3}{2}} \end{gathered}}$$

Proof. We can rewrite the equation (5) by using the equation (4) as follows:

$$B^*\left[B,p\right](t) = p + \langle B(t) - p, N(t)\rangle N(t). \tag{6}$$

By using the definition of curvature, the following equation is handled:

$$\kappa^*_{B^*[B,p]}(t) = \frac{\langle (B^*[B,p](t))'', J(B^*[B,p](t))'\rangle}{\|(B^*[B,p](t))'\|^3}. \tag{7}$$

After using the equations (2), (3) and (6) in (7), the proof is completed. ☐

Conclusion

In this work, the curvature of the pedal curve of a Bézier curve in Euclidean 2−space is studied.

References

[1] A. Y. Ceylan and M. Kara, On pedal and contrapedal curves of Bézier curves, Konuralp Journal of Mathematics, 9(2) (2021), 217–221.

[2] A. Gray, E. Abbena and S. Salamon, Modern differential geometry of curves and surfaces with mathematica, Chapman and Hall/CRC: Boca Raton, FL, USA, 2016.

[3] Y. Li and D. Pei, Pedal curves of frontals in the Euclidean plane, Math. Meth. Appl. Sci. 41 (2018), 1988–1997.

[4] D. Marsh, Applied geometry for computer graphics and CAD, London: Springer; 2006.

[5] J. Snchez-Reyes, p-Bézier curves, spirals, and sectrix curves, Computer Aided Geometric Design, 19(6) (2002), 445–464.

[6] O. O. Tuncer, H. Ceyhan, İ. Gök and F. N. Ekmekci, Notes on pedal and contrapedal curves of fronts in the Euclidean plane, Math. Meth. Appl. Sci., 41 (2018), 5096–5111.

[7] K. Ueda, A sequence of Bézier curves generated by successive pedal-point constructions. In: Le Mhaut, A., Rabut, C., Schumaker, L. (Eds.), Curves and surfaces with applications in CAGD. Vanderbilt University Press, (1997), 427–434.

[8] K. Ueda, Pedal curves and surfaces. In: Lyche, T., Schumaker, L. (Eds.), Mathematical methods for curves and surfaces. Vanderbilt University Press, (2000), 497–507.

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Joint model selection for Hidden Markov models with exponential family observations

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Hidden Markov models (HMMs) provide a useful method to characterize trajectories using underlying state transition patterns. HMMs assume that an unobserved sequence governs the observed sequence (the emissions), and that the hidden chain has the Markov property instead of the observed one. In this paper, we develop a model for HMMs when emissions are governed by an exponential family distribution, and extend it to incorporate covariates using generalized linear models (GLMs). We call it HMM-GLM, and propose a joint model selection method. The proposed selection criterion is tailored for HMM-GLM aiming at a more accurate approximation of the Kullback-Leibler divergence; we show that it provides an improvement over the widely-used AIC and BIC, especially for the more difficult case of small to medium sample sizes.

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On radical ideals associated with an ideal in a commutative ring

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This study focuses on radical ideals associated with an ideal in a commutative ring. Some properties of radical ideals associated with an ideal are examined and its examples are given. The connections between radical ideals and radicals associated with an ideal in a commutative ring are proved.

2010 Mathematics Subject Classifications: 16N40, 16N60
Keywords: Prime ideal, Radical ideal

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References

[1] D. D. Anderson and M. Bataineh, Generalizations of prime ideals, Communications in Algebra 36 (2008), 686–696.

[2] D. D. Anderson and E. Smith, Weakly prime ideals, Houston Journal of Mathematics 29 (2003), 831–840.

[3] M. F. Atiyah and I. G. MacDonald, Introduction to Commutative Algebra, London, U.K: Addison-Wesley; 1969.

[4] J. Dauns, Prime modules, J. Reine Angew Math. 298 (1978), 156–181.

[5] M. Ebrahimpour and R. Nekooei, On generalizations of prime ideals, Communications in Algebra 40 (2012), 1268–1279.

[6] J. Jenkins and P.F. Smith, On the prime radical of a module over a commutative ring, Comm. Algebra 20 (1992), 3593–3602.

[7] I. Kaplansky, Commutative rings, Chicago: University of Chicago; 1974.

[8] D. Kirby, Closure operations on ideals and submodules, J. London. Math. Soc. 44 (1969), 283–291.

[9] D. Kirby, Components of ideals in a commutative ring, Ann. Mat. Pure Appl. 4 (1966), 109–125.

[10] T. Y. Lam, A First Course in Noncommutative Rings, New York: Springer; 2001.

[11] T. Y. Lam and M. L. Reyes, A prime ideal principle in commutative algebra, Journal of Algebra 319 (2008), 3006–3027.

[12] O. Öneş, On prime ideals related to an ideal in a commutative ring, Gazi University Journal of Science Part A: Engineering and Innovation 8(4) (2021), 451–458.

[13] O. Öneş, A generalization of prime ideals in a commutative ring, Proceedings Book of the 2nd Mediterranean International Conference of Pure and Applied Mathematics and Related Areas (MICOPAM2019), Evry, Paris-France, (2019), 124–126.

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Highly accurate implicit schemes for the numerical computation of derivatives of the solution to heat equation

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Recently, hexagonal grid approximation of the solution to the first type boundary value problem of the heat equation on special polygons was given in [1]. Second order accurate implicit method for the approximation of the spatial derivatives of the solution of this problem on a rectangle by using hexagonal grids were obtained in [2]. In this study for the computation of the first order spatial derivatives and second order mixed derivatives involving the time derivative of the solution, implicit methods on hexagonal grids with $O(h^4+\tau)$ order of accuracy are given. Here, $h, \frac{\sqrt{3}}{2}h, \tau$ are the step sizes in x_1, x_2 (space) and t (time) variables, respectively. Numerical results are also presented.

2010 Mathematics Subject Classifications: 65M06, 65M12, 65M22
Keywords: Implicit methods, Hexagonal grids, Approximation of derivatives, Heat equation

References

[1] S. C. Buranay, N. Arshad, Hexagonal grid approximation of the solution of the heat equation on special polygons, Advances in Difference Equations 2020:309 (2020), 1-24.

[2] S. C. Buranay, A. H. Matan, N. Arshad, Two stage implicit method on hexagonal grids for approximating the first derivatives of the solution to the heat equation, Fractal Fract. 5(19) (2021), 1-26.

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On Fibonacci type numbers and polynomials derived from homogeneous linear recurrence relation

Yagmur Cetin [1], *Yilmaz Simsek* [2]

The main motivation source of this study, especially the homogeneous linear recurrence relation in the following Exercise 7.7.1 of the Charalambides's book (Ch. A. Charalambides, Enumerative Combinatorics, New York: Chapman & Hall/Crc Press Company; 2002), has an important place:

Let y_n be the number of n-permutations of set $\{0,1,2\}$ with the repetition and the restriction that no two zeros and no two ones are consecutive. Show that

$$y_{n+2} = 2y_{n+1} + y_n \tag{1}$$

with $y_0 = 1$and $y_1 = 3$ and $n \in \mathbb{N}$. Further, show that unique solution of this recurrence relation is given by

$$y_n = \frac{1}{2}\left\{\left(1+\sqrt{2}\right)^{n+1} + \left(1-\sqrt{2}\right)^{n+1}\right\}. \tag{2}$$

Dating back to the time of P. S. Laplace, who first introduced generating functions in the form of power series, these functions make great contributions and conveniences for a unified and modified solutions of combinatorial and probabilistic problems and also other areas involving mathematical physics and applied mathematics.

It is also well known that generating functions are used in solving the homogeneous or the complete linear recurrence relation of order k with $k \in \mathbb{N}$.

With the help of generating functions, this homogeneous linear recurrence relation is solved. Some interesting properties of related generating functions and the numbers y_n are investigated.

2010 Mathematics Subject Classifications: 05A15, 05A19, 11B39, 11B83
Keywords: Generating functions, Fibonacci type numbers and polynomials, Pell numbers, Combinatorial numbers

Introduction

Ozdemir and Simsek [4] gave the following generating functions for a new family of two variable polynomials $\mathcal{G}_n(x,y,k,m,v)$, which are generalization and unification of the Fibonacci polynomials and numbers, the Lucas polynomials and numbers, the Pell polynomials and numbers, the Chebyshev polynomials, the Jacobsthal polynomials, the Vieta-Lucas polynomials, the Humbert polynomials, the Geganbauer polynomials, etc:

$$\frac{1}{1-x^k t - y^m t^{m+n}} = \sum_{n=0}^{\infty} \mathcal{G}_n(x,y,k,m,v)\, t^n. \tag{3}$$

Ozdemir, Simsek, and Milovanović [5] also modified the equation (3). They gave many interesting and novel formulas involving not only some new generating functions for special numbers and polynomials, but also some explicit formulas for these numbers and polynomials.

Recently there are many papers and books for the Fibonacci type polynomials and numbers (*cf.* [1]-[6]).

Generating Functions For the numbers y_n

In this section, by using (1), we prove that

$$\frac{1+t}{1-2t-t^2}=\sum_{n=0}^{\infty} y_n t^n. \tag{4}$$

With the aid of (4), we arrive at the equation (2).

Conclusion

In this work, we gave a solution of the homogeneous linear recurrence relation in (1), given by Charalambides [1, Exercise 7.7.1, p. 267]. By using (1), we also found generating function for the numbers y_n, which is given by the equation (4).

Our future plan is to give some properties of the numbers y_n with the aid of the equation (4).

References

[1] Ch. A. Charalambides, Enumerative combinatorics, New York: Chapman & Hall/Crc Press Company; 2002.

[2] T. Koshy, Fibonacci and Lucas numbers with applications, New York: Wiley-Interscience; 2001.

[3] T. Koshy, Pell and Pell–Lucas numbers with applications, New York: Springer-Verlag; 2014.

[4] G. Ozdemir and Y. Simsek, Generating functions for two-variable polynomials related to a family of Fibonacci type polynomials and numbers, Filomat 30(4) (2016), 969–975.

[5] G. Ozdemir, Y. Simsek and G. V. Milovanović, Generating functions for special polynomials and numbers including Apostol-type and Humbert-type polynomials, Mediterr. J. Math. 14 (2017), 1–17; Article ID: 117.

[6] Y. Simsek, Miscellaneous formulae for the certain class of combinatorial sums and special numbers, Bulletin T. CLIV de l'Académie serbe des sciences et des arts 46 (2021), 151–167.

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Optimal control for multi-treatment of Dual listeriosis infection

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From the rising cases of high hospitalization versa-vise incessant fatality rates and the close affinity of listeriosis with HIV/AIDS infection, which often emanates from food-borne pathogens associated with listeria monocytogenes infection, this present paper seek and formulated as penultimate model, an 8-Dimensional classical mathematical equations which directly accounted for the biological interplay of dual listeriosis virions with dual set of population (human and animals). The model was studied under multiple chemotherapies (trimethoprimsulphamethoxazole with a combination of penicillin or ampicillin and/or gentamicin) and then transformed to an optimal control problem. The study explored classical Pontryagins Maximum Principle for the model optimality control system. Correlating the derived model with clinical implications, numerical validity of the model was conducted. Results indicated that under cogent and adherent to specify multiple chemotherapies, maximal recovery of both human and animal infected population was tremendously achieved with consequent rapid decline to near zero infection growth. The study therefore suggests further articulation of more treatment factors and early application at onset of infection for a visible elimination of listeriosis infection.

2010 Mathematics Subject Classifications: 93A30, 93C15, 65L07, 65K15, 49J15

Keywords: Listeria-monocytogenes, listeriosis, trimethoprim-sulphamethoxazole, maximal-recovery, gastrointestinal-infection, multiple-chemotherapy

References

[1] S. Osman, O. D. Makinde and D. M. Theuri, Stability analysis and modeling of listeriosis dynamics in human and animal populations, Global Journal of Pure and Applied Math. 14(1), (2018), 115-138.

[2] J. Pichler, G. Appl, A. Pietka and F. Allerberger , Lesson to be learned from an outbreak of foodborne listeriosis, Austria 2009-2010. Food Production Trends, 31(5), (2011), 268-273.

[3] M. I. Rossi, A. Paiva, M. Tornese, S. Chianelli and A. Troncoso, Listeria monocytogenes outbreaks: a review of the routes that favor bacterial presence, Revista chilena de infectologia: organo official de la sociedad chilena de infectologia, 35(5), (2008), 328-335.

[4] E. B. Bassey, Dynamic optimal control model for dual-pair treatment functions of dual delayed HIV-pathogen infections, Journal of Mathematical Sciences: Advances and Applications, 51(1), (2018), 1-50.

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Notes on statistical evaluations of the Bezier curves constructed on deformation hands

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A bezier curve is one of the widely used topic in computer graphics, engineering, medicine, mathematics, technology and science because of applications in many real-world problems. In recent years, bezier curves are used in medical studies as a tool to define a mathematical model for the relevant applications such as facial expression recogniton, modeling deformed versus healthy hands, imaging of ECG waveform [1, 3, 4]. In the initial study, our focus was providing mathematical modeling for the deformed hands and healthy hands using Beizer curves. In this study, we further develop our previous study and expand our results by applying statistical methods. We then make inferences from these mathematical models developed by the cubic Béizer curve.

References

[1] I. Kucukoglu, B. Simsek, Y. Simsek, Multidimensional Bernstein polynomials and Bezier curves: analysis of machine learning algorithm for facial expression recognition based on curvature, Appl Math Comput. 344, (2019), 150-162.

[2] V. Chutchavong, K. Nualon, O. Sangaroon, K. Janchitrapongvej, A mathematical model for ECG waveform using rational Bézier curves and Bernstein polynomials, 11th International Conference on Electrical Engineering/Electronics, Computer, Telecommunications and Information Technology (ECTI-CON), 2014.

[3] B. Simsek, Development of deformed hand model with Bezier curves, Master Thesis Akdeniz University, 2016.

[4] B. Simsek, A. Yardimci, Using Bezier curves in medical applications, Filomat 30(4), (2016), 937-943.

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Higher order general convex functions and variational inequalities

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Variational inequalities can be viewed as novel and significant generalizations of the variational principles, the origin of which can be traced back to Euler, Lagrange, Newton and Bernoulli brothers. A wide class of unrelated problems arising in various fields of pure and applied sciences are being studied in the general unified framework. We define and consider some new concepts of the convex functions with respect to an arbitrary function, called the general convex functions. Some properties of the general functions are investigated under suitable conditions. It is shown that the optimality conditions of the general convex functions are characterized by a class of variational inequalities. This result motivated us to introduce the higher order general variational inequality. Variational inequalities and complimentarily are obtained as special cases of higher order general variational inequality. It is observed projection methods and their variant forms can not used to find the approximate solution of the variational inequalities. To overcome these deficiencies, we apply the auxiliary principle technique to suggest several implicit and explicit inertial methods for solving higher order general variational inequalities. Convergence analysis of the proposed methods are investigated under suitable conditions. Parallelogram laws, which can be used to characterize the inner product and Banach spaces, are obtained as applications of higher order general convex functions. These parallelogram laws have applications in prediction theory and stochastic analysis. Some important and interesting special cases also discussed. Results obtained in this paper can be viewed as refinement and improvement of previously known results.

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On the search for universal laws for protein copy number fluctuations

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Modeling biological phenomena can be complicated because of potential differences in biophysical details in different contexts. Despite that variation, it is of interest to seek out general laws that may underlie certain processes. In this study, we are interested in the possible universality of protein copy number fluctuations. We seek a model to describe those fluctuations universally. We compare the fits of several models to those fluctuations using the Kullback-Leibler divergence as a measure of closeness. We consider the lognormal, generalized inverse Gaussian, and Fréchet models because they arise from different underlying mechanisms. The lognormal results from a large number of multiplicative process for exponential growth; the generalized inverse Gaussian arises as a first passage time for diffusions; and the Fréchet is an extreme value distribution. Our empirical findings show that the lognormal gives the best fit, and we discuss implications for underlying biophysical processes.

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Hybrid operators for solving the weakly singular Volterra integral equations: Theoretical analysis

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Sara Safarzadeh Falahhesar [1,c]

Our primary aim in this reaserch is the numerical solution of linear Volterra integral equations of second kind with weak singular kernels, particularly kernels of the form $(x-t)^{-v}\,\widetilde{K}\,(x,t)\,,\ 0<v<1$, where $\widetilde{K}$ is a smooth function. It is well known that the solutions in general possess singularities at the initial point, see [1]. Hence, we propose hybrid positive linear operators defined on certain subintervals of the domain $[0,1]$ by using the classical Bernstein-Kantorovich and Modified Bernstein-Kantorovich operators given in [2]. Further, we develop a combined method which uses the proposed hybrid operators and approximates the solution on the constructed subintervals. Theoretical analysis on asymptotic rate of convergence and error analysis are given.

2010 Mathematics Subject Classifications: 41A36, 41A25, 45E10, 45D05
Keywords: Weakly singular Volterra integral equations, Positive linear operators, Asymptotic rate of convergence, Error analysis

References

[1] R. K. Miller, A. Feldstein, Smoothness of solutions of Volterra integral equations with weakly singular kernels, SIAM J. Math. Anal. 2(2) (1971), 242-258.

[2] M. A. Özarslan, O. Duman, Smoothness properties of Modified Bernstein-Kantorovich operators, Numer. Funct. Anal. Optim. 37(1) (2016), 92-105.

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Some new inequalities for Berezin radius

Hamdullah Başaran [1] , *Verda Gürdal* [2]

The Berezin transform $\widetilde{A}$ and the Berezin radius of an operator A on the reproducing kernel Hilbert space over some set Ω with the reproducing kernel k_ξ are defined, respectively, by $\widetilde{A}(\xi) = \left\langle A\widehat{k}_\xi, \widehat{k}_\xi \right\rangle$, $\xi \in \Omega$ and $\mathrm{ber}(A) := \sup_{\xi\in\Omega} \left|\widetilde{A}(\xi)\right|$. We use the refinements of the Cauchy-Schwarz inequality via contractions to prove some new inequalities for Berezin norm and Berezin radius of reproducing kernel Hilbert space operators.

2010 Mathematics Subject Classifications: 47B32, 47B06, 47B34
Keywords: Cauchy-Schwarz inequality, Berezin transform, Berezin radius

References

[1] H. Başaran and M. Gürdal, Berezin number inequalities via Young inequality, Honam Mathematical J., 43(3) (2021), 523–537.

[2] H. Başaran, M. Gürdal and A. N. Güncan, Some operator inequalities associated with Kantorovich and Hölder-McCarthy inequalities and their applications, Turkish J. Math., 43(1) (2019), 523–532.

[3] F. A. Berezin, Covariant and contravariant symbols for operators, Math. USSR-Izv., 6 (1972), 1117–1151.

[4] M. T. Garayev, H. Guedri, M. Gürdal and G. M. Alsahli, On some problems for operators on the reproducing kernel Hilbert space, Linear Multilinear Algebra, 69(11) (2021), 2059–2077.

[5] M. Sababheh, H. R. Moradi, Z. Heydarbeygi, Buzano, Krein and Cauchy-Schwarz inequalities, arXiv:2010.02464v1 [math.FA] 6 Oct 2020.

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Hybrid operators for solving weakly singular Volterra integral equations: Numerical analysis

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Numerical solution of Fredholm and Volterra integral equations with continuous and square integrable kernels by using Modified Bernstein-Kantorovich Operators $K_{n,\alpha}$ where $n \in \mathbb{N}$ and $\alpha > 0$ is constant given [1] were studied in [2]. We consider linear Volterra integral equations of second kind with weakly singular kernels $(x-t)^{-v}\,\widetilde{K}(x,t)$, $0 < v < 1$, where $\widetilde{K}$ is a smooth function. For the approximate solutions two algorithms are developed depending on α, and given through a combined method of hybrid operators of Bernstein-Kantorovich and Modified Bernstein-Kantorovich operators defined on certain subintervals of $[0,1]$. Furthermore, examples are conducted from the literature and the results show the applicability, accuracy and the numerical stability of the given algorithms. Additionally, these algorithms are applied on first kind integral equations by first utilizing a regularization.

2010 Mathematics Subject Classifications: 41A36, 45E10, 45D05, 65D30, 65D15

Keywords: Weakly singular Volterra integral equations, linear positive operators, Numerical integration, algorithms

References

[1] M. A. Özarslan, O. Duman, Smoothness properties of Modified Bernstein-Kantorovich operators. Numer. Funct. Anal. Optim. 37(1) (2016), 92-105.

[2] S. C. Buranay, M. A. Özarslan, S. S. Falahhesar, Numerical solution of the Fredholm and Volterra integral equations by using Modified Bernstein-Kantorovich Operators, Mathematics 9, 1193, (2021), 1-32.

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Pidduck polynomials associated with the Umbral algebra

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In this work, we study the Pidduck polynomials belonging to the family of the Sheffer polynomials. By using the methods of the umbral algebra and the umbral calculus, we obtain some operator actions of these polynomials. Moreover, we give some identities and relationship between these polynomials and some other special polynomials.

2010 Mathematics Subject Classifications: 05A40, 11B83, 05A15
Keywords: Umbral algebra, generating fuctions, special polynomials

References

[1] R. P. Boas and R. C. Buck, Polynomial expansions of analytic functions, 2nd print., corr. New York: Academic Press; p. 38, 1964.

[2] R. Dere and Y. Simsek, Hermite base Bernoulli type polynomials on the umbral algebra, Russ. J. Math. Phys. 22(1) (2015), 1–5.

[3] R. Dere, Y. Simsek and H. M. Srivastava, A unified presentation of three families of generalized Apostol type polynomials based upon the theory of the umbral calculus and the umbral algebra, J. Number Theory 133 (2013), 3245–3263.

[4] D. S. Kim, T. Kim, D. V. Dolgy and S.-H. Rim, Some new identities of Bernoulli, Euler and Hermite polynomials arising from umbral calculus, Adv. Difference Equ. 73 (2013), 2013.

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